OPTIONAL PURE / VECTORS & MATRICES

Vector Product

Worked solutions 13 questions · 111 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · OCR A Y545 · AQA 7367 · OCR A H245 · OCR MEI H645 CIE 9231

Questions

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1. **(a)** The vectors

$$\mathbf{p} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} 2 \\ m \\ -1 \end{pmatrix}$$

span a plane with normal vector

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Given that $\mathbf{q} \times \mathbf{p}$ is in the same direction as $\mathbf{n}$, find m and the positive constant λ such that

$$\mathbf{q} \times \mathbf{p} = \lambda \mathbf{n}$$ **[3]**

(b) Using $\mathbf{q} \times \mathbf{p}$, determine the acute angle between $\mathbf{p}$ and $\mathbf{q}$. **[3]**

Solution

(a) Taking

$$\mathbf{q} = \begin{pmatrix} 2 \\ m \\ -1 \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

we find the vector product:

$$\mathbf{q} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & m & -1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= \mathbf{i}(2m + 3) - \mathbf{j}(4 + 1) + \mathbf{k}(6 - m)$$

$$= \begin{pmatrix} 2m + 3 \\ -5 \\ 6 - m \end{pmatrix}$$

Since $\mathbf{q} \times \mathbf{p}$ is in the same direction as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

we can write

$$\mathbf{q} \times \mathbf{p} = \lambda \mathbf{n} = \begin{pmatrix} \lambda \\ -\lambda \\ \lambda \end{pmatrix}$$

So

$$\begin{pmatrix} 2m + 3 \\ -5 \\ 6 - m \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \\ \lambda \end{pmatrix}$$

Equating components:

$$-5 = -\lambda \quad \Longrightarrow \quad \lambda = 5$$

Then

$$2m + 3 = 5 \quad \Longrightarrow \quad 2m = 2 \quad \Longrightarrow \quad m = 1$$

Hence

$$m = 1, \qquad \lambda = 5$$

(b) Using $m = 1$:

$$\mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Also

$$\mathbf{q} \times \mathbf{p} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$$

so

$$\begin{aligned} |\mathbf{q} \times \mathbf{p}| &= \sqrt{5^2 + (-5)^2 + 5^2} \\ &= \sqrt{75} \\ &= 5\sqrt{3} \end{aligned}$$

Now

$$|\mathbf{p}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

$$|\mathbf{q}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$

Using

$$|\mathbf{q} \times \mathbf{p}| = |\mathbf{q}|\,|\mathbf{p}| \sin\theta$$

gives

$$\begin{aligned} \sin\theta &= \frac{|\mathbf{q} \times \mathbf{p}|}{|\mathbf{q}|\,|\mathbf{p}|} \\ &= \frac{5\sqrt{3}}{\sqrt{6}\,\sqrt{14}} \\ &= \frac{5\sqrt{3}}{\sqrt{84}} \\ &= \frac{5}{2\sqrt{7}} \end{aligned}$$

So the acute angle is

$$\theta = \arcsin\left(\frac{5}{2\sqrt{7}}\right) \approx 70.9^\circ$$

2. Relative to the origin O, the points A, B and C have position vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

(a) Find, in its simplest exact form, the area of triangle ABC. **[4]**

(b) By calculating each side, show that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad \textbf{[3]}$$

Solution

(a) Find two sides of the triangle:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$$

The area of triangle ABC is

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

The vector product is

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -1 & 4 \\ -1 & -4 & 1 \end{vmatrix} \\ &= \big((-1) \times 1 - 4 \times (-4)\big)\mathbf{i} - \big((-4) \times 1 - 4 \times (-1)\big)\mathbf{j} + \big((-4) \times (-4) - (-1) \times (-1)\big)\mathbf{k} \\ &= \begin{pmatrix} 15 \\ 0 \\ 15 \end{pmatrix} \end{aligned}$$

so

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{15^2 + 0^2 + 15^2} = \sqrt{450} = 15\sqrt{2}$$

Hence

$$\text{Area} = \frac{1}{2} \times 15\sqrt{2} = \frac{15\sqrt{2}}{2}$$

(b) For the left-hand side, first find $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 0 \\ -1 & 1 & 4 \end{vmatrix} = \begin{pmatrix} 8 \\ -12 \\ 5 \end{pmatrix}$$

Then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -12 & 5 \\ 2 & -2 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ 2 \\ 8 \end{pmatrix}$$

For the right-hand side, first find $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 4 \\ 2 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}$$

Then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 0 \\ 9 & 9 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$$

Since

$$\begin{pmatrix} -2 \\ 2 \\ 8 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$$

it follows that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

3. You are given the fixed points $A(1,0,1)$ and $C(4,0,2)$, and the variable point $B(t,2,5-t)$, where t is a real parameter.

(a) Using only the geometrical properties of the vector product, explain why the statement "$\overrightarrow{AB}\times\overrightarrow{AC}=\mathbf{0}$" is false for all values of t. **[2]**

(b) (i) Use the vector product to find an expression, in terms of t, for the area of triangle ABC. **[4]**

(ii) Hence determine the value of t for which the area of triangle ABC is a minimum. **[2]**

Solution

(a)

$$\overrightarrow{AB}=\begin{pmatrix}t-1\\2\\4-t\end{pmatrix},\qquad \overrightarrow{AC}=\begin{pmatrix}3\\0\\1\end{pmatrix}$$

Geometrically, $\overrightarrow{AB}\times\overrightarrow{AC}=\mathbf{0}$ only if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel, or if one of them is the zero vector.

