OPTIONAL PURE / VECTORS & MATRICES

Shortest Distances

Worked solutions 10 questions · 112 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 CIE 9231

Instructions

Edexcel (9FM0): the vector (cross) product is Further Pure 1 content, not Core Pure. Questions 4 and 9 can be skipped; everything else can be done by scalar (dot) product methods, even where the solutions use the vector product.

Questions

Question 1	*5 marks*	2
Question 2	*8 marks*	3
Question 3	*13 marks*	5
Question 4	*18 marks*	8
Question 5	*8 marks*	12
Question 6	*10 marks*	15
Question 7	*8 marks*	18
Question 8	*17 marks*	21
Question 9	*9 marks*	26
Question 10	*16 marks*	29

1. The line ℓ has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

The point A has coordinates $(4, 1, 3)$.

(a) Find the coordinates of the point N on ℓ which is closest to A. **[4]**

(b) Hence find the shortest distance from A to ℓ. **[1]**

Solution

(a) A general point P on ℓ has position vector

$$\overrightarrow{OP} = \begin{pmatrix} 1 + 2\lambda \\ -2 + \lambda \\ 3 - 2\lambda \end{pmatrix}$$

so

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} 1 + 2\lambda \\ -2 + \lambda \\ 3 - 2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\lambda - 3 \\ \lambda - 3 \\ -2\lambda \end{pmatrix}$$

The point N closest to A is where $\overrightarrow{AP}$ is perpendicular to the direction of ℓ, so the scalar product with $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is zero:

$$\begin{pmatrix} 2\lambda - 3 \\ \lambda - 3 \\ -2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$2(2\lambda - 3) + (\lambda - 3) - 2(-2\lambda) = 0$$

$$4\lambda - 6 + \lambda - 3 + 4\lambda = 0$$

$$9\lambda - 9 = 0 \quad \Longrightarrow \quad \lambda = 1$$

Substituting $\lambda = 1$ into $\overrightarrow{OP}$:

$$\overrightarrow{ON} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

so N has coordinates $(3, -1, 1)$.

(b) Using part (a), with $\lambda = 1$:

$$\overrightarrow{AN} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

The shortest distance from A to ℓ is

$$\left|\overrightarrow{AN}\right| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

2. The planes Π_1 and Π_2 have equations $2x - y + 2z = 1$ and $4x - 2y + 4z = 11$ respectively.

(a) Show that Π_1 and Π_2 are parallel, and find the distance between them. **[3]**

The lines ℓ_1 and ℓ_2 have equations

$$\ell_1 : \ \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$\ell_2 : \ \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix}$$

(b) Show that ℓ_1 and ℓ_2 are parallel, and find the distance between them. **[5]**

Solution

(a) The normal vectors of Π_1 and Π_2 are

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

The normals are parallel, so Π_1 and Π_2 are parallel.

The point $(0, -1, 0)$ lies on Π_1, since $2 \times 0 - (-1) + 2 \times 0 = 1$. The distance between the planes is the distance from this point to Π_2, which is $4x - 2y + 4z - 11 = 0$.

The perpendicular distance from (α, β, γ) to $n_1 x + n_2 y + n_3 z + d = 0$ is

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Substituting $(\alpha, \beta, \gamma) = (0, -1, 0)$ and the coefficients of Π_2:

$$\begin{aligned} \text{dist} &= \frac{|4 \times 0 - 2 \times (-1) + 4 \times 0 - 11|}{\sqrt{4^2 + (-2)^2 + 4^2}} \\ &= \frac{|-9|}{6} \\ &= \frac{3}{2} \end{aligned}$$

(b) The direction vectors of ℓ_1 and ℓ_2 are

$$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

The direction vectors are parallel, so ℓ_1 and ℓ_2 are parallel.

Let B be the point $(4, 3, 2)$ on ℓ_2, and let P be the point on ℓ_1 closest to B. Then $\overrightarrow{BP}$ is perpendicular to ℓ_1. A general point on ℓ_1 has

$$\overrightarrow{OP} = \begin{pmatrix} 1 + \lambda \\ 2\lambda \\ 2 - 2\lambda \end{pmatrix}$$

so

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \begin{pmatrix} 1+\lambda \\ 2\lambda \\ 2-2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda - 3 \\ 2\lambda - 3 \\ -2\lambda \end{pmatrix}$$

For $\overrightarrow{BP}$ to be perpendicular to ℓ_1, its scalar product with the direction of ℓ_1 is zero:

$$\begin{pmatrix} \lambda - 3 \\ 2\lambda - 3 \\ -2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0$$

$$(\lambda - 3) + 2(2\lambda - 3) + 4\lambda = 0$$

$$9\lambda - 9 = 0$$

$$\lambda = 1$$

So

$$\overrightarrow{BP} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$$

and the distance between the lines is

$$\left|\overrightarrow{BP}\right| = \sqrt{(-2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

