OPTIONAL PURE / VECTORS & MATRICES

Scalar Triple Product

Worked solutions 10 questions · 105 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · OCR A Y545

Questions

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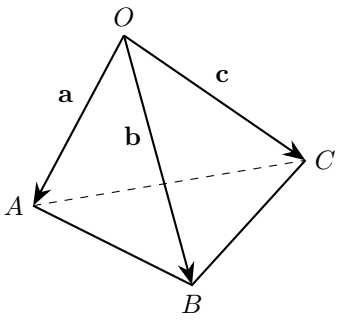

1. The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, relative to a fixed origin O, as shown in the diagram above.
It is given that, for a constant λ,

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{pmatrix} \lambda \\ -1 \\ 2 \end{pmatrix}$$

(a) Find $\mathbf{a} \times \mathbf{b}$. **[2]**

(b) Hence find the exact area of triangle OAB. **[2]**

(c) Find $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ in terms of λ. **[2]**

(d) Given that the volume of the tetrahedron $OABC$ is 1 cubic unit, find the two possible values of λ. **[2]**

Solution

(a) Writing the vector product as a determinant:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{vmatrix} \\ &= \big((-1) \times (-2) - 1 \times 3\big)\mathbf{i} - \big(2 \times (-2) - 1 \times 1\big)\mathbf{j} + \big(2 \times 3 - (-1) \times 1\big)\mathbf{k} \\ &= \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} \end{aligned}$$

(b) The area of triangle OAB is

$$\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

Using part (a):

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-1)^2 + 5^2 + 7^2} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area} = \frac{1}{2} \times 5\sqrt{3} = \frac{5\sqrt{3}}{2}$$

(c) Using part (a):

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} \lambda \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} = -\lambda - 5 + 14 = 9 - \lambda$$

(d) The volume of tetrahedron $OABC$ is

$$V = \frac{1}{6}|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$$

Using part (c), with $V = 1$:

$$\frac{1}{6}|9 - \lambda| = 1$$

$$|9 - \lambda| = 6$$

$$9 - \lambda = 6 \quad \text{or} \quad 9 - \lambda = -6$$

so $\lambda = 3$ or $\lambda = 15$.

2. The points A and B have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$ respectively, relative to the origin O. The line ℓ has equation

$$(\mathbf{r} - (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})) \times (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \mathbf{0}$$

The point C lies on ℓ such that OC is perpendicular to ℓ.

(a) Find the position vector of C. **[4]**

(b) Find the volume of the tetrahedron $OABC$. **[4]**

The transformation T is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} k & 1 & 0 \\ 3 & k & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

where k is a positive constant. The transformation T maps the points O, A, B and C to the points O, A', B' and C' respectively. The volume of the tetrahedron $OA'B'C'$ is 36.

(c) Use the determinant of $\mathbf{M}$ to find the value of k. **[4]**

Solution

(a) The vector product of $\mathbf{r} - (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$ with $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is $\mathbf{0}$, so $\mathbf{r} - (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$ is parallel to $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Hence ℓ has vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + \lambda \\ 2 - 2\lambda \\ 5 + 2\lambda \end{pmatrix}$$

Since C lies on ℓ, $\overrightarrow{OC}$ takes this form for some value of λ. Since OC is perpendicular to ℓ, the scalar product of $\overrightarrow{OC}$ with the direction of ℓ is zero:

$$\begin{pmatrix} 3 + \lambda \\ 2 - 2\lambda \\ 5 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 0$$

$$(3 + \lambda) - 2(2 - 2\lambda) + 2(5 + 2\lambda) = 0$$

$$9 + 9\lambda = 0$$

$$\lambda = -1$$

Substituting $\lambda = -1$ gives

$$\overrightarrow{OC} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

The position vector of C is $2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$.

(b) The volume of tetrahedron $OABC$ is

$$V = \frac{1}{6}\left|\overrightarrow{OC} \cdot \left(\overrightarrow{OA} \times \overrightarrow{OB}\right)\right|$$

First find the vector product:

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Then the scalar triple product is

$$\overrightarrow{OC} \cdot \left(\overrightarrow{OA} \times \overrightarrow{OB} \right) = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 4 - 4 + 18 = 18$$

so

$$V = \frac{1}{6} \times 18 = 3$$

(c) Expanding along the top row:

$$\begin{aligned} \det(\mathbf{M}) &= k \begin{vmatrix} k & 2 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + 0 \\ &= k(k-2) - 3 \\ &= k^2 - 2k - 3 \end{aligned}$$

The transformation T multiplies volumes by $|\det(\mathbf{M})|$, and the volume of $OABC$ is 3 by part (b), so

$$36 = 3|\det(\mathbf{M})| \quad \Longrightarrow \quad |\det(\mathbf{M})| = 12$$

If $\det(\mathbf{M}) = 12$, then

$$k^2 - 2k - 3 = 12$$

$$k^2 - 2k - 15 = 0$$

$$(k-5)(k+3) = 0$$

so $k = 5$ or $k = -3$.