Here $\overrightarrow{AC}\neq\mathbf{0}$, and $\overrightarrow{AB}$ can never be parallel to $\overrightarrow{AC}$ because

$$y\text{-component of }\overrightarrow{AB}=2,\qquad y\text{-component of }\overrightarrow{AC}=0$$

So $\overrightarrow{AB}$ is not a scalar multiple of $\overrightarrow{AC}$ for any real t.

Therefore $\overrightarrow{AB}\times\overrightarrow{AC}\neq\mathbf{0}$ for all values of t, so the statement is false for all t.

(b) (i) First find the two sides of the triangle:

$$\overrightarrow{AB}=\begin{pmatrix}t-1\\2\\4-t\end{pmatrix},\qquad \overrightarrow{AC}=\begin{pmatrix}3\\0\\1\end{pmatrix}$$

Now calculate the vector product:

$$\begin{aligned}\overrightarrow{AB}\times\overrightarrow{AC}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\t-1&2&4-t\\3&0&1\end{vmatrix}\\&=\mathbf{i}\big(2\times1-(4-t)\times0\big)-\mathbf{j}\big((t-1)\times1-(4-t)\times3\big)+\mathbf{k}\big((t-1)\times0-2\times3\big)\\&=2\mathbf{i}-\mathbf{j}\big(t-1-12+3t\big)-6\mathbf{k}\\&=2\mathbf{i}-(4t-13)\mathbf{j}-6\mathbf{k}\\&=\begin{pmatrix}2\\13-4t\\-6\end{pmatrix}\end{aligned}$$

The area of triangle ABC is

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

So

$$\begin{aligned}\text{Area} &= \frac{1}{2}\sqrt{2^2 + (13-4t)^2 + (-6)^2} \\ &= \frac{1}{2}\sqrt{4 + (13-4t)^2 + 36} \\ &= \frac{1}{2}\sqrt{(13-4t)^2 + 40}\end{aligned}$$

(ii) To minimise the area, it is enough to minimise

$$(13-4t)^2 + 40$$

Since $(13-4t)^2 \geq 0$, its smallest value is 0, which occurs when

$$13 - 4t = 0$$

Hence

$$t = \frac{13}{4}$$

4. With respect to a fixed origin O, the points A and B have coordinates $(1, 1, 0)$ and $(p, 3, 2)$ respectively, where p is a constant.

For each of the following, determine the possible values of p for which

(a) $\overrightarrow{AB}$ makes an angle of $60°$ with the positive y-axis. **[3]**

(b) $\overrightarrow{OA} \times \overrightarrow{AB}$ is parallel to

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$ **[3]**

(c) the area of triangle OAB is $\frac{3}{2}$. **[3]**

Solution

(a) First find the vector

$$\overrightarrow{AB} = \begin{pmatrix} p-1 \\ 3-1 \\ 2-0 \end{pmatrix} = \begin{pmatrix} p-1 \\ 2 \\ 2 \end{pmatrix}$$

The positive y-axis has direction vector

$$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Using the scalar product:

$$\cos 60° = \frac{\overrightarrow{AB} \cdot \mathbf{j}}{\left|\overrightarrow{AB}\right||\mathbf{j}|}$$

So

$$\frac{1}{2} = \frac{2}{\sqrt{(p-1)^2 + 2^2 + 2^2} \times 1}$$

$$\sqrt{(p-1)^2 + 8} = 4$$

$$(p-1)^2 + 8 = 16$$

$$(p-1)^2 = 8$$

Hence

$$p - 1 = \pm 2\sqrt{2}$$

Therefore the possible values of p are

$$p = 1 \pm 2\sqrt{2}$$

(b)

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \overrightarrow{AB} = \begin{pmatrix} p-1 \\ 2 \\ 2 \end{pmatrix}$$

Now calculate the vector product:

$$\overrightarrow{OA} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ p-1 & 2 & 2 \end{vmatrix}$$

$$= \mathbf{i}(1 \times 2 - 0 \times 2) - \mathbf{j}(1 \times 2 - 0 \times (p-1)) + \mathbf{k}(1 \times 2 - 1 \times (p-1))$$

$$= \begin{pmatrix} 2 \\ -2 \\ 3-p \end{pmatrix}$$

For this to be parallel to $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, we must have

$$\begin{pmatrix} 2 \\ -2 \\ 3-p \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Comparing first components:

$$2 = 2\lambda \quad \Longrightarrow \quad \lambda = 1$$

Then comparing third components:

$$3 - p = 1$$

So

$$p = 2$$

(c) The area of triangle OAB is

$$\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$$

Since

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

we have

$$\overrightarrow{OA} \times \overrightarrow{OB} = \overrightarrow{OA} \times (\overrightarrow{OA} + \overrightarrow{AB})$$

$$= \overrightarrow{OA} \times \overrightarrow{OA} + \overrightarrow{OA} \times \overrightarrow{AB}$$