3. The plane Π has equation $3x + y + 2z = -4$. The point P has coordinates $(5, -5, 3)$.

(a) Calculate the shortest distance from P to Π. **[2]**

The line L has vector equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

(b) Verify that P lies on L. **[2]**

(c) Find the coordinates of the point where L meets Π. **[3]**

(d) Determine the acute angle between L and Π. **[3]**

(e) Use the results of parts (b), (c) and (d) to verify your answer to part (a). **[3]**

Solution

(a) Using the Cartesian form of the point-to-plane distance formula:

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Here, the plane is

$$3x + y + 2z = -4$$

so $a = 3$, $b = 1$, $c = 2$, $d = -4$, and $P = (5, -5, 3)$.

Substituting into the formula:

$$\begin{aligned} \text{dist} &= \frac{|3(5) + 1(-5) + 2(3) - (-4)|}{\sqrt{3^2 + 1^2 + 2^2}} \\ &= \frac{|15 - 5 + 6 + 4|}{\sqrt{9 + 1 + 4}} \\ &= \frac{20}{\sqrt{14}} \\ &= \frac{10\sqrt{14}}{7} \end{aligned}$$

So the shortest distance from P to Π is $\frac{20}{\sqrt{14}} = \frac{10\sqrt{14}}{7}$.

(b) A general point on L has coordinates

$$(x, y, z) = (2 + \lambda,\ 1 - 2\lambda,\ -3 + 2\lambda)$$

For $P = (5, -5, 3)$ to lie on L, we need

$$2 + \lambda = 5$$

so

$$\lambda = 3$$

Now check the other coordinates:

$$y = 1 - 2(3) = 1 - 6 = -5$$
$$z = -3 + 2(3) = -3 + 6 = 3$$

These match the coordinates of P.

Hence P lies on L, with parameter value $\lambda = 3$.

(c) Let the point where L meets Π correspond to parameter λ.

From the line:

$$x = 2 + \lambda, \qquad y = 1 - 2\lambda, \qquad z = -3 + 2\lambda$$

Substitute into the plane equation $3x + y + 2z = -4$:

$$\begin{aligned} 3(2 + \lambda) + (1 - 2\lambda) + 2(-3 + 2\lambda) &= -4 \\ 6 + 3\lambda + 1 - 2\lambda - 6 + 4\lambda &= -4 \\ 1 + 5\lambda &= -4 \\ 5\lambda &= -5 \\ \lambda &= -1 \end{aligned}$$

Now substitute $\lambda = -1$ into the line:

$$x = 2 + (-1) = 1$$
$$y = 1 - 2(-1) = 3$$
$$z = -3 + 2(-1) = -5$$

So the line meets the plane at $(1, 3, -5)$.

(d) The direction vector of L is

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

and a normal vector to Π is

$$\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

For the acute angle θ between a line and a plane:

$$\sin\theta = \frac{|\mathbf{u} \cdot \mathbf{n}|}{|\mathbf{u}||\mathbf{n}|}$$

First find the scalar product:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 1(3) + (-2)(1) + 2(2) \\ &= 3 - 2 + 4 \\ &= 5 \end{aligned}$$

Now find the magnitudes:

$$|\mathbf{u}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$$

$$|\mathbf{n}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

Therefore

$$\sin\theta = \frac{5}{3\sqrt{14}}$$

So

$$\theta = \arcsin\left(\frac{5}{3\sqrt{14}}\right) \approx 26.5°$$

Hence the acute angle between L and Π is $\arcsin\left(\frac{5}{3\sqrt{14}}\right) \approx 26.5°$.

(e) From part (b), P corresponds to $\lambda = 3$ on the line.

From part (c), the point where L meets Π is $C(1, 3, -5)$, which corresponds to $\lambda = -1$.

So the change in parameter is

$$3 - (-1) = 4$$

Since the direction vector $\mathbf{u}$ of the line has magnitude

$$|\mathbf{u}| = 3$$

the length of the line segment from P to the plane along L is

$$PC = 4 \times 3 = 12$$

From part (d), the angle between the line and the plane is θ, where

$$\sin\theta = \frac{5}{3\sqrt{14}}$$

The perpendicular distance from P to the plane is the component of PC perpendicular to the plane:

$$\begin{aligned} \text{dist} &= PC\sin\theta \\ &= 12 \times \frac{5}{3\sqrt{14}} \\ &= \frac{60}{3\sqrt{14}} \\ &= \frac{20}{\sqrt{14}} \end{aligned}$$

This is exactly the same as the answer in part (a).

Hence the answer to part (a) is verified.

