OPTIONAL PURE / VECTORS & MATRICES

Reflections in Planes

Worked solutions 10 questions · 100 marks

EXAM BOARDS OCR MEI Extra Pure Y435 · Edexcel 9FM0 · AQA 7367 · OCR A H245

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1. The plane Π has equation $x + 2y - 2z = 6$ and the point A has coordinates $(5, 3, -2)$.

Find the coordinates of the reflection of A in Π. **[4]**

Solution

The reflection of the point with position vector $\mathbf{p}$ in the plane $\mathbf{r} \cdot \mathbf{n} = d$ has position vector

$$\mathbf{p}' = \mathbf{p} - 2\,\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\,\mathbf{n}$$

The equation $x + 2y - 2z = 6$ is $\mathbf{r} \cdot \mathbf{n} = 6$ with $\mathbf{n} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Here

$$\mathbf{p} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \qquad d = 6$$

so

$$\mathbf{p} \cdot \mathbf{n} = 5 + 6 + 4 = 15$$

$$|\mathbf{n}|^2 = 1 + 4 + 4 = 9$$

Substituting into the formula:

$$\begin{aligned} \mathbf{p}' &= \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} - 2 \times \frac{15 - 6}{9} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

The reflection of A in Π is $(3, -1, 2)$.

Alternative Solution:

We find the reflection by going from A along the normal to the plane and doubling the distance.

The line through A perpendicular to Π is

$$\mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

Substituting $(x, y, z) = (5 + t,\ 3 + 2t,\ -2 - 2t)$ into $x + 2y - 2z = 6$:

$$\begin{aligned} (5 + t) + 2(3 + 2t) - 2(-2 - 2t) &= 6 \\ 15 + 9t &= 6 \\ t &= -1 \end{aligned}$$

So this line meets Π at the point $M(4, 1, 0)$.

Since M is the midpoint of A and its reflection A', $\overrightarrow{AA'} = 2\overrightarrow{AM}$:

$$\begin{aligned}\overrightarrow{OA'} &= \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 4-5 \\ 1-3 \\ 0-(-2) \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\end{aligned}$$

So the reflection of A in Π is $(3, -1, 2)$, as before.

2. The plane Π has equation

$$\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2$$

The point P has position vector $5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$.

(a) Find the position vector of P', the reflection of P in Π. **[4]**

The line ℓ passes through P and is parallel to the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(b) Find the position vector of the point where ℓ meets Π. **[3]**

(c) Hence find a vector equation of the line ℓ', the reflection of ℓ in Π. **[2]**

Solution

(a) The reflection of the point with position vector $\mathbf{p}$ in the plane $\mathbf{r} \cdot \mathbf{n} = d$ has position vector

$$\mathbf{p}' = \mathbf{p} - 2\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\mathbf{n}$$

Here

$$\mathbf{p} = \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad d = 2$$

so

$$\mathbf{p} \cdot \mathbf{n} = 10 + 4 + 6 = 20$$

$$|\mathbf{n}|^2 = 4 + 1 + 4 = 9$$

Substituting into the formula:

$$\begin{aligned} \mathbf{p}' &= \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix} - 2 \times \frac{20 - 2}{9} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix} - 4 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} \end{aligned}$$

Alternative Solution:

We find P' by going from P along the normal to the plane and doubling the distance.

The line through P perpendicular to Π is

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Substituting into the equation of Π:

$$\begin{aligned} 2(5+2t)+(4+t)-2(-3-2t) &= 2 \\ 20+9t &= 2 \\ t &= -2 \end{aligned}$$

So this line meets Π at the point M with position vector

$$\overrightarrow{OM} = \begin{pmatrix} 5-4 \\ 4-2 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Since M is the midpoint of PP', $\overrightarrow{PP'} = 2\overrightarrow{PM}$:

$$\begin{aligned} \overrightarrow{OP'} &= \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix} + 2\begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} \end{aligned}$$

So $\overrightarrow{OP'} = -3\mathbf{i} + 5\mathbf{k}$, as before.

(b) The line ℓ has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

so a general point on ℓ has position vector

$$\begin{pmatrix} 5+\lambda \\ 4+2\lambda \\ -3-\lambda \end{pmatrix}$$

Substituting into the equation of Π:

$$\begin{aligned} 2(5+\lambda)+(4+2\lambda)-2(-3-\lambda) &= 2 \\ 20+6\lambda &= 2 \\ \lambda &= -3 \end{aligned}$$

So ℓ meets Π at the point Q with position vector

$$\overrightarrow{OQ} = \begin{pmatrix} 5-3 \\ 4-6 \\ -3+3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

(c) The point Q lies in Π, so it is its own reflection and lies on ℓ'. The point P lies on ℓ, so its reflection P' lies on ℓ'. Using parts (a) and (b), a direction vector of ℓ' is

$$\overrightarrow{QP'} = \overrightarrow{OP'} - \overrightarrow{OQ} = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 5 \end{pmatrix}$$

A vector equation of ℓ' is

$$\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ 5 \end{pmatrix}$$

3. The point $B(4, 5, 2)$ is the reflection of the point $A(0, 1, 4)$ in the plane Π.

(a) Find a Cartesian equation of Π. **[3]**

The line ℓ has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(b) Show that ℓ lies in Π, and hence write down an equation of the reflection of ℓ in Π. **[3]**

Solution

(a) Since B is the reflection of A in Π, the plane Π is perpendicular to AB and passes through the midpoint M of AB. First:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

so a normal to Π is

$$\mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

The midpoint of AB is given by

$$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) = \frac{1}{2}\begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

so M is $(2, 3, 3)$. The plane has equation $2x + 2y - z = d$, and substituting M gives

$$d = 2 \times 2 + 2 \times 3 - 3 = 7$$

Hence a Cartesian equation of Π is

$$2x + 2y - z = 7$$

(b) A general point of ℓ has position vector

$$\begin{pmatrix} 3 + \lambda \\ 1 \\ 1 + 2\lambda \end{pmatrix}$$

Substituting into the left-hand side of the equation of Π:

$$2(3 + \lambda) + 2(1) - (1 + 2\lambda) = 6 + 2\lambda + 2 - 1 - 2\lambda = 7$$

This equals 7 for every value of λ, so every point of ℓ lies in Π. Hence ℓ lies in Π, as required.