$$= \mathbf{0} + \overrightarrow{OA} \times \overrightarrow{AB}$$

$$= \begin{pmatrix} 2 \\ -2 \\ 3-p \end{pmatrix}$$

Therefore

$$\text{Area} = \frac{1}{2}\sqrt{2^2 + (-2)^2 + (3-p)^2} = \frac{3}{2}$$

$$\frac{1}{2}\sqrt{8 + (3-p)^2} = \frac{3}{2}$$

$$\sqrt{8 + (3-p)^2} = 3$$

$$8 + (3-p)^2 = 9$$

$$(3-p)^2 = 1$$

So

$$3 - p = \pm 1$$

Hence the possible values of p are

$$p = 2 \text{ or } p = 4$$

5. **(a)** Let

$$\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix}$$

The area of the parallelogram formed by $\mathbf{p}$ and $\mathbf{q}$ is $\sqrt{2}$.
It is also given that $\mathbf{p} \times \mathbf{q}$ is perpendicular to

$$\begin{pmatrix} 1 \\ 3 \\ m \end{pmatrix}$$

Find x and m. **[6]**

(b) Hence, using $|\mathbf{p} \times \mathbf{q}|$, determine the acute angle between $\mathbf{p}$ and $\mathbf{q}$. **[3]**

Solution

(a) For two vectors, the area of the parallelogram they form is

$$|\mathbf{p} \times \mathbf{q}|$$

First find $\mathbf{p} \times \mathbf{q}$:

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ x & 0 & 1 \end{vmatrix} \\ &= \mathbf{i}(1 \times 1 - 2 \times 0) - \mathbf{j}(1 \times 1 - 2x) + \mathbf{k}(1 \times 0 - 1 \times x) \\ &= \mathbf{i} - \mathbf{j}(1 - 2x) - x\mathbf{k} \\ &= \begin{pmatrix} 1 \\ 2x - 1 \\ -x \end{pmatrix} \end{aligned}$$

The area is $\sqrt{2}$, so

$$|\mathbf{p} \times \mathbf{q}| = \sqrt{2}$$

$$\therefore \ |\mathbf{p} \times \mathbf{q}|^2 = 2$$

So

$$\begin{aligned} 1^2 + (2x - 1)^2 + (-x)^2 &= 2 \\ 1 + (4x^2 - 4x + 1) + x^2 &= 2 \\ 5x^2 - 4x &= 0 \\ x(5x - 4) &= 0 \end{aligned}$$

Hence

$$x = 0 \quad \text{or} \quad x = \frac{4}{5}$$

Now use the condition that $\mathbf{p} \times \mathbf{q}$ is perpendicular to $\mathbf{i} + 3\mathbf{j} + m\mathbf{k}$.

Perpendicular vectors have zero scalar product, so

$$\begin{pmatrix} 1 \\ 2x-1 \\ -x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ m \end{pmatrix} = 0$$

$$1 + 3(2x-1) + (-x)m = 0$$

$$1 + 6x - 3 - mx = 0$$

$$x(6-m) = 2$$

This shows $x \neq 0$, so the value $x = 0$ must be rejected.

Therefore

$$x = \frac{4}{5}$$

Substitute into $x(6-m) = 2$:

$$\frac{4}{5}(6-m) = 2$$

$$6 - m = \frac{5}{2}$$

$$m = 6 - \frac{5}{2} = \frac{7}{2}$$

Therefore

$$x = \frac{4}{5}, \qquad m = \frac{7}{2}$$

(b) Using

$$|\mathbf{p} \times \mathbf{q}| = |\mathbf{p}|\,|\mathbf{q}| \sin\theta$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$.

From part (a):

$$|\mathbf{p} \times \mathbf{q}| = \sqrt{2}$$

Also

$$|\mathbf{p}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

and with $x = \frac{4}{5}$:

$$|\mathbf{q}| = \sqrt{\left(\frac{4}{5}\right)^2 + 0^2 + 1^2}$$

$$= \sqrt{\frac{16}{25} + 1} = \sqrt{\frac{41}{25}} = \frac{\sqrt{41}}{5}$$

So

$$\sqrt{2} = \sqrt{6} \times \frac{\sqrt{41}}{5} \sin\theta$$

$$\sin\theta = \frac{\sqrt{2}}{\sqrt{6}(\sqrt{41}/5)}$$

$$= \frac{5\sqrt{2}}{\sqrt{246}}$$

$$= \frac{5}{\sqrt{123}}$$

Hence the acute angle is

$$\theta = \arcsin\left(\frac{5}{\sqrt{123}}\right) \approx 26.8^\circ$$

6. The point P has coordinates $P(2,-1,1)$ and the midpoint of PR is $M(1,0,2)$. The vector $\overrightarrow{PQ}$ is given by

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

(a) Use a vector product to show that the area of triangle PQR is $\sqrt{14}$. **[5]**

(b) Find a vector equation of the plane containing P, Q and R in the form $\mathbf{r} \cdot \mathbf{n} = k$. **[2]**

(c) Hence find the exact distance of the plane from the origin. **[2]**

Solution

(a) First find the coordinates of Q and R.

Using $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$:

$$\overrightarrow{OQ} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

so Q is the point $(3,1,1)$.