4. The points C and D have coordinates $(0, 0, 2)$ and $(3, 3, -1)$ respectively.

The planes P_1 and P_2 have equations $x + y + z = 3$ and $2x - y = 1$ respectively.

(a) Find the acute angle between the line CD and the plane P_1. **[4]**

(b) Show that the line CD meets P_1 and P_2 in a common point, and find its coordinates. **[5]**

(c) (i) Find $(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{j})$. **[2]**

(ii) Hence find the acute angle between the planes P_1 and P_2. **[3]**

(iii) Find the shortest distance between the point C and the line of intersection of the planes P_1 and P_2. **[4]**

Solution

(a) The direction vector of the line CD is

$$\overrightarrow{CD} = \begin{pmatrix} 3-0 \\ 3-0 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

So we may use the parallel direction vector

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

For the plane $P_1 : x + y + z = 3$, a normal vector is

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

If α is the acute angle between a line and a plane, then

$$\sin\alpha = \frac{|\mathbf{u} \cdot \mathbf{n}_1|}{|\mathbf{u}||\mathbf{n}_1|}$$

Now

$$\mathbf{u} \cdot \mathbf{n}_1 = (1)(1) + (1)(1) + (-1)(1) = 1$$

$$|\mathbf{u}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$|\mathbf{n}_1| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Hence

$$\sin\alpha = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$$

So

$$\alpha = \arcsin\left(\frac{1}{3}\right) \approx 19.5^\circ$$

The acute angle between CD and P_1 is $\arcsin\left(\frac{1}{3}\right) \approx 19.5°$.

(b) The line CD through $C = (0, 0, 2)$ with direction vector $\mathbf{u}$ has equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

So a general point on CD is

$$(x, y, z) = (t, t, 2 - t)$$

First, substitute into $P_1 : x + y + z = 3$:

$$\begin{aligned} t + t + (2 - t) &= 3 \\ t + 2 &= 3 \\ t &= 1 \end{aligned}$$

Now substitute into $P_2 : 2x - y = 1$:

$$\begin{aligned} 2t - t &= 1 \\ t &= 1 \end{aligned}$$

Both planes are met when $t = 1$, so the line CD meets P_1 and P_2 at the same point.

When $t = 1$:

$$(x, y, z) = (1, 1, 2 - 1) = (1, 1, 1)$$

Therefore the common point is $(1, 1, 1)$.

(c) (i)

$$(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$$

Expanding:

$$\begin{aligned} &= \mathbf{i}(1 \times 0 - 1 \times (-1)) - \mathbf{j}(1 \times 0 - 1 \times 2) + \mathbf{k}(1 \times (-1) - 1 \times 2) \\ &= \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \end{aligned}$$

Hence

$$(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{j}) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

(ii) The acute angle between two planes is the acute angle between their normal vectors.

For

$$P_1 : x + y + z = 3 \qquad \text{and} \qquad P_2 : 2x - y = 1$$

normal vectors are

$$\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} - \mathbf{j}$$

If θ is the acute angle between the planes, then $|\mathbf{n}_1 \times \mathbf{n}_2| = |\mathbf{n}_1||\mathbf{n}_2| \sin\theta$, so

$$\sin\theta = \frac{|\mathbf{n}_1 \times \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

From part (c)(i), $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, so

$$|\mathbf{n}_1 \times \mathbf{n}_2| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

$$|\mathbf{n}_1| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\mathbf{n}_2| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$$

So

$$\sin\theta = \frac{\sqrt{14}}{\sqrt{3}\sqrt{5}} = \sqrt{\frac{14}{15}}$$

Hence

$$\theta = \arcsin\left(\sqrt{\frac{14}{15}}\right) \approx 75.0^\circ$$

The acute angle between the planes is $\arcsin\left(\sqrt{\frac{14}{15}}\right) \approx 75.0^\circ$.

(iii) From part (b), a point on both planes, and so on their line of intersection, is

$$A = (1, 1, 1)$$

From part (c)(i), a direction vector for their line of intersection is

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

So the line of intersection is

$$\ell : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

We need the shortest distance from $C = (0, 0, 2)$ to ℓ.

Using the formula

$$\text{dist} = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{b}|}{|\mathbf{b}|}$$

with

$$\mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

we have

$$\mathbf{p} - \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Now find the vector product:

$$(\mathbf{p}-\mathbf{a})\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&-1&1\\1&2&-3\end{vmatrix}$$

So

$$\begin{aligned}(\mathbf{p}-\mathbf{a})\times\mathbf{b}&=\mathbf{i}\big((-1)(-3)-1\times 2\big)-\mathbf{j}\big((-1)(-3)-1\times 1\big)+\mathbf{k}\big((-1)\times 2-(-1)\times 1\big)\\&=\begin{pmatrix}1\\-2\\-1\end{pmatrix}\end{aligned}$$

Hence

$$|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|=\sqrt{1^2+(-2)^2+(-1)^2}=\sqrt{6}$$

$$|\mathbf{b}|=\sqrt{1^2+2^2+(-3)^2}=\sqrt{14}$$

Therefore

$$\text{dist}=\frac{\sqrt{6}}{\sqrt{14}}=\sqrt{\frac{3}{7}}$$

So the shortest distance is

$$\sqrt{\frac{3}{7}}\approx 0.655$$

5. The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is perpendicular to Π. **[2]**

(b) Hence find a Cartesian equation of Π. **[2]**

The line ℓ has equation

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where t is a scalar parameter.