If $\det(\mathbf{M}) = -12$, then

$$k^2 - 2k + 9 = 0$$

whose discriminant is $b^2 - 4ac = (-2)^2 - 4 \times 1 \times 9 = -32 < 0$, so there are no real values of k in this case.

Since k is positive, $k = 5$.

3. The points A and B have position vectors $\mathbf{a} = p\mathbf{i} + 2\mathbf{j} + q\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ respectively, relative to the origin O.

(a) Given that $\mathbf{a} \times \mathbf{b}$ is parallel to $-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and that $\mathbf{a} \cdot \mathbf{b} = -15$, find the value of p and the value of q. **[4]**

(b) The point C has coordinates (d, e, f) and the tetrahedron $OABC$ has volume 4.

(i) Using your values of p and q, find the two possible equations linking d, e and f. **[3]**

(ii) Describe the geometrical significance of these equations, stating the perpendicular distance of C from the plane OAB. **[2]**

Solution

(a) The vector product is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & 2 & q \\ 1 & -2 & 2 \end{vmatrix}$$

$$= (4 + 2q)\mathbf{i} - (2p - q)\mathbf{j} + (-2p - 2)\mathbf{k}$$

$$= \begin{pmatrix} 4 + 2q \\ q - 2p \\ -2p - 2 \end{pmatrix}$$

Since $\mathbf{a} \times \mathbf{b}$ is parallel to $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, there is a scalar λ with

$$\begin{pmatrix} 4 + 2q \\ q - 2p \\ -2p - 2 \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

The third component gives $\lambda = -p - 1$. Substituting into the second component:

$$q - 2p = -p - 1 \quad \Longrightarrow \quad q = p - 1$$

The first component, $4 + 2q = 2p + 2$, gives the same relation. The scalar product condition is

$$\mathbf{a} \cdot \mathbf{b} = p \times 1 + 2 \times (-2) + q \times 2 = p - 4 + 2q = -15$$

so $p + 2q = -11$. Substituting $q = p - 1$:

$$p + 2(p - 1) = -11 \quad \Longrightarrow \quad 3p = -9$$

so

$$p = -3, \qquad q = -4$$

(b) (i) With $p = -3$ and $q = -4$, part (a) gives

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

The volume of tetrahedron $OABC$ is

$$V = \frac{1}{6}|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$$

where $\mathbf{c} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$. Then

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = -4d + 2e + 4f$$

so $V = 4$ gives

$$\frac{1}{6}|-4d + 2e + 4f| = 4 \quad \Longrightarrow \quad |-2d + e + 2f| = 12$$

The two possible equations are

$$-2d + e + 2f = 12 \qquad \text{or} \qquad -2d + e + 2f = -12$$

(ii) The plane OAB passes through O and has normal $\mathbf{a} \times \mathbf{b}$, which is parallel to $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, so its equation is $-2x + y + 2z = 0$. Each equation in part (b)(i) says that C lies in a plane $-2x + y + 2z = \pm 12$ with the same normal, so these are two planes parallel to plane OAB, one on each side of it, and C lies in one of them.

Since $|-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}| = \sqrt{4 + 1 + 4} = 3$, dividing by 3 puts the planes in the form

$$\mathbf{r} \cdot \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \pm 4$$

with a unit normal, so each plane is a distance 4 from the origin, and so from the parallel plane OAB through O. The perpendicular distance of C from the plane OAB is 4.

4. The points A, B, C and D have coordinates $(2,-1,1)$, $(3,p,0)$, $(p,1,3)$ and $(4,0,2)$ respectively, where p is a constant.