The midpoint of PR is $M(1,0,2)$, so

$$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OR}\right)$$

so

$$\overrightarrow{OR} = 2\overrightarrow{OM} - \overrightarrow{OP} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Now

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

Now calculate the vector product:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= \mathbf{i}(2 \times 2 - 0 \times 2) - \mathbf{j}\big(1 \times 2 - 0 \times (-2)\big) + \mathbf{k}\big(1 \times 2 - 2 \times (-2)\big)$$

$$= \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$$

Its magnitude is

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{4^2 + (-2)^2 + 6^2} = \sqrt{16 + 4 + 36} = \sqrt{56} = 2\sqrt{14}$$

The area of triangle PQR is

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

so

$$\text{Area} = \frac{1}{2}\left(2\sqrt{14}\right) = \sqrt{14}$$

as required.

(b) A normal vector to the plane is

$$\mathbf{n} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$$

which can be simplified to

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Using the point $P(2, -1, 1)$, with position vector $\mathbf{p}$:

$$k = \mathbf{p} \cdot \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 4 + 1 + 3 = 8$$

So the plane is

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8$$

(c) For a plane $\mathbf{r} \cdot \mathbf{n} = k$, the distance from a point with position vector $\mathbf{a}$ is

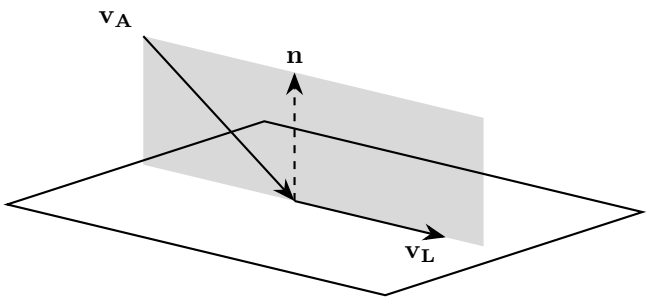
$$\text{dist} = \frac{|\mathbf{a} \cdot \mathbf{n} - k|}{|\mathbf{n}|}$$

Here the point is the origin, so $\mathbf{a} = \mathbf{0}$, and the plane is

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8$$

Therefore

$$\text{dist} = \frac{|0 - 8|}{\sqrt{2^2 + (-1)^2 + 3^2}} = \frac{8}{\sqrt{14}} = \frac{4\sqrt{14}}{7}$$



7. A small aircraft is landing in a field.
In a model for the landing, the aircraft travels in different straight lines before and after it lands, as shown in the diagram above.
The vector $\mathbf{v_A}$ is in the direction of travel of the aircraft as it approaches the field, and the vector $\mathbf{v_L}$ is in the direction of travel of the aircraft after it lands.
With respect to a fixed origin, the field is modelled as the plane with vector equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 10 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 20 \\ -1 \end{pmatrix}$$

and

$$\mathbf{v_A} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$$

(a) Show that the vector $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 20\mathbf{k}$ is normal to the field. **[2]**

(b) Show that

$$\mathbf{n} \times \mathbf{v_A} = \lambda \begin{pmatrix} 3 \\ 14 \\ -1 \end{pmatrix}$$

where λ is a constant to be determined. **[2]**

When the aircraft lands, it remains in contact with the field and travels in the direction $\mathbf{v_L}$.
The vector $\mathbf{v_L}$ lies in the same plane as both $\mathbf{v_A}$ and $\mathbf{n}$, as shown in the diagram above.

(c) Determine a vector which has the same direction as $\mathbf{v_L}$. **[3]**

Solution

(a) The direction vectors of the plane are $10\mathbf{i} - \mathbf{k}$ and $20\mathbf{j} - \mathbf{k}$. Taking the scalar product of $\mathbf{n}$ with each:

$$\begin{pmatrix} 2 \\ 1 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 0 \\ -1 \end{pmatrix} = 20 + 0 - 20 = 0$$

$$\begin{pmatrix} 2 \\ 1 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 20 \\ -1 \end{pmatrix} = 0 + 20 - 20 = 0$$

So $\mathbf{n}$ is perpendicular to two non-parallel vectors in the field, and hence $\mathbf{n}$ is normal to the field, as required.

(b) Working the vector product as a determinant:

$$\begin{aligned}\mathbf{n}\times\mathbf{v_A} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 20\\ 4 & -1 & -2\end{vmatrix}\\ &= \big(1\times(-2)-20\times(-1)\big)\mathbf{i}-\big(2\times(-2)-20\times 4\big)\mathbf{j}+\big(2\times(-1)-1\times 4\big)\mathbf{k}\\ &= \begin{pmatrix}18\\84\\-6\end{pmatrix}\\ &= 6\begin{pmatrix}3\\14\\-1\end{pmatrix}\end{aligned}$$

This is in the required form, with $\lambda = 6$.

(c) Since $\mathbf{v_L}$, $\mathbf{v_A}$ and $\mathbf{n}$ lie in the same plane, their scalar triple product is zero:

$$\mathbf{v_L}\cdot(\mathbf{n}\times\mathbf{v_A}) = 0$$

Using part (b), $\mathbf{v_L}$ is therefore perpendicular to $3\mathbf{i}+14\mathbf{j}-\mathbf{k}$. Also, the aircraft travels along the field after landing, so $\mathbf{v_L}$ is perpendicular to the normal $\mathbf{n}$.