The point A lies on ℓ. Given that the shortest distance between A and Π is $\frac{9}{\sqrt{14}}$

(c) determine the possible coordinates of A. **[4]**

Solution

(a) Let

$$\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

The plane Π has direction vectors

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$$

Check the scalar products:

$$\mathbf{n} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1(1) + (-2)(2) + 3(1) = 1 - 4 + 3 = 0$$

$$\mathbf{n} \cdot \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix} = 1(4) + (-2)(-1) + 3(-2) = 4 + 2 - 6 = 0$$

Since $\mathbf{n}$ is perpendicular to both direction vectors of the plane, $\mathbf{n}$ is perpendicular to Π.

Therefore $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is perpendicular to Π.

(b) Using the normal vector $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and the point $(2, -1, 3)$ on Π, an equation of Π is

$$\mathbf{n} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right) = 0$$

So

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y+1 \\ z-3 \end{pmatrix} = 0$$

$$(x-2) - 2(y+1) + 3(z-3) = 0$$

$$x - 2 - 2y - 2 + 3z - 9 = 0$$

$$x - 2y + 3z - 13 = 0$$

Therefore a Cartesian equation of Π is

$$x - 2y + 3z = 13$$

(c) A lies on the line ℓ with equation

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

so the coordinates of A are

$$A = (-2+t,\ 1+t,\ 4+t)$$

From part (b), the plane is

$$x - 2y + 3z = 13$$

Using the point-to-plane distance formula

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

with $a = 1$, $b = -2$, $c = 3$, $d = 13$, we get

$$\frac{|(-2+t) - 2(1+t) + 3(4+t) - 13|}{\sqrt{1^2 + (-2)^2 + 3^2}} = \frac{9}{\sqrt{14}}$$

$$\frac{|-2 + t - 2 - 2t + 12 + 3t - 13|}{\sqrt{14}} = \frac{9}{\sqrt{14}}$$

$$\frac{|2t - 5|}{\sqrt{14}} = \frac{9}{\sqrt{14}}$$

Hence

$$|2t - 5| = 9$$

So

$$2t - 5 = 9 \quad \text{or} \quad 2t - 5 = -9$$

giving

$$t = 7 \quad \text{or} \quad t = -2$$

Substitute back into the line:

$$t = 7 \quad \Longrightarrow \quad A = (5, 8, 11)$$

$$t = -2 \quad \Longrightarrow \quad A = (-4, -1, 2)$$

Therefore the possible coordinates of A are $(5, 8, 11)$ or $(-4, -1, 2)$.

6. **(a)** The plane Π passes through the points $A(1,-2,3)$, $B(2,0,3)$ and $C(2,-2,2)$.

The point D has coordinates $(4,1,0)$. Find the shortest distance between D and Π. **[6]**

(b) Lines ℓ_1 and ℓ_2 have the following equations.

$$\ell_1 : \ \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\ell_2 : \ \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Find, in exact form, the distance between ℓ_1 and ℓ_2. **[4]**

Solution

(a) First find two direction vectors in the plane:

$$\overrightarrow{AB} = \begin{pmatrix} 2-1 \\ 0-(-2) \\ 3-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \overrightarrow{AC} = \begin{pmatrix} 2-1 \\ -2-(-2) \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

A normal vector to the plane is

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} \\ &= \mathbf{i}(2 \times (-1) - 0 \times 0) - \mathbf{j}(1 \times (-1) - 0 \times 1) + \mathbf{k}(1 \times 0 - 2 \times 1) \\ &= \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} \end{aligned}$$

So we may take the normal as

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

The plane has equation

$$2x - y + 2z = d$$

It passes through $A(1,-2,3)$, so

$$\begin{aligned} d &= 2(1) - (-2) + 2(3) \\ &= 2 + 2 + 6 \\ &= 10 \end{aligned}$$

Hence the plane is

$$\Pi : \ 2x - y + 2z = 10$$

Using the point-to-plane distance formula

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

with $a = 2$, $b = -1$, $c = 2$, $d = 10$ and $D(4, 1, 0)$:

$$\begin{aligned}\text{dist} &= \frac{|2(4) - 1 + 2(0) - 10|}{\sqrt{2^2 + (-1)^2 + 2^2}} \\ &= \frac{|8 - 1 - 10|}{\sqrt{4 + 1 + 4}} \\ &= \frac{3}{3} \\ &= 1\end{aligned}$$

The shortest distance between D and Π is 1.