(a) Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of p. **[2]**

(b) Show that

$$\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=(6-p)(p+2)$$ **[5]**

(c) Find the values of p for which the points A, B, C and D are coplanar, justifying your answer. **[2]**

(d) Given that the tetrahedron $ABCD$ has volume 8, find the possible values of p. **[4]**

Solution

(a) Using $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$, and similarly for C and D:

$$\overrightarrow{AB}=\begin{pmatrix}3\\p\\0\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\p+1\\-1\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}p\\1\\3\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}p-2\\2\\2\end{pmatrix}$$

$$\overrightarrow{AD}=\begin{pmatrix}4\\0\\2\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix}$$

(b) First find the vector product:

$$\begin{aligned}\overrightarrow{AB}\times\overrightarrow{AC}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&p+1&-1\\p-2&2&2\end{vmatrix}\\&=\big(2(p+1)+2\big)\mathbf{i}-\big(2+(p-2)\big)\mathbf{j}+\big(2-(p+1)(p-2)\big)\mathbf{k}\\&=\begin{pmatrix}2p+4\\-p\\-p^2+p+4\end{pmatrix}\end{aligned}$$

Now take the scalar product with $\overrightarrow{AD}$:

$$\begin{aligned}\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)&=\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2p+4\\-p\\-p^2+p+4\end{pmatrix}\\&=2(2p+4)-p+\left(-p^2+p+4\right)\\&=-p^2+4p+12\\&=(6-p)(p+2)\end{aligned}$$

as required.

(c) The points A, B, C and D are coplanar exactly when $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ lie in one plane, so that

the parallelepiped they define has zero volume. This happens when the scalar triple product is zero. Using part (b):

$$(6-p)(p+2)=0$$

so $p=6$ or $p=-2$.

(d) The volume of tetrahedron $ABCD$ is

$$V=\frac{1}{6}\left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|$$

so, using part (b):

$$\frac{1}{6}|(6-p)(p+2)|=8 \quad\Longrightarrow\quad -p^2+4p+12=\pm 48$$

Taking $-p^2+4p+12=48$ gives

$$p^2-4p+36=0$$

which has discriminant $b^2-4ac=(-4)^2-4\times 1\times 36=-128<0$, so there are no real solutions.

Taking $-p^2+4p+12=-48$ gives

$$p^2-4p-60=0$$

$$(p-10)(p+6)=0$$

so $p=10$ or $p=-6$.

5. The parallelogram P has vertices A, B, C and D, in that order. The vertices A, B and D have coordinates

$$A(3,-2,1) \qquad B(5,-3,0) \qquad D(0,-6,2)$$

(a) Find the coordinates of C, and determine the exact area of P, giving your answer as a simplified surd. **[4]**

The line ℓ passes through C and is perpendicular to the plane containing P.

(b) Determine an equation for ℓ, giving your answer in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. **[2]**

The parallelogram P is one face of a parallelepiped. The opposite face of the parallelepiped lies in the plane Π, and the point $Q(k,k,4)$, where k is a constant, lies in Π.

(c) Given that the volume of the parallelepiped is 36, determine the two possible values of k. **[5]**

Solution

(a) The two sides from A are

$$\overrightarrow{AB}=\begin{pmatrix}5\\-3\\0\end{pmatrix}-\begin{pmatrix}3\\-2\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\-1\end{pmatrix}, \qquad \overrightarrow{AD}=\begin{pmatrix}0\\-6\\2\end{pmatrix}-\begin{pmatrix}3\\-2\\1\end{pmatrix}=\begin{pmatrix}-3\\-4\\1\end{pmatrix}$$

Since $ABCD$ is a parallelogram, $\overrightarrow{DC}=\overrightarrow{AB}$, so

$$\overrightarrow{OC}=\overrightarrow{OD}+\overrightarrow{AB}=\begin{pmatrix}0\\-6\\2\end{pmatrix}+\begin{pmatrix}2\\-1\\-1\end{pmatrix}=\begin{pmatrix}2\\-7\\1\end{pmatrix}$$

Hence C has coordinates $(2,-7,1)$.

