OPTIONAL PURE / VECTORS & MATRICES

Plane Intersections

Worked solutions 10 questions · 83 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 CIE 9231

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1. Three planes have equations

$$\begin{aligned} x + y + z &= 4 \\ 2x - y + z &= 8 \\ x + 3y - 2z &= -7 \end{aligned}$$

(a) Show that the three planes meet at a single point. **[2]**

(b) Find the coordinates of this point. **[3]**

Solution

(a) Write the system in matrix form as

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{b}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ -7 \end{pmatrix}$$

Expanding along the top row:

$$\begin{aligned} \det(\mathbf{M}) &= 1\big((-1) \times (-2) - 1 \times 3\big) - 1\big(2 \times (-2) - 1 \times 1\big) + 1\big(2 \times 3 - (-1) \times 1\big) \\ &= -1 + 5 + 7 \\ &= 11 \end{aligned}$$

Since $\det(\mathbf{M}) = 11 \neq 0$, $\mathbf{M}$ is non-singular, so the system has a unique solution. Hence the three planes meet at a single point.

(b) Since $\mathbf{M}$ is non-singular, multiplying both sides on the left by $\mathbf{M}^{-1}$ gives the solution. Using a calculator, the inverse is

$$\mathbf{M}^{-1} = \frac{1}{11}\begin{pmatrix} -1 & 5 & 2 \\ 5 & -3 & 1 \\ 7 & -2 & -3 \end{pmatrix}$$

so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -1 & 5 & 2 \\ 5 & -3 & 1 \\ 7 & -2 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \\ -7 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 22 \\ -11 \\ 33 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

The planes meet at the point $(2, -1, 3)$.

2. The planes Π_1 and Π_2 have equations $x+2y+z=4$ and $2x-y+3z=3$ respectively, and meet in the line L.

(a) Find a vector equation of L. **[4]**

The plane Π_3 has equation $3x-4y+5z=2$.

(b) Show that L lies in Π_3. **[2]**

(c) Hence describe the geometrical arrangement of the planes Π_1, Π_2 and Π_3. **[1]**

Solution

(a) The line L lies in both planes, so its direction is perpendicular to both normals. The normals are

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

so a direction of L is given by the vector product:

$$\begin{aligned} \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} \\ &= (2\times 3 - 1\times(-1))\mathbf{i} - (1\times 3 - 1\times 2)\mathbf{j} + (1\times(-1) - 2\times 2)\mathbf{k} \\ &= \begin{pmatrix} 7 \\ -1 \\ -5 \end{pmatrix} \end{aligned}$$

To find a point on L, set $z=0$ in both plane equations:

$$x+2y=4, \qquad 2x-y=3$$

Solving these simultaneously gives $x=2$, $y=1$, so $(2,1,0)$ lies on L. A vector equation of L is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -1 \\ -5 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

(b) A general point on L has coordinates $(2+7\lambda,\ 1-\lambda,\ -5\lambda)$. Substituting into the left-hand side of the equation of Π_3:

$$\begin{aligned} 3x-4y+5z &= 3(2+7\lambda) - 4(1-\lambda) + 5(-5\lambda) \\ &= 6 + 21\lambda - 4 + 4\lambda - 25\lambda \\ &= 2 \end{aligned}$$

This holds for every value of λ, so every point of L lies in Π_3. Hence L lies in Π_3, as required.

(c) No two of the normals $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3 = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ are multiples of each other, so the three planes are distinct and no two are parallel. By part (b), all three planes contain the line L, so the planes form a sheaf: they meet in the common line L.

3. Three planes have equations

$$\begin{aligned} \Pi_1 : &\quad 2x - \ \ y + 3z = 4 \\ \Pi_2 : &\quad 4x - 2y + 6z = 5 \\ \Pi_3 : &\quad \ \ x + \ \ y - \ \ z = 2 \end{aligned}$$

(a) Show that Π_1 and Π_2 are parallel and distinct. **[1]**

(b) Hence explain why the three equations have no common solution, and describe the geometrical arrangement of the three planes. **[2]**

The equation of Π_2 is now changed to $4x - 2y + 6z = k$, where k is a constant.