(b) Write the lines as

$$\ell_1 : \ \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1, \qquad \mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\ell_2 : \ \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2, \qquad \mathbf{a}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For the distance between two skew lines, use

$$\text{dist} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

First find the common normal direction:

$$\begin{aligned}\mathbf{b}_1 \times \mathbf{b}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} \\ &= \mathbf{i}(1 \times 1 - 2 \times (-2)) - \mathbf{j}(2 \times 1 - 2 \times 1) + \mathbf{k}(2 \times (-2) - 1 \times 1) \\ &= \begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}\end{aligned}$$

Also

$$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} 3 - 1 \\ -1 - 2 \\ 4 - (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

Now calculate the numerator:

$$\begin{aligned}(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) &= \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix} \\ &= 2(5) + (-3)(0) + 5(-5) \\ &= 10 - 25 \\ &= -15\end{aligned}$$

So

$$|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)| = 15$$

And the denominator is

$$\begin{aligned}|\mathbf{b}_1 \times \mathbf{b}_2| &= \sqrt{5^2 + 0^2 + (-5)^2} \\ &= \sqrt{25 + 25} \\ &= 5\sqrt{2}\end{aligned}$$

Therefore

$$\begin{aligned}\text{dist} &= \frac{15}{5\sqrt{2}} \\ &= \frac{3}{\sqrt{2}} \\ &= \frac{3\sqrt{2}}{2}\end{aligned}$$

The distance between ℓ_1 and ℓ_2 is $\frac{3\sqrt{2}}{2}$.

7. The line ℓ_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

The line ℓ_2 is the line of intersection of the planes

$$x - y + z = 6$$

and

$$2x + y - z = 3$$

(i) Find the shortest distance between ℓ_1 and ℓ_2. **[6]**

(ii) Find a Cartesian equation of the plane which contains ℓ_1 and is parallel to ℓ_2. **[2]**

Solution

(i) First find a vector equation for ℓ_2.

Let $z = t$. Then the plane equations become

$$x - y + t = 6$$

$$2x + y - t = 3$$

Adding these equations gives

$$3x = 9$$

so

$$x = 3$$

Substituting into $x - y + t = 6$:

$$3 - y + t = 6$$

$$-y = 3 - t$$

$$y = t - 3$$

Hence ℓ_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So we can take

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

For the shortest distance between two skew lines:

$$\text{dist} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

First calculate the vector product:

$$\begin{aligned}\mathbf{b}_1 \times \mathbf{b}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i}\big(2(1) - (-1)(1)\big) - \mathbf{j}\big(1(1) - (-1)(0)\big) + \mathbf{k}\big(1(1) - 2(0)\big) \\ &= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}\end{aligned}$$

Also

$$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

Now find the scalar triple product:

$$\begin{aligned}(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) &= \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \\ &= 1(3) + (-2)(-1) + (-3)(1) \\ &= 3 + 2 - 3 \\ &= 2\end{aligned}$$

And

$$\begin{aligned}|\mathbf{b}_1 \times \mathbf{b}_2| &= \sqrt{3^2 + (-1)^2 + 1^2} \\ &= \sqrt{11}\end{aligned}$$

Therefore

$$\text{dist} = \frac{|2|}{\sqrt{11}} = \frac{2}{\sqrt{11}}$$

The shortest distance between ℓ_1 and ℓ_2 is $\frac{2}{\sqrt{11}}$.

(ii) A plane containing ℓ_1 and parallel to ℓ_2 must contain both direction vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So a normal vector to the plane is

$$\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

The plane passes through the point $(2, -1, 3)$ on ℓ_1. Hence its equation is

$$\mathbf{n} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right) = 0$$

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y+1 \\ z-3 \end{pmatrix} = 0$$

$$3(x-2) - (y+1) + (z-3) = 0$$

Simplifying:

$$3x - 6 - y - 1 + z - 3 = 0$$

$$3x - y + z - 10 = 0$$

So a Cartesian equation of the plane is

$$3x - y + z = 10$$

8. **(a)** Show that the three planes with equations

$$\begin{aligned} x + \lambda y + z &= 4 \\ 2x - y + 2z &= 3 \\ 2x + y - z &= 5 \end{aligned} \qquad \textbf{[3]}$$

where λ is a constant, meet at a unique point except for one value of λ, which is to be determined.