A vector perpendicular to both is their vector product:

$$\begin{pmatrix}3\\14\\-1\end{pmatrix}\times\begin{pmatrix}2\\1\\20\end{pmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 14 & -1\\ 2 & 1 & 20\end{vmatrix} = \begin{pmatrix}281\\-62\\-25\end{pmatrix}$$

The aircraft keeps moving forwards, so $\mathbf{v_L}$ should make an acute angle with $\mathbf{v_A}$. Here

$$\begin{pmatrix}4\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}281\\-62\\-25\end{pmatrix} = 1124+62+50 = 1236 > 0$$

so the angle is acute, and a vector with the same direction as $\mathbf{v_L}$ is

$$\begin{pmatrix}281\\-62\\-25\end{pmatrix}$$

8. The points P, Q and R have position vectors $\mathbf{p} = 2\mathbf{i} - 2\mathbf{k}$, $\mathbf{q} = 3\mathbf{j} + 3\mathbf{k}$ and $\mathbf{r} = t\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively, relative to the origin O.

Determine the value(s) of t in each of the following cases.

(a) The line OR is parallel to a normal to the plane OPQ. **[3]**

(b) The volume of tetrahedron $OPQR$ is 9. **[4]**

Solution

(a) The plane OPQ contains the vectors

$$\mathbf{p} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

A normal to the plane is $\mathbf{p} \times \mathbf{q}$:

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 0 & 3 & 3 \end{vmatrix} \\ &= \mathbf{i}(0 \times 3 - (-2) \times 3) - \mathbf{j}(2 \times 3 - (-2) \times 0) + \mathbf{k}(2 \times 3 - 0 \times 0) \\ &= \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix} \\ &= 6\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

The line OR has direction vector

$$\mathbf{r} = \begin{pmatrix} t \\ -1 \\ 1 \end{pmatrix}$$

If OR is parallel to a normal to the plane, then $\mathbf{r}$ must be parallel to $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

So

$$\begin{pmatrix} t \\ -1 \\ 1 \end{pmatrix} \parallel \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Comparing components gives $t = 1$.

(b) The volume of tetrahedron $OPQR$ is

$$V = \frac{1}{6}|\mathbf{r} \cdot (\mathbf{p} \times \mathbf{q})|$$

From part (a):

$$\mathbf{p} \times \mathbf{q} = 6\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Also

$$\mathbf{r} = \begin{pmatrix} t \\ -1 \\ 1 \end{pmatrix}$$

So the scalar triple product is

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{p} \times \mathbf{q}) &= 6 \begin{pmatrix} t \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= 6(t + 1 + 1) \\ &= 6(t + 2) \end{aligned}$$

Therefore

$$\begin{aligned} V &= \frac{1}{6}|6(t + 2)| \\ &= |t + 2| \end{aligned}$$

The volume is 9, so

$$|t + 2| = 9$$

So

$$\begin{aligned} t + 2 = 9 \quad \text{or} \quad t + 2 &= -9 \\ t = 7 \quad \text{or} \quad t &= -11 \end{aligned}$$

9. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}1\\2\\-1\end{pmatrix},\qquad \mathbf{b}\times\mathbf{c}=\begin{pmatrix}0\\3\\1\end{pmatrix}\qquad\text{and}\qquad \mathbf{c}\times\mathbf{a}=\begin{pmatrix}2\\-1\\4\end{pmatrix}$$

Work out $(\mathbf{a}+2\mathbf{b}-\mathbf{c})\times(3\mathbf{a}-\mathbf{b}+\mathbf{c})$. **[5]**

Solution

The vector product distributes over addition, and for any vectors $\mathbf{u}$ and $\mathbf{v}$ and scalars λ and μ:

$$(\lambda\mathbf{u})\times(\mu\mathbf{v})=\lambda\mu\,(\mathbf{u}\times\mathbf{v}),\qquad \mathbf{u}\times\mathbf{u}=\mathbf{0},\qquad \mathbf{v}\times\mathbf{u}=-\,\mathbf{u}\times\mathbf{v}$$

Expanding the brackets term by term:

$$\mathbf{a}\times(3\mathbf{a}-\mathbf{b}+\mathbf{c})=3\,\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=-\,\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}$$

$$2\mathbf{b}\times(3\mathbf{a}-\mathbf{b}+\mathbf{c})=6\,\mathbf{b}\times\mathbf{a}-2\,\mathbf{b}\times\mathbf{b}+2\,\mathbf{b}\times\mathbf{c}=-6\,\mathbf{a}\times\mathbf{b}+2\,\mathbf{b}\times\mathbf{c}$$

$$-\mathbf{c}\times(3\mathbf{a}-\mathbf{b}+\mathbf{c})=-3\,\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{c}=-3\,\mathbf{c}\times\mathbf{a}-\mathbf{b}\times\mathbf{c}$$

Adding these three results:

$$(\mathbf{a}+2\mathbf{b}-\mathbf{c})\times(3\mathbf{a}-\mathbf{b}+\mathbf{c})=-7\,\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-4\,\mathbf{c}\times\mathbf{a}$$

Substituting the given vector products:

$$\begin{aligned}(\mathbf{a}+2\mathbf{b}-\mathbf{c})\times(3\mathbf{a}-\mathbf{b}+\mathbf{c})&=-7\begin{pmatrix}1\\2\\-1\end{pmatrix}+\begin{pmatrix}0\\3\\1\end{pmatrix}-4\begin{pmatrix}2\\-1\\4\end{pmatrix}\\&=\begin{pmatrix}-7\\-14\\7\end{pmatrix}+\begin{pmatrix}0\\3\\1\end{pmatrix}+\begin{pmatrix}-8\\4\\-16\end{pmatrix}\\&=\begin{pmatrix}-15\\-7\\-8\end{pmatrix}\end{aligned}$$

10. Points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively, relative to origin O.

It is given that $(\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=2\mathbf{p}$ and that $|\mathbf{p}|=5$.