The area of the parallelogram is $\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$, where

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&-1\\-3&-4&1\end{vmatrix}=\begin{pmatrix}-5\\1\\-11\end{pmatrix}$$

so the area of P is

$$\sqrt{(-5)^2+1^2+(-11)^2}=\sqrt{147}=7\sqrt{3}$$

(b) The line ℓ is perpendicular to the plane containing P, so its direction is that of the normal $\overrightarrow{AB}\times\overrightarrow{AD}$, or any multiple of it, such as $5\mathbf{i}-\mathbf{j}+11\mathbf{k}$. The line passes through $C(2,-7,1)$, so an equation for ℓ is

$$\left(\mathbf{r}-\begin{pmatrix}2\\-7\\1\end{pmatrix}\right)\times\begin{pmatrix}5\\-1\\11\end{pmatrix}=\mathbf{0}$$

(c) The volume of the parallelepiped is the area of the face P multiplied by the perpendicular distance between the plane of P and Π. Since Q lies in Π, this distance is the component of $\overrightarrow{AQ}$ in the direction of the normal $\mathbf{n}=\overrightarrow{AB}\times\overrightarrow{AD}$, so

$$V=|\mathbf{n}|\times\frac{\left|\overrightarrow{AQ}\cdot\mathbf{n}\right|}{|\mathbf{n}|}=\left|\overrightarrow{AQ}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AD}\right)\right|$$

which is the modulus of the scalar triple product of $\overrightarrow{AQ}$, $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Here

$$\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \begin{pmatrix} k \\ k \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} k-3 \\ k+2 \\ 3 \end{pmatrix}$$

Using part (a):

$$\begin{aligned} \overrightarrow{AQ} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AD} \right) &= \begin{pmatrix} k-3 \\ k+2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 1 \\ -11 \end{pmatrix} \\ &= -5(k-3) + (k+2) - 33 \\ &= -4k - 16 \end{aligned}$$

The volume is 36, so

$$|-4k - 16| = 36 \quad \Longrightarrow \quad 4k + 16 = 36 \quad \text{or} \quad 4k + 16 = -36$$

giving

$$k = 5 \quad \text{or} \quad k = -13$$

6. With respect to a fixed origin O, the points A, B and C have coordinates $(1, 1, 4)$, $(4, 1, -2)$ and $(1, -3, 0)$ respectively.
The plane Π has equation $2x - y + 2z = 7$.
The line segment AB meets Π at the point P, and the line segment AC meets Π at the point Q.

(a) Find the exact area of the quadrilateral $PBCQ$. **[6]**

The point D has coordinates $(-2, 4, 7)$. The line segment AD meets Π at the point R.

(b) Use the scalar triple product to find the volume of the tetrahedron $APQR$. **[6]**

(c) The plane Π divides the tetrahedron $ABCD$ into two pieces. Find the ratio of the volume of the smaller piece to the volume of the larger piece. **[3]**

Solution

(a) The direction of AB is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

so a point on the line AB has position vector

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 + 3t \\ 1 \\ 4 - 6t \end{pmatrix}$$

Substituting into the equation of Π:

$$\begin{aligned} 2(1 + 3t) - 1 + 2(4 - 6t) &= 7 \\ 9 - 6t &= 7 \\ t &= \frac{1}{3} \end{aligned}$$

Since $0 \leq t \leq 1$, this point lies on the segment AB, and

$$\overrightarrow{OP} = \begin{pmatrix} 1 + 1 \\ 1 \\ 4 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

so P is $(2, 1, 2)$.

The direction of AC is

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix}$$

so a point on the line AC has position vector

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + s \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - 4s \\ 4 - 4s \end{pmatrix}$$

Substituting into the equation of Π:

$$2-(1-4s)+2(4-4s)=7$$
$$9-4s=7$$
$$s=\frac{1}{2}$$

Since $0 \leq s \leq 1$, this point lies on the segment AC, and

$$\overrightarrow{OQ}=\begin{pmatrix}1\\1-2\\4-2\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

so Q is $(1,-1,2)$.

As P lies on AB and Q lies on AC, the quadrilateral $PBCQ$ is the triangle ABC with the triangle APQ removed.

The area of triangle ABC is

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

where

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&0&-6\\0&-4&-4\end{vmatrix}=\begin{pmatrix}-24\\12\\-12\end{pmatrix}$$

so

$$\text{Area of } ABC=\frac{1}{2}\sqrt{(-24)^2+12^2+(-12)^2}=\frac{1}{2}\sqrt{864}=\frac{1}{2}\times 12\sqrt{6}=6\sqrt{6}$$