(b) In the case $\lambda = 2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

The line ℓ has equation

$$\frac{x+1}{1} = \frac{y-1}{-1} = \frac{z}{1}$$

(c) Find a vector equation of ℓ. **[2]**

(d) Find the shortest distance between the point P and ℓ. **[4]**

(e) (i) Show that ℓ is parallel to the plane $2x + y - z = 5$. **[3]**

(ii) Find the distance between ℓ and the plane $2x + y - z = 5$. **[2]**

Solution

(a) Let the coefficient matrix be

$$\mathbf{A} = \begin{pmatrix} 1 & \lambda & 1 \\ 2 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

The three planes meet at a unique point if and only if the simultaneous equations have a unique solution, that is, if and only if $\det(\mathbf{A}) \neq 0$.

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A}) &= 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - \lambda\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} \\ &= 1\big((-1)(-1) - 2(1)\big) - \lambda\big(2(-1) - 2(2)\big) + \big(2(1) - (-1)(2)\big) \\ &= (1-2) - \lambda(-2-4) + (2+2) \\ &= -1 + 6\lambda + 4 \\ &= 6\lambda + 3 \\ &= 3(2\lambda + 1) \end{aligned}$$

So the planes meet at a unique point whenever

$$3(2\lambda + 1) \neq 0$$

that is

$$\lambda \neq -\frac{1}{2}$$

Hence the exceptional value is

$$\lambda = -\frac{1}{2}$$

For $\lambda = -\frac{1}{2}$, the first plane is

$$x - \frac{1}{2}y + z = 4$$

and multiplying by 2 gives

$$2x - y + 2z = 8$$

but the second plane is

$$2x - y + 2z = 3$$

so the system is inconsistent. Therefore there is no common point in this case.

Hence the three planes meet at a unique point for all λ except $\lambda = -\frac{1}{2}$.

(b) When $\lambda = 2$, the equations are

$$x + 2y + z = 4$$
$$2x - y + 2z = 3$$
$$2x + y - z = 5$$

Write this system as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$$

Since $\det(\mathbf{A}) \neq 0$, the matrix is invertible, so

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Using a calculator to find the inverse:

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{15} & \frac{1}{5} & \frac{1}{3} \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ \frac{4}{15} & \frac{1}{5} & -\frac{1}{3} \end{pmatrix}$$

Hence

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{15} & \frac{1}{5} & \frac{1}{3} \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ \frac{4}{15} & \frac{1}{5} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

So the point of intersection is

$$P = (2, 1, 0)$$

(c) Let the common parameter be t. Then

$$\frac{x+1}{1} = \frac{y-1}{-1} = \frac{z}{1} = t$$

So

$$x+1 = t, \qquad y-1 = -t, \qquad z = t$$

hence

$$x = t-1, \qquad y = 1-t, \qquad z = t$$

Therefore a vector equation of ℓ is

$$\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(d) From part (b), $P = (2, 1, 0)$.

Take the point

$$Q = (-1, 1, 0)$$

on the line ℓ, and the direction vector of ℓ as

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Then

$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

Using

$$\text{dist} = \frac{\left|\overrightarrow{QP} \times \mathbf{u}\right|}{|\mathbf{u}|}$$

First find the vector product:

$$\overrightarrow{QP} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$$

So

$$\left|\overrightarrow{QP} \times \mathbf{u}\right| = \sqrt{0^2 + (-3)^2 + (-3)^2}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

Also

$$|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

Hence

$$\begin{aligned} \text{dist} &= \frac{3\sqrt{2}}{\sqrt{3}} \\ &= \sqrt{6} \end{aligned}$$

Therefore the shortest distance between P and ℓ is

$$\sqrt{6}$$

(e) (i) The plane

$$2x + y - z = 5$$

has normal vector

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

The line ℓ has direction vector

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Now

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= 1(2) + (-1)(1) + 1(-1) \\ &= 2 - 1 - 1 \\ &= 0 \end{aligned}$$

So ℓ is either parallel to the plane or lies in the plane.

Take the point $Q = (-1, 1, 0)$ on ℓ. Substituting into the plane equation:

$$2(-1) + 1 - 0 = -1$$

which is not equal to 5.

So Q is not on the plane, hence ℓ does not lie in the plane.

Therefore ℓ is parallel to the plane $2x + y - z = 5$.

(ii) Since ℓ is parallel to the plane, the distance from ℓ to the plane is the distance from any point on ℓ to the plane.

Using $Q = (-1, 1, 0)$ and the formula

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

for the plane $2x + y - z = 5$:

$$\begin{aligned}\text{dist} &= \frac{|2(-1) + 1(1) + (-1)(0) - 5|}{\sqrt{2^2 + 1^2 + (-1)^2}} \\ &= \frac{|-2 + 1 - 5|}{\sqrt{6}} \\ &= \frac{6}{\sqrt{6}} \\ &= \sqrt{6}\end{aligned}$$

Therefore the distance between ℓ and the plane is

$$\sqrt{6}$$

9. **(a)** Given that $\mathbf{u} = \lambda\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, find the following, giving your answers in terms of λ.