A point in Π is its own reflection in Π, so each point of ℓ is unchanged by the reflection. The reflection of ℓ in Π is therefore ℓ itself, with equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

4. A plane passes through the points $C(1,-1,2)$, $D(3,0,1)$ and $E(2,2,2)$.

The point A has coordinates $(4,-2,7)$ and the point B is the reflection of A in the plane.

Find the coordinates of the point B. **[8]**

Solution

Let the plane be Π.

Two direction vectors in the plane are

$$\overrightarrow{CD} = \begin{pmatrix} 3-1 \\ 0-(-1) \\ 1-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

and

$$\overrightarrow{CE} = \begin{pmatrix} 2-1 \\ 2-(-1) \\ 2-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

So a normal vector to the plane is

$$\begin{aligned} \mathbf{n} &= \overrightarrow{CD} \times \overrightarrow{CE} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 3 & 0 \end{vmatrix} \\ &= \mathbf{i}(1 \times 0 - (-1) \times 3) - \mathbf{j}(2 \times 0 - (-1) \times 1) + \mathbf{k}(2 \times 3 - 1 \times 1) \\ &= \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \end{aligned}$$

Using the form $\mathbf{r} \cdot \mathbf{n} = d$, we find d from point $C(1,-1,2)$:

$$\begin{aligned} d &= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \\ &= 1 \times 3 + (-1) \times (-1) + 2 \times 5 \\ &= 3 + 1 + 10 \\ &= 14 \end{aligned}$$

Hence the plane is

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 14$$

so a Cartesian equation of the plane is

$$3x - y + 5z = 14$$

Now use the reflection formula for a point in the plane $\mathbf{r} \cdot \mathbf{n} = d$:

$$\mathbf{p}' = \mathbf{p} - 2\,\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\,\mathbf{n}$$

Here

$$\mathbf{p} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \qquad d = 14$$

First calculate $\mathbf{p} \cdot \mathbf{n} - d$:

$$\begin{aligned} \mathbf{p} \cdot \mathbf{n} - d &= \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - 14 \\ &= 4 \times 3 + (-2) \times (-1) + 7 \times 5 - 14 \\ &= 12 + 2 + 35 - 14 \\ &= 35 \end{aligned}$$

Also

$$|\mathbf{n}|^2 = 3^2 + (-1)^2 + 5^2 = 9 + 1 + 25 = 35$$

Substituting into the reflection formula:

$$\begin{aligned} \mathbf{p}' &= \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} - 2\left(\frac{35}{35}\right)\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 10 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix} \end{aligned}$$

Therefore the coordinates of B are

$$B = (-2, 0, -3)$$

Alternative Solution:

We find B by going from A along the normal to the plane and doubling the distance.

The line through A perpendicular to Π is

$$\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

Substituting $(x, y, z) = (4 + 3\lambda,\ -2 - \lambda,\ 7 + 5\lambda)$ into $3x - y + 5z = 14$:

$$\begin{aligned} 3(4 + 3\lambda) - (-2 - \lambda) + 5(7 + 5\lambda) &= 14 \\ 49 + 35\lambda &= 14 \\ \lambda &= -1 \end{aligned}$$

So this line meets Π at $(1, -1, 2)$, which is the point C.

Since C is the midpoint of AB, $\overrightarrow{AB} = 2\overrightarrow{AC}$:

$$\begin{aligned}\overrightarrow{OB} &= \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 1-4 \\ -1-(-2) \\ 2-7 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + 2\begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}\end{aligned}$$

So $B = (-2, 0, -3)$, as before.

5. A line ℓ has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Another line m has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

A plane Π contains m and is parallel to ℓ.

(i) Find an equation of Π, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = p$. **[4]**

(ii) Find the distance between ℓ and Π. **[2]**

(iii) Find an equation of the line which is the reflection of ℓ in Π, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[5]**

Solution

(i) Let the direction vector of line ℓ be

$$\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

and the direction vector of line m be

$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Since Π contains m and is parallel to ℓ, both $\mathbf{u}$ and $\mathbf{b}$ lie in the plane, so a normal vector is

$$\begin{aligned} \mathbf{n} &= \mathbf{u} \times \mathbf{b} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} \\ &= \mathbf{i}(1 \times 1 - 0 \times (-1)) - \mathbf{j}(2 \times 1 - 0 \times 2) + \mathbf{k}(2 \times (-1) - 1 \times 2) \\ &= \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} \end{aligned}$$

The point $(1, 1, 2)$ lies on m, so for the plane equation $\mathbf{r} \cdot \mathbf{n} = p$:

$$\begin{aligned} p &= \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\ &= 1 - 2 - 8 \\ &= -9 \end{aligned}$$

Hence an equation of Π is

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} = -9$$

(ii) Since ℓ is parallel to Π, take the point $P = (4, 2, -3)$ on ℓ. Using the point-to-plane distance formula

$$\text{dist} = \frac{|\mathbf{p} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$$

with $\mathbf{p} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ and $d = p = -9$:

$$\begin{aligned} \mathbf{p} \cdot \mathbf{n} &= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} \\ &= 4 - 4 + 12 \\ &= 12 \end{aligned}$$

Also

$$\begin{aligned} |\mathbf{n}| &= \sqrt{1^2 + (-2)^2 + (-4)^2} \\ &= \sqrt{1 + 4 + 16} \\ &= \sqrt{21} \end{aligned}$$

Therefore

$$\begin{aligned} \text{dist} &= \frac{|12 - (-9)|}{\sqrt{21}} \\ &= \frac{21}{\sqrt{21}} \\ &= \sqrt{21} \end{aligned}$$

(iii) Use the reflection formulae for a plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$.