(a) Determine each of the following as either a single vector or a scalar quantity.

(i) $(\mathbf{r}-\mathbf{q})\times(\mathbf{q}-\mathbf{p})$ **[1]**

(ii) $\mathbf{p}\times\big((\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\big)$ **[2]**

(iii) $\mathbf{p}\cdot\big((\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\big)$ **[2]**

(b) Describe a geometrical relationship between the points O, P, Q and R which can be deduced from

(i) the statement $(\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=2\mathbf{p}$. **[1]**

(ii) the result of part (a)(iii). **[1]**

Solution

(a) (i) Using the fact that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$:

$$(\mathbf{r}-\mathbf{q})\times(\mathbf{q}-\mathbf{p})=-\big((\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\big)=-2\mathbf{p}$$

So the vector is $-2\mathbf{p}$.

(ii) Substitute the given result directly:

$$\mathbf{p}\times\big((\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\big)=\mathbf{p}\times 2\mathbf{p}=2(\mathbf{p}\times\mathbf{p})=\mathbf{0}$$

So the vector is $\mathbf{0}$.

(iii) Again using the given result:

$$\begin{aligned}\mathbf{p}\cdot\big((\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\big)&=\mathbf{p}\cdot 2\mathbf{p}=2(\mathbf{p}\cdot\mathbf{p})\\&=2|\mathbf{p}|^2=2(5^2)=50\end{aligned}$$

So the scalar is 50.

(b) (i) The vector $\mathbf{q}-\mathbf{p}$ is in the direction of PQ, and $\mathbf{r}-\mathbf{q}$ is in the direction of QR.

Their vector product is perpendicular to both of these vectors. Since

$$(\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=2\mathbf{p}$$

the vector $\mathbf{p}$ is parallel to this perpendicular direction.

But $\mathbf{p}$ is the position vector of P, so it is the direction of OP. Therefore OP is perpendicular to both PQ and QR, so OP is normal to the plane PQR.

(ii) From part (a)(iii):

$$\mathbf{p}\cdot\big((\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\big)=50\neq 0$$

A non-zero scalar triple product means the three vectors involved are not coplanar.

Therefore the points O, P, Q and R are not coplanar.

11. The points A, B and C lie on the plane Π.

The position vectors of A and B are $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$ respectively.

The centroid G of triangle ABC has position vector $4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. **[4]**

(b) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = p$. **[2]**

The point D has position vector $\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

(c) Determine the volume of the tetrahedron $ABCD$. **[4]**

Solution

(a) The centroid of triangle ABC has position vector

$$\mathbf{g} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$$

So

$$\mathbf{c} = 3\mathbf{g} - \mathbf{a} - \mathbf{b}$$

Using

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

gives

$$\begin{aligned} \mathbf{c} &= 3\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 12 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \end{aligned}$$

Now

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

Hence

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= \mathbf{i}(2 \times 0 - (-3) \times 4) - \mathbf{j}(3 \times 0 - (-3) \times 3) + \mathbf{k}(3 \times 4 - 2 \times 3)$$

$$= 12\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$$

(b) A normal vector to the plane is $\overrightarrow{AB} \times \overrightarrow{AC}$, which is

$$\begin{pmatrix} 12 \\ -9 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

so we may take the simpler normal vector

$$\mathbf{n} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

The plane has equation $\mathbf{r} \cdot \mathbf{n} = p$. The point A lies on the plane, so

$$p = \mathbf{a} \cdot \mathbf{n}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

$$= 2(4) + (-1)(-3) + 3(2)$$

$$= 8 + 3 + 6 = 17$$

Therefore the equation of Π is

$$\mathbf{r} \cdot (4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = 17$$

(c) First find

$$\mathbf{d} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

so

$$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$$

The volume of tetrahedron $ABCD$ is

$$V = \frac{1}{6}\left|\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right|$$

Using the result from part (a):

$$
\begin{aligned}
V &= \frac{1}{6}\left|\begin{pmatrix}-1\\6\\1\end{pmatrix}\cdot\begin{pmatrix}12\\-9\\6\end{pmatrix}\right| \\
&= \frac{1}{6}|(-1)(12)+6(-9)+1(6)| \\
&= \frac{1}{6}|-12-54+6| \\
&= \frac{1}{6}|-60| \\
&= 10
\end{aligned}
$$

12. The lines L_1, L_2 and L_3 are given by

$$L_1: \quad \mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$L_2: \quad x = 9 - 4\mu,\ y = -2 + 3\mu,\ z = -1 + 3\mu$$

$$L_3: \quad \frac{x-3}{1} = \frac{y+6}{-5} = \frac{z-1}{-2}$$

The pairwise intersections of these lines are the vertices of a triangle.