The area of triangle APQ is

$$\frac{1}{2}\left|\overrightarrow{AP}\times\overrightarrow{AQ}\right|$$

where

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\begin{pmatrix}1\\0\\-2\end{pmatrix},\qquad \overrightarrow{AQ}=\overrightarrow{OQ}-\overrightarrow{OA}=\begin{pmatrix}0\\-2\\-2\end{pmatrix}$$

and

$$\overrightarrow{AP}\times\overrightarrow{AQ}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&-2\\0&-2&-2\end{vmatrix}=\begin{pmatrix}-4\\2\\-2\end{pmatrix}$$

so

$$\text{Area of } APQ=\frac{1}{2}\sqrt{(-4)^2+2^2+(-2)^2}=\frac{1}{2}\sqrt{24}=\frac{1}{2}\times 2\sqrt{6}=\sqrt{6}$$

Hence

$$\text{Area of } PBCQ=6\sqrt{6}-\sqrt{6}=5\sqrt{6}$$

(b) The direction of AD is

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

so a point on the line AD has position vector

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + u \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 - 3u \\ 1 + 3u \\ 4 + 3u \end{pmatrix}$$

Substituting into the equation of Π:

$$\begin{aligned} 2(1 - 3u) - (1 + 3u) + 2(4 + 3u) &= 7 \\ 9 - 3u &= 7 \\ u &= \frac{2}{3} \end{aligned}$$

Since $0 \leq u \leq 1$, this point lies on the segment AD, and

$$\overrightarrow{OR} = \begin{pmatrix} 1 - 2 \\ 1 + 2 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}, \qquad \overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

The volume of tetrahedron $APQR$ is

$$V = \frac{1}{6} \left| \overrightarrow{AR} \cdot \left(\overrightarrow{AP} \times \overrightarrow{AQ} \right) \right|$$

Using $\overrightarrow{AP} \times \overrightarrow{AQ}$ from part (a):

$$\overrightarrow{AR} \cdot \left(\overrightarrow{AP} \times \overrightarrow{AQ} \right) = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = (-2) \times (-4) + 2 \times 2 + 2 \times (-2) = 8$$

so

$$V = \frac{1}{6} \times 8 = \frac{4}{3}$$

(c) Substituting the vertices into $2x - y + 2z$:

$$\begin{aligned} A: \ & 2 - 1 + 8 = 9 > 7 \\ B: \ & 8 - 1 - 4 = 3 < 7 \\ C: \ & 2 + 3 + 0 = 5 < 7 \\ D: \ & -4 - 4 + 14 = 6 < 7 \end{aligned}$$

So A lies on one side of Π and B, C, D on the other. The plane meets the edges AB, AC, AD at P, Q, R, so one piece is the tetrahedron $APQR$, of volume $\frac{4}{3}$ from part (b).

The volume of tetrahedron $ABCD$ is

$$\text{Volume of } ABCD = \frac{1}{6} \left| \overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC} \right) \right|$$

Using $\overrightarrow{AB} \times \overrightarrow{AC}$ from part (a):

$$\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -24 \\ 12 \\ -12 \end{pmatrix} = (-3) \times (-24) + 3 \times 12 + 3 \times (-12) = 72$$

so Volume of $ABCD = \frac{1}{6} \times 72 = 12$, and the other piece has volume

$$12 - \frac{4}{3} = \frac{32}{3}$$

The smaller piece is $APQR$, so the ratio of the volumes is

$$\frac{4}{3} : \frac{32}{3} = 1 : 8$$

7. The vectors $\mathbf{b}$ and $\mathbf{d}$ are non-zero and not parallel. The lines ℓ_1 and ℓ_2 have vector equations

$$\ell_1 : \ \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R} \qquad \text{and} \qquad \ell_2 : \ (\mathbf{r} - \mathbf{c}) \times \mathbf{d} = \mathbf{0}$$

The lines ℓ_1 and ℓ_2 intersect.

(a) Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{d})$, giving a reason for each step of your argument. **[4]**

(b) The plane Π contains both ℓ_1 and ℓ_2, and has equation $\mathbf{r} \cdot \mathbf{n} = p$.
Write down a suitable vector $\mathbf{n}$ in terms of $\mathbf{b}$ and $\mathbf{d}$. Give two different expressions for p, one in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ and the other in terms of $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, explaining your reasoning. **[3]**