For a point:

$$\mathbf{p}' = \mathbf{p} - 2\,\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\,\mathbf{n}$$

For a direction vector:

$$\mathbf{b}' = \mathbf{b} - 2\,\frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}$$

Here

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}, \qquad d = -9$$

First reflect the direction vector of ℓ. Since ℓ is parallel to Π, $\mathbf{b} \cdot \mathbf{n} = 0$, so

$$\begin{aligned}\mathbf{b}' &= \mathbf{b} - 2\,\frac{0}{|\mathbf{n}|^2}\,\mathbf{n} \\ &= \mathbf{b} \\ &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\end{aligned}$$

So the direction is unchanged.

Now reflect the point $P = (4, 2, -3)$ on ℓ:

$$|\mathbf{n}|^2 = 1^2 + (-2)^2 + (-4)^2 = 21$$

$$\mathbf{p} \cdot \mathbf{n} - d = 12 - (-9) = 21$$

Hence

$$\begin{aligned}\mathbf{p}' &= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} - 2\left(\frac{21}{21}\right)\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}\end{aligned}$$

Alternative Solution:

Find the image of P directly, using the line through P perpendicular to Π:

$$\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 + s \\ 2 - 2s \\ -3 - 4s \end{pmatrix}$$

Substituting into the plane equation $\mathbf{r} \cdot \mathbf{n} = -9$:

$$\begin{aligned}(4 + s) - 2(2 - 2s) - 4(-3 - 4s) &= -9 \\ 12 + 21s &= -9 \\ s &= -1\end{aligned}$$

So this line meets Π at $Q(3,4,1)$. Since P' is as far beyond Q as P is before it, $\overrightarrow{PP'} = 2\overrightarrow{PQ}$:

$$\begin{aligned}\mathbf{p}' &= \mathbf{p} + 2(\mathbf{q} - \mathbf{p}) \\ &= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}\end{aligned}$$

as before.

Therefore an equation of the reflected line is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

6. The line ℓ_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

The plane Π_1 passes through the point $(2, 0, 0)$ and has normal vector

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

(a) Find the point of intersection of ℓ_1 and Π_1. **[3]**

The line ℓ_2 is the reflection of the line ℓ_1 in the plane Π_1.

(b) Show that a vector equation for the line ℓ_2 is

$$\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

where μ is a scalar parameter. **[6]**

The plane Π_2 contains the line ℓ_1 and the line ℓ_2.

(c) Determine a vector equation for the line of intersection of Π_1 and Π_2. **[2]**

Solution

(a) The plane Π_1 passes through $(2, 0, 0)$ and has normal vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, so its equation is

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right) = 0$$

Hence

$$3(x - 2) - y + 2z = 0$$

so

$$3x - y + 2z = 6$$

A general point on ℓ_1 is

$$(x, y, z) = (1 + 2\lambda,\ 4 + \lambda,\ \lambda)$$

Substituting into the plane equation:

$$\begin{aligned} 3(1+2\lambda) - (4+\lambda) + 2\lambda &= 6 \\ 3 + 6\lambda - 4 - \lambda + 2\lambda &= 6 \\ 7\lambda - 1 &= 6 \\ 7\lambda &= 7 \\ \lambda &= 1 \end{aligned}$$

So the point of intersection is

$$(1+2,\ 4+1,\ 1) = (3, 5, 1)$$

(b) Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

so that

$$\ell_1 : \ \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$$

Since Π_1 passes through $(2, 0, 0)$, its equation is

$$\Pi_1 : \ \mathbf{r} \cdot \mathbf{n} = 6$$

because

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 6$$

The reflection formulae for a plane $\mathbf{r} \cdot \mathbf{n} = d$ are

$$\mathbf{p}' = \mathbf{p} - 2\,\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\,\mathbf{n}$$

for a point, and

$$\mathbf{b}' = \mathbf{b} - 2\,\frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}$$

for a direction vector. Here $d = 6$ and

$$|\mathbf{n}|^2 = 3^2 + (-1)^2 + 2^2 = 14$$

First reflect the point with position vector $\mathbf{a}$, taking $\mathbf{p} = \mathbf{a}$:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{n} &= \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ &= 3 - 4 + 0 \\ &= -1 \end{aligned}$$

So

$$\begin{aligned}\mathbf{a}' &= \mathbf{a} - 2\,\frac{\mathbf{a}\cdot\mathbf{n} - 6}{|\mathbf{n}|^2}\,\mathbf{n} \\ &= \mathbf{a} - 2\,\frac{-1-6}{14}\,\mathbf{n} \\ &= \mathbf{a} - 2\left(-\frac{7}{14}\right)\mathbf{n} \\ &= \mathbf{a} + \mathbf{n} \\ &= \begin{pmatrix}1\\4\\0\end{pmatrix} + \begin{pmatrix}3\\-1\\2\end{pmatrix} \\ &= \begin{pmatrix}4\\3\\2\end{pmatrix}\end{aligned}$$

Alternative Solution:

Reflect $\mathbf{a}$ by finding where the line through $\mathbf{a}$ in the direction of $\mathbf{n}$ meets Π_1.

This line is

$$\mathbf{r} = \begin{pmatrix}1\\4\\0\end{pmatrix} + s\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

so a general point on it is $(1+3s,\ 4-s,\ 2s)$.