(i) $\mathbf{u} \cdot \mathbf{v}$ **[1]**

(ii) $\mathbf{u} \times \mathbf{v}$ **[2]**

(b) Hence determine

(i) the acute angle between the planes $2x - 2y + 2z = 11$ and $x + y - z = 7$ **[3]**

(ii) the shortest distance between the lines

$$\mathbf{r} = \mathbf{i} - \mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

and

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

giving your answer as a multiple of $\sqrt{2}$. **[3]**

Solution

(a) (i)

$$\mathbf{u} \cdot \mathbf{v} = (\lambda)(1) + (-2)(1) + (2)(-1) = \lambda - 2 - 2 = \lambda - 4$$

(ii)

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & -2 & 2 \\ 1 & 1 & -1 \end{vmatrix} \\
&= \mathbf{i}\big((-2)(-1) - 2(1)\big) - \mathbf{j}\big(\lambda(-1) - 2(1)\big) + \mathbf{k}\big(\lambda(1) - (-2)(1)\big) \\
&= (\lambda + 2)\mathbf{j} + (\lambda + 2)\mathbf{k} \\
&= (\lambda + 2)(\mathbf{j} + \mathbf{k})
\end{aligned}$$

(b) (i) The acute angle between two planes is the acute angle between their normal vectors.

For the planes

$$2x - 2y + 2z = 11 \qquad \text{and} \qquad x + y - z = 7$$

the normal vectors are

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \qquad \mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Using part (a)(i), $\mathbf{n}_1 \cdot \mathbf{n}_2$ is $\mathbf{u} \cdot \mathbf{v}$ with $\lambda = 2$, so

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 4 = -2$$

Also

$$|\mathbf{n}_1| = \sqrt{2^2 + (-2)^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$$

$$|\mathbf{n}_2| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

Using

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}$$

gives

$$\begin{aligned}\cos\theta &= \frac{|-2|}{(2\sqrt{3})(\sqrt{3})} \\ &= \frac{2}{6} \\ &= \frac{1}{3}\end{aligned}$$

Therefore

$$\theta = \arccos\left(\frac{1}{3}\right) \approx 70.5^\circ$$

Hence the acute angle between the planes is $\arccos\left(\frac{1}{3}\right) \approx 70.5^\circ$.

(ii) Write the lines as

$$\ell_1 : \ \mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1, \qquad \ell_2 : \ \mathbf{r} = \mathbf{a}_2 + t\mathbf{b}_2$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Using the formula for the distance between two skew lines:

$$\text{dist} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

From part (a)(ii), with $\lambda = 1$:

$$\mathbf{b}_1 \times \mathbf{b}_2 = (1+2)(\mathbf{j} + \mathbf{k}) = 3(\mathbf{j} + \mathbf{k}) = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

Also

$$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} 2-1 \\ 3-0 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Now

$$\begin{aligned}(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) &= \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \\ &= 1 \times 0 + 3 \times 3 + 4 \times 3 \\ &= 21\end{aligned}$$

and

$$|\mathbf{b}_1 \times \mathbf{b}_2| = \sqrt{0^2 + 3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

Therefore

$$\begin{aligned}\text{dist} &= \frac{21}{3\sqrt{2}} \\ &= \frac{7}{\sqrt{2}} \\ &= \frac{7\sqrt{2}}{2}\end{aligned}$$

Hence the shortest distance is

$$\frac{7\sqrt{2}}{2}$$

10. The position vectors of the points A, B, C, D are

$$\mathbf{i}+2\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{j}, \quad \mathbf{i}+3\mathbf{j}+2\mathbf{k}, \quad 2\mathbf{i}+3\mathbf{j}+m\mathbf{k}$$

respectively, where m is an integer.

It is given that the shortest distance between the line through A and D and the line through B and C is $\frac{1}{\sqrt{14}}$.

(a) Show that the only possible value of m is 1. **[7]**

(b) Find the shortest distance of C from the line through A and D. **[4]**

(c) Show that the acute angle between the planes ABD and ACD is $\arccos\left(\frac{\sqrt{3}}{2}\right)$. **[5]**

Solution

(a) Writing the points in coordinates:

$$A=(1,2,0), \quad B=(2,2,0), \quad C=(1,3,2), \quad D=(2,3,m)$$

So the direction vectors of the two lines are

$$\overrightarrow{AD}=\begin{pmatrix}1\\1\\m\end{pmatrix}, \qquad \overrightarrow{BC}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