Solution

(a) The vector product of $\mathbf{r} - \mathbf{c}$ with the non-zero vector $\mathbf{d}$ is $\mathbf{0}$ exactly when $\mathbf{r} - \mathbf{c}$ is parallel to $\mathbf{d}$ (or is $\mathbf{0}$), that is, when $\mathbf{r} - \mathbf{c}$ is a scalar multiple of $\mathbf{d}$. So ℓ_2 has the vector equation

$$\ell_2 : \ \mathbf{r} = \mathbf{c} + \mu\mathbf{d}, \ \mu \in \mathbb{R}$$

The lines intersect, so there are particular values λ^* and μ^* of the parameters that give the same point:

$$\mathbf{a} + \lambda^*\mathbf{b} = \mathbf{c} + \mu^*\mathbf{d}$$

Take the scalar product of both sides with $\mathbf{b} \times \mathbf{d}$. The scalar product distributes over addition and scalars can be taken outside, so

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) + \lambda^* \, \mathbf{b} \cdot (\mathbf{b} \times \mathbf{d}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{d}) + \mu^* \, \mathbf{d} \cdot (\mathbf{b} \times \mathbf{d})$$

The vector $\mathbf{b} \times \mathbf{d}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{d}$, so its scalar product with each of them is zero:

$$\mathbf{b} \cdot (\mathbf{b} \times \mathbf{d}) = 0, \qquad \mathbf{d} \cdot (\mathbf{b} \times \mathbf{d}) = 0$$

The terms in λ^* and μ^* therefore vanish, leaving

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{d})$$

as required.

(b) The plane Π contains ℓ_1 and ℓ_2, so it contains their directions $\mathbf{b}$ and $\mathbf{d}$, and a normal to Π must be perpendicular to both. The vector $\mathbf{b} \times \mathbf{d}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{d}$, and it is non-zero because $\mathbf{b}$ and $\mathbf{d}$ are non-zero and not parallel. A suitable normal is

$$\mathbf{n} = \mathbf{b} \times \mathbf{d}$$

The point with position vector $\mathbf{a}$ lies on ℓ_1 (taking $\lambda = 0$), so it lies in Π and satisfies $\mathbf{a} \cdot \mathbf{n} = p$. The point with position vector $\mathbf{c}$ lies on ℓ_2, since $(\mathbf{c} - \mathbf{c}) \times \mathbf{d} = \mathbf{0}$, so it also lies in Π and satisfies $\mathbf{c} \cdot \mathbf{n} = p$. Hence

$$p = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) \qquad \text{and} \qquad p = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{d})$$

8. The points P, Q and R have position vectors $p\mathbf{i}$, $q\mathbf{j}$ and $r\mathbf{k}$ respectively, relative to the origin O, where p, q and r are positive constants. The points O, P, Q and R are the vertices of a tetrahedron.

(a) Use the scalar triple product to find the volume of the tetrahedron $OPQR$ in terms of p, q and r. **[3]**

(b) Find $\overrightarrow{RP} \times \overrightarrow{RQ}$ in terms of p, q and r, and hence show that the area of triangle PQR is

$$\frac{1}{2}\sqrt{q^2r^2 + r^2p^2 + p^2q^2}$$ **[4]**

(c) The perpendicular distance from O to the plane PQR is h. By finding the volume of the tetrahedron in a second way, show that

$$\frac{1}{h^2} = \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$$

Hence find the exact value of h when $p = 2$, $q = 3$ and $r = 6$. **[6]**

Solution

(a) The volume of tetrahedron $OPQR$ is

$$V = \frac{1}{6}\left|\overrightarrow{OR} \cdot \left(\overrightarrow{OP} \times \overrightarrow{OQ}\right)\right|$$

where

$$\overrightarrow{OP} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \qquad \overrightarrow{OQ} = \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix}, \qquad \overrightarrow{OR} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

The vector product is

$$\begin{aligned} \overrightarrow{OP} \times \overrightarrow{OQ} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & 0 & 0 \\ 0 & q & 0 \end{vmatrix} \\ &= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (pq-0)\mathbf{k} \\ &= \begin{pmatrix} 0 \\ 0 \\ pq \end{pmatrix} \end{aligned}$$

so the scalar triple product is

$$\overrightarrow{OR} \cdot \left(\overrightarrow{OP} \times \overrightarrow{OQ}\right) = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ pq \end{pmatrix} = pqr$$

Since p, q and r are positive, $pqr > 0$, so

$$V = \frac{pqr}{6}$$

(b) Using $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$ and $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$:

$$\overrightarrow{RP} = \begin{pmatrix} p \\ 0 \\ -r \end{pmatrix}, \qquad \overrightarrow{RQ} = \begin{pmatrix} 0 \\ q \\ -r \end{pmatrix}$$

The vector product is

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & 0 & -r \\ 0 & q & -r \end{vmatrix}$$
$$= \big(0 - (-rq)\big)\mathbf{i} - \big(-pr - 0\big)\mathbf{j} + (pq - 0)\mathbf{k}$$
$$= \begin{pmatrix} qr \\ rp \\ pq \end{pmatrix}$$

The area of triangle PQR is

$$\frac{1}{2}\left|\overrightarrow{RP} \times \overrightarrow{RQ}\right|$$

so

$$\text{Area} = \frac{1}{2}\sqrt{q^2r^2 + r^2p^2 + p^2q^2}$$

as required.