Substituting into the plane equation $3x - y + 2z = 6$:

$$\begin{aligned}3(1+3s) - (4-s) + 2(2s) &= 6 \\ 14s - 1 &= 6 \\ s &= \tfrac{1}{2}\end{aligned}$$

So the line meets Π_1 at $Q\left(\frac{5}{2}, \frac{7}{2}, 1\right)$. Q is halfway between $(1,4,0)$ and its image, so

$$\begin{aligned}\mathbf{a}' &= \mathbf{a} + 2(\mathbf{q} - \mathbf{a}) \\ &= \begin{pmatrix}1\\4\\0\end{pmatrix} + 2\begin{pmatrix}\frac{3}{2}\\-\frac{1}{2}\\1\end{pmatrix} \\ &= \begin{pmatrix}4\\3\\2\end{pmatrix}\end{aligned}$$

as before.

Now reflect the direction vector $\mathbf{b}$:

$$\begin{aligned}\mathbf{b}\cdot\mathbf{n} &= \begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix} \\ &= 6 - 1 + 2 \\ &= 7\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{b}' &= \mathbf{b} - 2\,\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}\\ &= \mathbf{b} - 2\,\frac{7}{14}\,\mathbf{n}\\ &= \mathbf{b} - \mathbf{n}\\ &= \begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix}\\ &= \begin{pmatrix}-1\\2\\-1\end{pmatrix}\end{aligned}$$

Alternative Solution:

The direction of ℓ_2 can also be found by joining two points on it.

From part (a), ℓ_1 meets Π_1 at $P(3,5,1)$. Since P lies on Π_1, the reflection does not move it, so P lies on ℓ_2.

The point $(1,4,0)$ on ℓ_1 is not on Π_1, since $\mathbf{a}\cdot\mathbf{n} = -1 \neq 6$. Its image $A'(4,3,2)$, found above, also lies on ℓ_2.

So a direction vector of ℓ_2 is

$$\overrightarrow{PA'} = \begin{pmatrix}4\\3\\2\end{pmatrix} - \begin{pmatrix}3\\5\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix} = -\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

This is parallel to $\mathbf{b}'$, as before.

Therefore the reflected line ℓ_2 has vector equation

$$\mathbf{r} = \begin{pmatrix}4\\3\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

which is the required result.

Alternative Solution:

The point found in part (a) is left fixed by the reflection, so it can be used as the point of ℓ_2.

From part (a), ℓ_1 meets Π_1 at $P(3,5,1)$. Since P lies on Π_1, the reflection does not move it, so P lies on ℓ_2.

The direction of ℓ_2 is the reflection of $\mathbf{b}$. As above, $\mathbf{b}\cdot\mathbf{n} = 7$ and $|\mathbf{n}|^2 = 14$, so

$$\mathbf{b}' = \mathbf{b} - 2\,\frac{7}{14}\,\mathbf{n} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

The direction can also be found by reflecting a second point of ℓ_1, such as $(1,4,0)$, to $A'(4,3,2)$ and taking $\overrightarrow{PA'} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, a multiple of $\mathbf{b}'$.

So ℓ_2 passes through P in the direction $\begin{pmatrix}-1\\2\\-1\end{pmatrix}$. The printed line has the same direction, and with $\mu = 1$ it gives

$$\begin{pmatrix}4\\3\\2\end{pmatrix} + \begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\5\\1\end{pmatrix}$$

so it also passes through P. Two lines through the same point in the same direction are the same line, so ℓ_2 is

$$\mathbf{r} = \begin{pmatrix}4\\3\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

as required.

(c) From part (a), the point

$$P = (3, 5, 1)$$

lies on Π_1 and on ℓ_1.

Since ℓ_1 lies in Π_2, the point P also lies on Π_2. So the line of intersection of Π_1 and Π_2 passes through P.

A direction vector of ℓ_1 is

$$\mathbf{b} = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

and a direction vector of ℓ_2 is

$$\mathbf{b}' = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

Because Π_2 contains both lines, it contains both direction vectors, so it also contains

$$\mathbf{b} + \mathbf{b}' = \begin{pmatrix}2\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix}$$

To check that this is also parallel to Π_1, take the scalar product with the normal to Π_1:

$$\begin{pmatrix}1\\3\\0\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 3 - 3 + 0 = 0$$

So $\begin{pmatrix}1\\3\\0\end{pmatrix}$ is a direction vector for the line of intersection of Π_1 and Π_2.

Hence a vector equation for the line of intersection is

$$\mathbf{r} = \begin{pmatrix}3\\5\\1\end{pmatrix} + t\begin{pmatrix}1\\3\\0\end{pmatrix}$$

where t is a scalar parameter.

7. The triangle ABC lies in the plane Π. The vertices are

$$A(1,0,0),\ B(2,2,1) \text{ and } C(1,1,c)$$

where c is a constant.

(a) Find, in terms of c, $\overrightarrow{BA} \times \overrightarrow{BC}$. **[2]**

Given that $\overrightarrow{BA} \times \overrightarrow{BC} = d\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, where d is a constant,

(b) find the value of c and show that $d = -3$. **[2]**

(c) find an equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$, where p is a constant. **[2]**

The point S has position vector $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The point S' is the image of S under reflection in Π.