Also

$$\overrightarrow{AB}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Using the formula for the distance between two skew lines:

$$\text{dist}=\frac{\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AD}\times\overrightarrow{BC}\right)\right|}{\left|\overrightarrow{AD}\times\overrightarrow{BC}\right|}$$

First find the vector product:

$$\begin{aligned}\overrightarrow{AD}\times\overrightarrow{BC}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&m\\-1&1&2\end{vmatrix}\\&=\mathbf{i}(1\times 2-m\times 1)-\mathbf{j}(1\times 2-m\times(-1))+\mathbf{k}(1\times 1-1\times(-1))\\&=(2-m)\mathbf{i}-(2+m)\mathbf{j}+2\mathbf{k}\end{aligned}$$

Now the scalar triple product is

$$\begin{aligned}\overrightarrow{AB}\cdot\left(\overrightarrow{AD}\times\overrightarrow{BC}\right)&=\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}2-m\\-2-m\\2\end{pmatrix}\\&=2-m\end{aligned}$$

Next

$$\left|\overrightarrow{AD} \times \overrightarrow{BC}\right|^2 = (2-m)^2 + (-2-m)^2 + 2^2$$
$$= (m^2 - 4m + 4) + (m^2 + 4m + 4) + 4$$
$$= 2m^2 + 12$$

So

$$\left|\overrightarrow{AD} \times \overrightarrow{BC}\right| = \sqrt{2m^2 + 12}$$

Therefore the shortest distance is

$$\frac{|2-m|}{\sqrt{2m^2+12}}$$

This is given to be $\frac{1}{\sqrt{14}}$, so

$$\frac{|2-m|}{\sqrt{2m^2+12}} = \frac{1}{\sqrt{14}}$$

Squaring both sides:

$$14(2-m)^2 = 2m^2 + 12$$
$$14(m^2 - 4m + 4) = 2m^2 + 12$$
$$14m^2 - 56m + 56 = 2m^2 + 12$$
$$12m^2 - 56m + 44 = 0$$
$$3m^2 - 14m + 11 = 0$$

Factorising:

$$3m^2 - 14m + 11 = (3m - 11)(m - 1)$$

So

$$(3m - 11)(m - 1) = 0$$

Hence

$$m = \frac{11}{3} \quad \text{or} \quad m = 1$$

Since m is an integer, the only possible value is

$$m = 1$$

(b) Using $m = 1$:

$$D = (2, 3, 1)$$

For the line through A and D:

$$\overrightarrow{AD} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Also

$$\overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Using the point-to-line distance formula:

$$\text{dist} = \frac{\left|\overrightarrow{AC} \times \overrightarrow{AD}\right|}{\left|\overrightarrow{AD}\right|}$$

Now

$$\begin{aligned} \overrightarrow{AC} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i}(1 \times 1 - 2 \times 1) - \mathbf{j}(0 \times 1 - 2 \times 1) + \mathbf{k}(0 \times 1 - 1 \times 1) \\ &= -\mathbf{i} + 2\mathbf{j} - \mathbf{k} \end{aligned}$$

So

$$\left|\overrightarrow{AC} \times \overrightarrow{AD}\right| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$$

and

$$\left|\overrightarrow{AD}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Hence the shortest distance is

$$\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

So the shortest distance of C from the line through A and D is

$$\sqrt{2}$$

(c) With $m = 1$:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

A normal to plane ABD is

$$\begin{aligned} \mathbf{n}_{ABD} &= \overrightarrow{AB} \times \overrightarrow{AD} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 0\mathbf{i} - \mathbf{j} + \mathbf{k} \end{aligned}$$

A normal to plane ACD is

$$\begin{aligned}\mathbf{n}_{ACD} &= \overrightarrow{AC} \times \overrightarrow{AD} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= -\mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

Using the acute angle formula for two planes:

$$\cos\theta = \frac{|\mathbf{n}_{ABD} \cdot \mathbf{n}_{ACD}|}{|\mathbf{n}_{ABD}||\mathbf{n}_{ACD}|}$$

Now

$$\begin{aligned}\mathbf{n}_{ABD} \cdot \mathbf{n}_{ACD} &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \\ &= 0 - 2 - 1 \\ &= -3\end{aligned}$$

Hence

$$|\mathbf{n}_{ABD} \cdot \mathbf{n}_{ACD}| = 3$$

Also

$$|\mathbf{n}_{ABD}| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$$

and

$$|\mathbf{n}_{ACD}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$$

Therefore

$$\begin{aligned}\cos\theta &= \frac{3}{\sqrt{2}\sqrt{6}} \\ &= \frac{3}{\sqrt{12}} \\ &= \frac{3}{2\sqrt{3}} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

So the acute angle between the planes ABD and ACD is

$$\theta = \arccos\left(\frac{\sqrt{3}}{2}\right)$$

That is

$$\theta = \frac{\pi}{6} = 30^\circ$$