(c) Taking triangle PQR as the base, the perpendicular height of the tetrahedron is h, so its volume is

$$V = \frac{1}{3} \times \text{base area} \times \text{height} = \frac{1}{3} \times \frac{1}{2}\sqrt{q^2r^2 + r^2p^2 + p^2q^2} \times h$$

Equating this to the volume from part (a):

$$\frac{h}{6}\sqrt{q^2r^2 + r^2p^2 + p^2q^2} = \frac{pqr}{6}$$
$$h = \frac{pqr}{\sqrt{q^2r^2 + r^2p^2 + p^2q^2}}$$
$$\frac{1}{h^2} = \frac{q^2r^2 + r^2p^2 + p^2q^2}{p^2q^2r^2}$$
$$= \frac{q^2r^2}{p^2q^2r^2} + \frac{r^2p^2}{p^2q^2r^2} + \frac{p^2q^2}{p^2q^2r^2}$$
$$= \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$$

as required.

When $p = 2$, $q = 3$ and $r = 6$:

$$\frac{1}{h^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{36} = \frac{9 + 4 + 1}{36} = \frac{14}{36} = \frac{7}{18}$$

Since $h > 0$, taking the positive square root gives

$$h = \sqrt{\frac{18}{7}} = \frac{3\sqrt{2}}{\sqrt{7}} = \frac{3\sqrt{14}}{7}$$

9. The vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are given by

$$\mathbf{p} = \begin{pmatrix} 2 \\ -1 \\ k \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

where k is a constant.
It is given that $\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = -21$.

(a) Find the value of k, and hence calculate $\mathbf{p} \times (\mathbf{q} \times \mathbf{r})$ and $(\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$. **[7]**

(b) Using your answers to part (a), state whether the vector product is associative for three-dimensional column vectors with real components. Justify your answer. **[1]**

Throughout the rest of this question, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three-dimensional column vectors with real components.

(c) Explain geometrically why the vector $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ must be expressible in the form $\lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are scalar constants. **[2]**

It is given that the following relationship holds for all such $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \qquad (*)$$

(d) Use $(*)$ to show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$ **[3]**

Solution

(a) First find $\mathbf{q} \times \mathbf{r}$:

$$\mathbf{q} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 4 & 0 & 1 \end{vmatrix} = (3-0)\mathbf{i} - (1+8)\mathbf{j} + (0-12)\mathbf{k} = \begin{pmatrix} 3 \\ -9 \\ -12 \end{pmatrix}$$

Then the scalar triple product is

$$\begin{aligned} \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) &= \begin{pmatrix} 2 \\ -1 \\ k \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -9 \\ -12 \end{pmatrix} \\ &= 2 \times 3 + (-1) \times (-9) + k \times (-12) \\ &= 15 - 12k \end{aligned}$$

Setting this equal to -21:

$$15 - 12k = -21 \implies 12k = 36 \implies k = 3$$

So $\mathbf{p} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and

$$\begin{aligned}
\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 3 & -9 & -12 \end{vmatrix} \\
&= \big((-1) \times (-12) - 3 \times (-9)\big)\mathbf{i} - \big(2 \times (-12) - 3 \times 3\big)\mathbf{j} + \big(2 \times (-9) - (-1) \times 3\big)\mathbf{k} \\
&= \begin{pmatrix} 39 \\ 33 \\ -15 \end{pmatrix}
\end{aligned}$$

For the other bracketing, first find $\mathbf{p} \times \mathbf{q}$:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = (2-9)\mathbf{i} - (-4-3)\mathbf{j} + (6+1)\mathbf{k} = \begin{pmatrix} -7 \\ 7 \\ 7 \end{pmatrix}$$

Then

$$\begin{aligned}
(\mathbf{p} \times \mathbf{q}) \times \mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 7 & 7 \\ 4 & 0 & 1 \end{vmatrix} \\
&= (7-0)\mathbf{i} - (-7-28)\mathbf{j} + (0-28)\mathbf{k} \\
&= \begin{pmatrix} 7 \\ 35 \\ -28 \end{pmatrix}
\end{aligned}$$

(b) The vector product is not associative. Part (a) gives a counterexample:

$$\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) = \begin{pmatrix} 39 \\ 33 \\ -15 \end{pmatrix} \neq \begin{pmatrix} 7 \\ 35 \\ -28 \end{pmatrix} = (\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$$

(c) If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so it is a normal to the plane through the origin containing $\mathbf{a}$ and $\mathbf{b}$.