(d) Find the position vector of S'. **[4]**

Solution

(a) First find the two side vectors from B:

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}1\\1\\c\end{pmatrix} - \begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\c-1\end{pmatrix}$$

Now take the vector product:

$$\begin{aligned}\overrightarrow{BA} \times \overrightarrow{BC} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -2 & -1\\ -1 & -1 & c-1\end{vmatrix}\\ &= \mathbf{i}\big((-2)(c-1) - (-1)(-1)\big) - \mathbf{j}\big((-1)(c-1) - (-1)(-1)\big) + \mathbf{k}\big((-1)(-1) - (-2)(-1)\big)\\ &= \mathbf{i}\big(-2c + 2 - 1\big) - \mathbf{j}\big(-c + 1 - 1\big) + \mathbf{k}(1-2)\\ &= (1-2c)\mathbf{i} + c\mathbf{j} - \mathbf{k}\end{aligned}$$

(b) We are given that

$$\overrightarrow{BA} \times \overrightarrow{BC} = d\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

From part (a):

$$\overrightarrow{BA} \times \overrightarrow{BC} = (1-2c)\mathbf{i} + c\mathbf{j} - \mathbf{k}$$

Comparing coefficients of $\mathbf{j}$:

$$c = 2$$

Then comparing coefficients of $\mathbf{i}$:

$$d = 1 - 2c = 1 - 2(2) = 1 - 4 = -3$$

Hence

$$c = 2 \quad \text{and} \quad d = -3$$

(c) A normal vector to the plane Π is

$$\mathbf{n} = \overrightarrow{BA} \times \overrightarrow{BC}$$

Using $c = 2$, part (a) gives

$$\mathbf{n} = \begin{pmatrix} 1 - 2(2) \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

So we can take

$$\mathbf{n} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

The plane has equation $\mathbf{r} \cdot \mathbf{n} = p$. Substituting $A(1, 0, 0)$, which lies on Π:

$$\begin{aligned} p &= \overrightarrow{OA} \cdot \mathbf{n} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} \\ &= -3 \end{aligned}$$

Therefore the equation of Π is

$$\mathbf{r} \cdot (-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -3$$

Equivalently, a Cartesian equation of Π is

$$-3x + 2y - z = -3$$

(d) Let the position vector of S be

$$\mathbf{s} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

The reflection formula for a point in the plane $\mathbf{r} \cdot \mathbf{n} = p$ is

$$\mathbf{s}' = \mathbf{s} - 2\,\frac{\mathbf{s} \cdot \mathbf{n} - p}{|\mathbf{n}|^2}\,\mathbf{n}$$

Here, from part (c):

$$\mathbf{n} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \qquad p = -3$$

We now calculate the required quantities:

$$\begin{aligned}\mathbf{s}\cdot\mathbf{n} &= \begin{pmatrix}3\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\-1\end{pmatrix}\\ &= 3(-3)+1(2)+2(-1)\\ &= -9+2-2\\ &= -9\end{aligned}$$

$$\begin{aligned}|\mathbf{n}|^2 &= (-3)^2+2^2+(-1)^2\\ &= 9+4+1\\ &= 14\end{aligned}$$

So

$$\begin{aligned}\mathbf{s}' &= \mathbf{s}-2\,\frac{-9-(-3)}{14}\,\mathbf{n}\\ &= \mathbf{s}-2\left(-\frac{6}{14}\right)\mathbf{n}\\ &= \mathbf{s}+\frac{12}{14}\,\mathbf{n}\\ &= \mathbf{s}+\frac{6}{7}\,\mathbf{n}\end{aligned}$$

Substituting for $\mathbf{s}$ and $\mathbf{n}$:

$$\begin{aligned}\mathbf{s}' &= \begin{pmatrix}3\\1\\2\end{pmatrix}+\frac{6}{7}\begin{pmatrix}-3\\2\\-1\end{pmatrix}\\ &= \begin{pmatrix}3-\frac{18}{7}\\1+\frac{12}{7}\\2-\frac{6}{7}\end{pmatrix}\\ &= \begin{pmatrix}\frac{21}{7}-\frac{18}{7}\\\frac{7}{7}+\frac{12}{7}\\\frac{14}{7}-\frac{6}{7}\end{pmatrix}\\ &= \begin{pmatrix}\frac{3}{7}\\\frac{19}{7}\\\frac{8}{7}\end{pmatrix}\end{aligned}$$

Hence the position vector of S' is

$$\frac{3}{7}\mathbf{i}+\frac{19}{7}\mathbf{j}+\frac{8}{7}\mathbf{k}$$

Alternative Solution:

We find S' using the line through S along the normal to Π. This line is

$$\mathbf{r} = \begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\-1\end{pmatrix} = \begin{pmatrix}3-3\lambda\\1+2\lambda\\2-\lambda\end{pmatrix}$$

Substituting into the plane equation $-3x + 2y - z = -3$:

$$-3(3 - 3\lambda) + 2(1 + 2\lambda) - (2 - \lambda) = -3$$

$$-9 + 14\lambda = -3$$

$$\lambda = \frac{3}{7}$$

So this line meets Π at the point $M\left(\frac{12}{7}, \frac{13}{7}, \frac{11}{7}\right)$, and

$$\overrightarrow{SM} = \frac{3}{7}\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{9}{7} \\ \frac{6}{7} \\ -\frac{3}{7} \end{pmatrix}$$

M is the midpoint of SS', so $\overrightarrow{SS'} = 2\overrightarrow{SM}$ and

$$\mathbf{s}' = \mathbf{s} + 2\overrightarrow{SM} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{18}{7} \\ \frac{12}{7} \\ -\frac{6}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{19}{7} \\ \frac{8}{7} \end{pmatrix}$$

as before.

8. The line ℓ has equation

$$\mathbf{r} = (\mathbf{i} + \mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

where λ is a scalar parameter, and the plane Π has equation

$$\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 9$$

(a) Find the coordinates of the point of intersection of ℓ and Π. **[3]**

The perpendicular to Π from the point $A\,(0, 4, 1)$ meets Π at the point B.

(b) Verify that the coordinates of B are $(4, 2, 3)$. **[3]**

The point $A\,(0, 4, 1)$ is reflected in the plane Π to give the image point A'.