Find the area of this triangle. **[9]**

Solution

Write the lines in parametric form:

$$L_1: \ (x, y, z) = (8 + 3\lambda,\ 3 + 2\lambda,\ 1 - \lambda)$$

$$L_2: \ (x, y, z) = (9 - 4\mu,\ -2 + 3\mu,\ -1 + 3\mu)$$

and for L_3, let the common value be t. Then

$$\frac{x-3}{1} = \frac{y+6}{-5} = \frac{z-1}{-2} = t$$

so

$$L_3: \ (x, y, z) = (3 + t,\ -6 - 5t,\ 1 - 2t)$$

Let the three vertices of the triangle be

$$A = L_1 \cap L_2, \qquad B = L_1 \cap L_3, \qquad C = L_2 \cap L_3$$

For $A = L_1 \cap L_2$, equate coordinates:

$$8 + 3\lambda = 9 - 4\mu$$

$$3 + 2\lambda = -2 + 3\mu$$

$$1 - \lambda = -1 + 3\mu$$

From the third equation:

$$2 - \lambda = 3\mu$$

so

$$\mu = \frac{2 - \lambda}{3}$$

Substitute into the first equation:

$$8 + 3\lambda = 9 - \frac{4(2-\lambda)}{3}$$

$$24 + 9\lambda = 27 - 8 + 4\lambda$$

$$24 + 9\lambda = 19 + 4\lambda$$

$$5\lambda = -5$$

$$\lambda = -1$$

Hence

$$\mu = \frac{2-(-1)}{3} = 1$$

So

$$A = (8 + 3(-1),\ 3 + 2(-1),\ 1 - (-1)) = (5, 1, 2)$$

For $B = L_1 \cap L_3$, equate coordinates:

$$8 + 3\lambda = 3 + t$$

$$3 + 2\lambda = -6 - 5t$$

$$1 - \lambda = 1 - 2t$$

From the third equation:

$$\lambda = 2t$$

Substitute into the first equation:

$$8 + 3(2t) = 3 + t$$

$$8 + 6t = 3 + t$$

$$5t = -5$$

$$t = -1$$

Hence

$$\lambda = 2(-1) = -2$$

So

$$B = (8 + 3(-2),\ 3 + 2(-2),\ 1 - (-2)) = (2, -1, 3)$$

For $C = L_2 \cap L_3$, equate coordinates:

$$9 - 4\mu = 3 + t$$

$$-2 + 3\mu = -6 - 5t$$

$$-1 + 3\mu = 1 - 2t$$

From the first equation:

$$t = 6 - 4\mu$$

Substitute into the third equation:

$$\begin{aligned} -1+3\mu &= 1-2(6-4\mu) \\ -1+3\mu &= 1-12+8\mu \\ -1+3\mu &= -11+8\mu \\ 10 &= 5\mu \\ \mu &= 2 \end{aligned}$$

Hence

$$t = 6-4(2) = -2$$

So

$$C = (9-4(2),\ -2+3(2),\ -1+3(2)) = (1,4,5)$$

The area of triangle ABC is

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

Take

$$\overrightarrow{AB} = \begin{pmatrix} 2-5 \\ -1-1 \\ 3-2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}, \qquad \overrightarrow{AC} = \begin{pmatrix} 1-5 \\ 4-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}$$

Then

$$\begin{aligned} \overrightarrow{AB}\times\overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -4 & 3 & 3 \end{vmatrix} \\ &= \mathbf{i}\big((-2)(3)-(1)(3)\big) - \mathbf{j}\big((-3)(3)-(1)(-4)\big) + \mathbf{k}\big((-3)(3)-(-2)(-4)\big) \\ &= \begin{pmatrix} -9 \\ 5 \\ -17 \end{pmatrix} \end{aligned}$$

So

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{(-9)^2+5^2+(-17)^2} = \sqrt{81+25+289} = \sqrt{395}$$

Therefore

$$\text{Area} = \frac{1}{2}\sqrt{395}$$

13. With respect to a fixed origin O the point A has coordinates $(9, 10, -1)$ and the line ℓ is the intersection of the planes $x - z = -2$ and $x + 2y = 5$.

The points B and C lie on ℓ such that $AB = AC = 15$.

Given that A does not lie on ℓ and that the z coordinate of B is negative

(a) determine the coordinates of B and the coordinates of C. **[6]**

(b) Hence determine a Cartesian equation of the plane containing the points A, B and C. **[4]**

The point D has coordinates $(8, 4, \alpha)$, where α is a constant.

Given that the volume of the tetrahedron $ABCD$ is 156

(c) determine the possible values of α. **[4]**

Given that $\alpha > 0$

(d) determine the shortest distance between the line ℓ and the line passing through the points A and D, giving your answer to 2 significant figures. **[4]**

Solution

(a) Since ℓ is the intersection of

$$x - z = -2 \qquad \text{and} \qquad x + 2y = 5$$

let $z = t$. Then

$$x = t - 2$$

and substituting into $x + 2y = 5$ gives

$$t - 2 + 2y = 5$$

so

$$2y = 7 - t \quad \Longrightarrow \quad y = \frac{7 - t}{2}$$

Hence a general point P on ℓ is

$$P\left(t - 2, \frac{7 - t}{2}, t\right)$$

Now $AB = AC = 15$, so points B and C on ℓ satisfy $AP = 15$.