The vector $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$, so it is perpendicular to this normal and therefore lies in the plane containing $\mathbf{a}$ and $\mathbf{b}$. Every vector in that plane can be written as $\lambda\mathbf{a} + \mu\mathbf{b}$.

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, so $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0} = 0\mathbf{a} + 0\mathbf{b}$, which is also of the required form.

(d) Applying $(*)$ to each term, with the letters replaced cyclically for the second and third terms:

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \\
\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) &= (\mathbf{b} \cdot \mathbf{a})\,\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} \\
\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= (\mathbf{c} \cdot \mathbf{b})\,\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\,\mathbf{b}
\end{aligned}$$

The scalar product is commutative, so $\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{c} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{c}$. Adding the three lines and collecting the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

$$\begin{aligned}
&\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\
&\quad = \big(\mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c}\big)\mathbf{a} + \big(\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c}\big)\mathbf{b} + \big(\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}\big)\mathbf{c} \\
&\quad = \mathbf{0}
\end{aligned}$$

10. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = \begin{pmatrix} p - 2r + 4 \\ q + r - 6 \\ pq - 8 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} p - 2q \\ q^2 - r \\ qr - 2p \end{pmatrix}$$

where p, q and r are real numbers.
It is given that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ for every vector $\mathbf{c}$, and that $\mathbf{a}$ is **not** a multiple of $\mathbf{b}$.
Determine all possible sets of values of p, q and r. **[4]**

Solution

The scalar triple product is the determinant with rows $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Swapping two rows twice leaves the determinant unchanged, so

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

This is zero for every vector $\mathbf{c}$. In particular, taking $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ gives

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = |\mathbf{a} \times \mathbf{b}|^2 = 0 \quad \Longrightarrow \quad \mathbf{a} \times \mathbf{b} = \mathbf{0}$$

So $\mathbf{a}$ and $\mathbf{b}$ are parallel, or one of them is $\mathbf{0}$. If $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a} = k\mathbf{b}$ for some scalar k (with $k = 0$ when $\mathbf{a} = \mathbf{0}$), which is not allowed. Hence $\mathbf{b} = \mathbf{0}$, and also $\mathbf{a} \neq \mathbf{0}$, since $\mathbf{0} = 1\mathbf{b}$ would be a multiple of $\mathbf{b}$.

Setting each component of $\mathbf{b}$ to zero:

$$p - 2q = 0 \quad \Longrightarrow \quad p = 2q$$

$$q^2 - r = 0 \quad \Longrightarrow \quad r = q^2$$

$$qr - 2p = 0$$

Substituting the first two into the third:

$$q^3 - 4q = 0 \quad \Longrightarrow \quad q(q-2)(q+2) = 0 \quad \Longrightarrow \quad q = 0,\ 2 \text{ or } -2$$

Now check that $\mathbf{a} \neq \mathbf{0}$ in each case.

When $q = 0$: $p = 0$, $r = 0$, and

$$\mathbf{a} = \begin{pmatrix} 4 \\ -6 \\ -8 \end{pmatrix} \neq \mathbf{0}$$

When $q = 2$: $p = 4$, $r = 4$, and

$$\mathbf{a} = \begin{pmatrix} 4 - 8 + 4 \\ 2 + 4 - 6 \\ 8 - 8 \end{pmatrix} = \mathbf{0}$$

so this case is rejected.

When $q = -2$: $p = -4$, $r = 4$, and

$$\mathbf{a} = \begin{pmatrix} -4 - 8 + 4 \\ -2 + 4 - 6 \\ 8 - 8 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 0 \end{pmatrix} \neq \mathbf{0}$$

Hence the possible sets of values are $p = 0$, $q = 0$, $r = 0$, or $p = -4$, $q = -2$, $r = 4$.