(c) Find the coordinates of the point A'. **[2]**

(d) Find an equation for the line obtained by reflecting the line ℓ in the plane Π, giving your answer in the form

$$\mathbf{r} \times \mathbf{a} = \mathbf{b}$$

[8]

Solution

(a) Write the line ℓ in component form:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

so

$$(x, y, z) = (1 + \lambda,\, 2\lambda,\, 1 + 3\lambda)$$

The plane Π has equation

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 9$$

so a Cartesian equation of Π is

$$2x - y + z = 9$$

Substituting the line coordinates into the plane equation:

$$\begin{aligned} 2(1 + \lambda) - 2\lambda + (1 + 3\lambda) &= 9 \\ 2 + 2\lambda - 2\lambda + 1 + 3\lambda &= 9 \\ 3 + 3\lambda &= 9 \\ 3\lambda &= 6 \\ \lambda &= 2 \end{aligned}$$

Hence the point of intersection is

$$(x, y, z) = (1 + 2,\ 2 \times 2,\ 1 + 3 \times 2) = (3, 4, 7)$$

(b) A normal vector to the plane Π is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

So the perpendicular through $A(0, 4, 1)$ has equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Hence points on this line have coordinates

$$(x, y, z) = (2t,\ 4 - t,\ 1 + t)$$

Substituting into the plane equation $2x - y + z = 9$:

$$\begin{aligned} 2(2t) - (4 - t) + (1 + t) &= 9 \\ 4t - 4 + t + 1 + t &= 9 \\ 6t - 3 &= 9 \\ 6t &= 12 \\ t &= 2 \end{aligned}$$

So

$$B = (2 \times 2,\ 4 - 2,\ 1 + 2) = (4, 2, 3)$$

Hence the coordinates of B are verified to be $(4, 2, 3)$.

(c) The reflection formula for a point in the plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$ is

$$\mathbf{p}' = \mathbf{p} - 2\,\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\,\mathbf{n}$$

Here

$$\mathbf{p} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad d = 9$$

First calculate:

$$\mathbf{p} \cdot \mathbf{n} = 0 \times 2 + 4(-1) + 1 \times 1 = -3, \qquad |\mathbf{n}|^2 = 2^2 + (-1)^2 + 1^2 = 6$$

Therefore

$$\mathbf{p}' = \begin{pmatrix}0\\4\\1\end{pmatrix} - 2\,\frac{-3-9}{6}\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\4\\1\end{pmatrix} - 2\left(-\frac{12}{6}\right)\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\4\\1\end{pmatrix} + 4\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}8\\0\\5\end{pmatrix}$$

So the image point is $A' = (8, 0, 5)$.

Alternative Solution:

Find A' using the point B from part (b).

From part (b), the perpendicular from A meets Π at $B(4, 2, 3)$, so

$$\overrightarrow{AB} = \begin{pmatrix}4\\2\\3\end{pmatrix} - \begin{pmatrix}0\\4\\1\end{pmatrix} = \begin{pmatrix}4\\-2\\2\end{pmatrix}$$

Since B is the midpoint of AA', $\overrightarrow{AA'} = 2\overrightarrow{AB}$, so

$$\overrightarrow{OA'} = \begin{pmatrix}0\\4\\1\end{pmatrix} + 2\begin{pmatrix}4\\-2\\2\end{pmatrix} = \begin{pmatrix}8\\0\\5\end{pmatrix}$$

So $A' = (8, 0, 5)$, as before.

(d) Let P be the point where ℓ meets Π. From part (a):

$$P = (3, 4, 7)$$

Since P lies in the mirror plane, it stays fixed under reflection.

The direction vector of ℓ is

$$\mathbf{u} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Using the reflection formula for a direction vector:

$$\mathbf{u}' = \mathbf{u} - 2\,\frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}$$

Here $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

First calculate:

$$\mathbf{u}\cdot\mathbf{n} = 1\times 2 + 2(-1) + 3\times 1 = 3, \qquad |\mathbf{n}|^2 = 6$$

So

$$\mathbf{u}' = \begin{pmatrix}1\\2\\3\end{pmatrix} - 2\left(\frac{3}{6}\right)\begin{pmatrix}2\\-1\\1\end{pmatrix}$$
$$= \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\1\end{pmatrix}$$
$$= \begin{pmatrix}-1\\3\\2\end{pmatrix}$$

Alternative Solution:

Find the reflected direction by reflecting a point of ℓ and joining its image to P.

Take $C(1,0,1)$ on ℓ ($\lambda = 0$). It is not on Π, since $2(1) - 0 + 1 = 3 \neq 9$. The line through C along $\mathbf{n}$ is

$$\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + s\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

so points on it have coordinates $(1+2s,\ -s,\ 1+s)$.

Substituting into $2x - y + z = 9$:

$$2(1+2s) + s + (1+s) = 9$$
$$3 + 6s = 9$$
$$s = 1$$

so this line meets Π at $Q(3,-1,2)$.

Since $\overrightarrow{CC'} = 2\overrightarrow{CQ}$, the position vector of C' is

$$\mathbf{c}' = \mathbf{c} + 2(\mathbf{q} - \mathbf{c}) = \begin{pmatrix}1\\0\\1\end{pmatrix} + 2\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}5\\-2\\3\end{pmatrix}$$

P is fixed by the reflection, so the reflected line passes through $P(3,4,7)$ and $C'(5,-2,3)$:

$$\overrightarrow{PC'} = \begin{pmatrix}2\\-6\\-4\end{pmatrix} = -2\begin{pmatrix}-1\\3\\2\end{pmatrix}$$

This is parallel to $\mathbf{u}'$, the same direction as above.