Using $A = (9, 10, -1)$:

$$AP^2 = (t - 2 - 9)^2 + \left(\frac{7 - t}{2} - 10\right)^2 + (t - (-1))^2$$

$$= (t - 11)^2 + \left(\frac{-t - 13}{2}\right)^2 + (t + 1)^2$$

Setting $AP = 15$:

$$(t - 11)^2 + \left(\frac{t + 13}{2}\right)^2 + (t + 1)^2 = 225$$

Multiply by 4:

$$4(t-11)^2 + (t+13)^2 + 4(t+1)^2 = 900$$
$$4(t^2 - 22t + 121) + (t^2 + 26t + 169) + 4(t^2 + 2t + 1) = 900$$
$$9t^2 - 54t + 657 = 900$$
$$9t^2 - 54t - 243 = 0$$
$$t^2 - 6t - 27 = 0$$
$$(t-9)(t+3) = 0$$

So

$$t = 9 \quad \text{or} \quad t = -3$$

If $t = -3$, then

$$P = (-5, 5, -3)$$

If $t = 9$, then

$$P = (7, -1, 9)$$

Given that the z-coordinate of B is negative, we take

$$B = (-5, 5, -3)$$

and

$$C = (7, -1, 9)$$

(b) The plane through A, B and C has normal vector

$$\overrightarrow{AB} \times \overrightarrow{AC}$$

First find the two vectors:

$$\overrightarrow{AB} = \begin{pmatrix} -5-9 \\ 5-10 \\ -3-(-1) \end{pmatrix} = \begin{pmatrix} -14 \\ -5 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 7-9 \\ -1-10 \\ 9-(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ -11 \\ 10 \end{pmatrix}$$

Now

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -14 & -5 & -2 \\ -2 & -11 & 10 \end{vmatrix}$$
$$= \begin{pmatrix} -72 \\ 144 \\ 144 \end{pmatrix}$$

So a normal vector is

$$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Using the point $A = (9, 10, -1)$, the plane equation is

$$-(x-9) + 2(y-10) + 2(z+1) = 0$$

Simplifying:

$$-x + 9 + 2y - 20 + 2z + 2 = 0$$
$$-x + 2y + 2z - 9 = 0$$

Hence

$$x - 2y - 2z + 9 = 0$$

(c) From part (b):

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -72 \\ 144 \\ 144 \end{pmatrix}$$

Also

$$D = (8, 4, \alpha)$$

so

$$\overrightarrow{AD} = \begin{pmatrix} 8-9 \\ 4-10 \\ \alpha-(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ \alpha+1 \end{pmatrix}$$

The volume of tetrahedron $ABCD$ is

$$V = \frac{1}{6}\left|\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right|$$

So

$$\begin{aligned} \overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) &= \begin{pmatrix} -1 \\ -6 \\ \alpha+1 \end{pmatrix} \cdot \begin{pmatrix} -72 \\ 144 \\ 144 \end{pmatrix} \\ &= 72 - 864 + 144(\alpha+1) \\ &= 72 - 864 + 144\alpha + 144 \\ &= 144\alpha - 648 \end{aligned}$$

The volume is 156, so

$$\frac{1}{6}|144\alpha - 648| = 156$$

So

$$|144\alpha - 648| = 936$$

Divide by 72:

$$|2\alpha - 9| = 13$$

Hence

$$2\alpha - 9 = 13 \quad \text{or} \quad 2\alpha - 9 = -13$$

giving

$$\alpha = 11 \quad \text{or} \quad \alpha = -2$$

(d) Since $\alpha > 0$, from part (c) we use

$$\alpha = 11$$

So

$$D = (8, 4, 11)$$

A direction vector for ℓ is obtained from its parametric form:

$$(x, y, z) = \left(t - 2, \frac{7 - t}{2}, t\right)$$

so a direction vector is

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

and hence we can use

$$\mathbf{b}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

The line through A and D has direction vector

$$\mathbf{b}_2 = \overrightarrow{AD} = \begin{pmatrix} 8 - 9 \\ 4 - 10 \\ 11 - (-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 12 \end{pmatrix}$$

Using $B = (-5, 5, -3)$ on ℓ, we have

$$\overrightarrow{BA} = \begin{pmatrix} 9 - (-5) \\ 10 - 5 \\ -1 - (-3) \end{pmatrix} = \begin{pmatrix} 14 \\ 5 \\ 2 \end{pmatrix}$$

For two skew lines, the shortest distance is

$$\text{dist} = \frac{\left| \overrightarrow{BA} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) \right|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Now find the vector product:

$$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ -1 & -6 & 12 \end{vmatrix}$$
$$= \begin{pmatrix} 0 \\ -26 \\ -13 \end{pmatrix}$$

Its magnitude is

$$|\mathbf{b}_1 \times \mathbf{b}_2| = \sqrt{0^2 + (-26)^2 + (-13)^2} = \sqrt{845} = 13\sqrt{5}$$

Also

$$\overrightarrow{BA} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = \begin{pmatrix} 14 \\ 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -26 \\ -13 \end{pmatrix}$$
$$= 0 - 130 - 26$$
$$= -156$$

So

$$\text{dist} = \frac{|-156|}{13\sqrt{5}}$$
$$= \frac{156}{13\sqrt{5}}$$
$$= \frac{12}{\sqrt{5}}$$
$$= \frac{12\sqrt{5}}{5}$$
$$\approx 5.4 \qquad (2 \text{ s.f.})$$