Hence the reflected line ℓ' is

$$\mathbf{r} = \begin{pmatrix}3\\4\\7\end{pmatrix} + \mu\begin{pmatrix}-1\\3\\2\end{pmatrix}$$

To write this in the form $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, take

$$\mathbf{a} = \begin{pmatrix}-1\\3\\2\end{pmatrix}$$

Then

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} + \mu\mathbf{a}$$

implies

$$\mathbf{r} \times \mathbf{a} = \left(\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} + \mu\mathbf{a} \right) \times \mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \times \mathbf{a}$$

since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

Now calculate

$$\begin{aligned} \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 7 \\ -1 & 3 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 \times 2 - 7 \times 3) - \mathbf{j}(3 \times 2 - 7 \times (-1)) + \mathbf{k}(3 \times 3 - 4 \times (-1)) \\ &= -13\mathbf{i} - 13\mathbf{j} + 13\mathbf{k} \end{aligned}$$

Therefore an equation of the reflected line is

$$\mathbf{r} \times (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = -13\mathbf{i} - 13\mathbf{j} + 13\mathbf{k}$$

So

$$\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -13 \\ -13 \\ 13 \end{pmatrix}$$

9. The plane Π has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The line ℓ_1 passes through the point $P(4, 1, 2)$ and is perpendicular to Π.
The line ℓ_1 meets Π at the point Q.

(a) Find the coordinates of Q. **[7]**

Given that the point $R(3, 1, 0)$ lies on Π,

(b) find a Cartesian equation of the plane containing P, Q and R. **[4]**

The line ℓ_2 passes through P and R.
The line ℓ_3 is the reflection of ℓ_2 in Π.

(c) Find a vector equation of ℓ_3. **[4]**

Solution

(a) The plane Π has direction vectors

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

So a normal vector to Π is

$$\begin{aligned} \mathbf{n} &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\ &= \mathbf{i}((-1) \times 0 - 1 \times 1) - \mathbf{j}(2 \times 0 - 1 \times 1) + \mathbf{k}(2 \times 1 - (-1) \times 1) \\ &= \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

Using the point $(1, 2, -1)$ on Π:

$$\begin{aligned} d &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \\ &= -1 + 2 - 3 \\ &= -2 \end{aligned}$$

Hence Π is

$$\mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = -2$$

so a Cartesian equation of Π is

$$-x + y + 3z = -2$$

Since ℓ_1 is perpendicular to Π, it has direction vector $\mathbf{n}$. Therefore

$$\ell_1 : \ \mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

Substituting $(x, y, z) = (4 - \lambda,\, 1 + \lambda,\, 2 + 3\lambda)$ into $-x + y + 3z = -2$:

$$-(4 - \lambda) + (1 + \lambda) + 3(2 + 3\lambda) = -2$$

$$3 + 11\lambda = -2$$

$$\lambda = -\frac{5}{11}$$

So

$$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \frac{5}{11} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + \frac{5}{11} \\ 1 - \frac{5}{11} \\ 2 - \frac{15}{11} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{49}{11} \\ \frac{6}{11} \\ \frac{7}{11} \end{pmatrix}$$

Hence

$$Q\left(\frac{49}{11}, \frac{6}{11}, \frac{7}{11}\right)$$

(b) A plane containing P, Q, R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

First

$$\overrightarrow{PQ} = \begin{pmatrix} \frac{49}{11} \\ \frac{6}{11} \\ \frac{7}{11} \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} \\ -\frac{5}{11} \\ -\frac{15}{11} \end{pmatrix} = \frac{5}{11} \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

Also

$$\overrightarrow{QR} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{49}{11} \\ \frac{6}{11} \\ \frac{7}{11} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{16}{11} \\ \frac{5}{11} \\ -\frac{7}{11} \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -16 \\ 5 \\ -7 \end{pmatrix}$$

So a normal vector to the plane PQR is

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -16 \\ 5 \\ -7 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -3 \\ -16 & 5 & -7 \end{vmatrix}$$

$$= \mathbf{i}((-1) \times (-7) - (-3) \times 5) - \mathbf{j}(1 \times (-7) - (-3) \times (-16)) + \mathbf{k}(1 \times 5 - (-1) \times (-16))$$

$$= \begin{pmatrix} 22 \\ 55 \\ -11 \end{pmatrix}$$

$$= 11 \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

Hence we may take the normal vector as

$$\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

Using the point $P(4, 1, 2)$, the plane equation is

$$\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \right) = 0$$

So

$$2(x - 4) + 5(y - 1) - (z - 2) = 0$$

$$2x - 8 + 5y - 5 - z + 2 = 0$$

$$2x + 5y - z = 11$$

(c) The line ℓ_2 passes through $R(3, 1, 0)$ and $P(4, 1, 2)$, so

$$\ell_2 : \ \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Write Π in the form $\mathbf{r} \cdot \mathbf{n} = d$, where

$$\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \qquad d = -2$$

Since R lies on Π, it is unchanged by reflection.

So to find ℓ_3, we reflect the direction vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Using the reflection formula for a direction vector:

$$\mathbf{b}' = \mathbf{b} - 2\,\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}$$

Now

$$\begin{aligned}\mathbf{b}\cdot\mathbf{n} &= \begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\3\end{pmatrix}\\ &= -1+0+6\\ &= 5\end{aligned}$$

and

$$\begin{aligned}|\mathbf{n}|^2 &= \begin{pmatrix}-1\\1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\3\end{pmatrix}\\ &= 1+1+9\\ &= 11\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{b}' &= \begin{pmatrix}1\\0\\2\end{pmatrix} - 2\times\frac{5}{11}\begin{pmatrix}-1\\1\\3\end{pmatrix}\\ &= \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}-\frac{10}{11}\\\frac{10}{11}\\\frac{30}{11}\end{pmatrix}\\ &= \begin{pmatrix}\frac{21}{11}\\-\frac{10}{11}\\-\frac{8}{11}\end{pmatrix}\end{aligned}$$

Alternative Solution:

We can instead find the direction of ℓ_3 by reflecting the point P, which lies on ℓ_2 but not on Π, and joining its image P' to R.

From part (a), ℓ_1 is the line through P along $\mathbf{n}$, and it meets Π at Q. From part (b):

$$\overrightarrow{PQ} = \begin{pmatrix}\frac{5}{11}\\-\frac{5}{11}\\-\frac{15}{11}\end{pmatrix}$$

Since $\overrightarrow{PP'} = 2\overrightarrow{PQ}$, the position vector of P' is

$$\begin{aligned}\overrightarrow{OP'} &= \begin{pmatrix}4\\1\\2\end{pmatrix} + 2\begin{pmatrix}\frac{5}{11}\\-\frac{5}{11}\\-\frac{15}{11}\end{pmatrix}\\ &= \begin{pmatrix}\frac{54}{11}\\\frac{1}{11}\\-\frac{8}{11}\end{pmatrix}\end{aligned}$$

R is fixed by the reflection, so ℓ_3 passes through R and P', and

$$\overrightarrow{RP'} = \begin{pmatrix} \frac{54}{11} \\ \frac{1}{11} \\ -\frac{8}{11} \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{21}{11} \\ -\frac{10}{11} \\ -\frac{8}{11} \end{pmatrix} = \mathbf{b}'$$

This is parallel to $\begin{pmatrix} 21 \\ -10 \\ -8 \end{pmatrix}$. Hence

$$\ell_3 : \ \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 21 \\ -10 \\ -8 \end{pmatrix}$$

10. The line ℓ_1 is the intersection of the planes

$$x + y + z = 1$$

and

$$2x - y + z = 4$$

The plane Π has equation

$$x + y + 2z = 3$$

The line ℓ_2 is the reflection of the line ℓ_1 in the plane Π.

Find a vector equation of the line ℓ_2. **[10]**

Solution

Write the reflecting plane in the form

$$\Pi : \ \mathbf{r} \cdot \mathbf{n} = d$$

so here

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad d = 3$$

To find the reflection of the line ℓ_1, we need a point on ℓ_2 and a direction vector of ℓ_2.

First find a direction vector for ℓ_1. Since ℓ_1 is the intersection of the planes

$$x + y + z = 1 \qquad \text{and} \qquad 2x - y + z = 4$$

their normal vectors are

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

So a direction vector of ℓ_1 is

$$\begin{aligned} \mathbf{u} &= \mathbf{n}_1 \times \mathbf{n}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \end{aligned}$$

Now find a convenient point on ℓ_1. The best choice is the point where ℓ_1 meets Π, since a point on the mirror plane is unchanged by reflection.

So solve

$$x + y + z = 1$$

$$2x - y + z = 4$$

$$x + y + 2z = 3$$

Subtracting the first equation from the third gives

$$z = 2$$

Substituting $z = 2$ into the first two equations:

$$x + y + 2 = 1$$

$$2x - y + 2 = 4$$

so

$$x + y = -1$$

$$2x - y = 2$$

Adding these equations:

$$3x = 1$$

$$x = \frac{1}{3}$$

Then

$$y = -1 - \frac{1}{3} = -\frac{4}{3}$$

Hence ℓ_1 meets Π at the point with position vector

$$\mathbf{q} = \begin{pmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 2 \end{pmatrix}$$

This point lies on Π, so it is unchanged by the reflection, and ℓ_2 also passes through it.

Now reflect $\mathbf{u}$ with the reflection formula for a direction vector:

$$\mathbf{u}' = \mathbf{u} - 2\,\frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}$$

Compute $\mathbf{u}\cdot\mathbf{n}$ and $|\mathbf{n}|^2$:

$$\mathbf{u}\cdot\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2 + 1 - 6 = -3$$

$$|\mathbf{n}|^2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 + 1 + 4 = 6$$

Hence

$$\begin{aligned} \mathbf{u}' &= \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - 2\left(-\frac{3}{6}\right)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \end{aligned}$$

Alternative Solution:

We can also find the direction of ℓ_2 by reflecting a second point of ℓ_1.

Putting $y = 0$ in the equations of the two planes gives $x + z = 1$ and $2x + z = 4$, so $x = 3$ and $z = -2$. Hence $A(3, 0, -2)$ lies on ℓ_1, and since $3 + 0 - 4 = -1 \neq 3$, A is not on Π.

The line through A perpendicular to Π is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Substituting into the equation of Π:

$$\begin{aligned} (3 + \lambda) + \lambda + 2(-2 + 2\lambda) &= 3 \\ 6\lambda - 1 &= 3 \\ \lambda &= \frac{2}{3} \end{aligned}$$

so this line meets Π at $B\left(\frac{11}{3}, \frac{2}{3}, -\frac{2}{3}\right)$.

Let A' be the reflection of A. Then $\overrightarrow{AA'} = 2\overrightarrow{AB}$, so, with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}'$:

$$\begin{aligned} \mathbf{a}' &= \mathbf{a} + 2(\mathbf{b} - \mathbf{a}) \\ &= \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + 2\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{4}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{13}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{pmatrix} \end{aligned}$$

Both A' and the point with position vector $\mathbf{q}$, where ℓ_1 meets Π, lie on ℓ_2. So a direction vector of ℓ_2 is

$$\mathbf{a}' - \mathbf{q} = \begin{pmatrix} 4 \\ \frac{8}{3} \\ -\frac{4}{3} \end{pmatrix} = \frac{4}{3}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

which is parallel to $\mathbf{u}'$, as before.

Therefore a vector equation of the reflected line ℓ_2 is

$$\mathbf{r} = \begin{pmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \qquad t \in \mathbb{R}$$