(c) Find the value of k for which the three equations have infinitely many solutions, and describe the geometrical arrangement of the planes in this case. **[2]**

Solution

(a) The normal vectors of Π_1 and Π_2 are

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad \mathbf{n}_2 = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2\mathbf{n}_1$$

so Π_1 and Π_2 are parallel. Dividing the equation of Π_2 by 2 gives

$$2x - y + 3z = \frac{5}{2}$$

which has the same left-hand side as Π_1 but $\frac{5}{2} \neq 4$, so no point satisfies both equations. Hence Π_1 and Π_2 are parallel and distinct.

(b) A common solution of the three equations would be a point lying on all three planes. Since Π_1 and Π_2 are parallel and distinct, no point lies on both of them, so the three equations have no common solution.

The normal of Π_3 is

$$\mathbf{n}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

which is not a multiple of $\mathbf{n}_1$, so Π_3 is not parallel to Π_1 or Π_2. The geometrical arrangement is two parallel planes Π_1 and Π_2 cut by the third plane Π_3, which meets them in two parallel lines.

(c) Dividing the new equation of Π_2 by 2 gives

$$2x - y + 3z = \frac{k}{2}$$

so Π_2 is still parallel to Π_1. If $\frac{k}{2} \neq 4$, the planes are distinct and, as in part (b), there is no common solution. For infinitely many solutions Π_1 and Π_2 must therefore be the same plane:

$$\frac{k}{2} = 4 \quad \Longrightarrow \quad k = 8$$

With $k = 8$, the equations of Π_1 and Π_2 describe the same plane, and Π_3 is not parallel to it, so Π_3 meets it in a line and every point of that line satisfies all three equations. The geometrical arrangement is that Π_1 and Π_2 coincide, and Π_3 meets them in a single line.

4. **(a)** Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -4 \\ 7 & 0 & -7 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$$

Show that $\mathbf{A}$ is singular, and deduce that the system does not have a unique solution. **[2]**

(b) Show that the system is consistent. Interpret the solution set geometrically. **[3]**

Solution

(a) Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A}) &= 3\begin{vmatrix} 1 & -4 \\ 0 & -7 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -4 \\ 7 & -7 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 7 & 0 \end{vmatrix} \\ &= 3(-7-0) + 2(-14+28) + 1(0-7) \\ &= -21 + 28 - 7 \\ &= 0 \end{aligned}$$

Therefore $\mathbf{A}$ is singular.

Since a unique solution to $\mathbf{Ax} = \mathbf{b}$ would require $\mathbf{A}$ to be non-singular, the system does not have a unique solution.

(b) The system is

$$\begin{aligned} 3x - 2y + z &= 5 \\ 2x + y - 4z &= 1 \\ 7x - 7z &= 7 \end{aligned}$$

First check consistency. Take the first equation and add twice the second:

$$\begin{aligned} (3x - 2y + z) + 2(2x + y - 4z) &= 5 + 2(1) \\ 7x - 7z &= 7 \end{aligned}$$

This is exactly the third equation, so the system is consistent.

Now solve using the first two equations. From

$$2x + y - 4z = 1$$

we get

$$y = 1 - 2x + 4z$$

Substitute into $3x - 2y + z = 5$:

$$\begin{aligned} 3x - 2(1 - 2x + 4z) + z &= 5 \\ 3x - 2 + 4x - 8z + z &= 5 \\ 7x - 7z - 2 &= 5 \\ 7x - 7z &= 7 \\ x - z &= 1 \\ x &= z + 1 \end{aligned}$$

Then

$$\begin{aligned} y &= 1 - 2x + 4z \\ &= 1 - 2(z + 1) + 4z \\ &= 1 - 2z - 2 + 4z \\ &= -1 + 2z \end{aligned}$$

Let $z = \lambda$, where $\lambda \in \mathbb{R}$. Then

$$x = 1 + \lambda, \qquad y = -1 + 2\lambda, \qquad z = \lambda$$

Geometrically, each equation represents a plane. Since the third equation is the first equation plus twice the second, all three planes intersect in the line

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

Hence the solution set is a line, so the planes form a **sheaf**.

5. **(a)** Show that the system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - y + 4z &= 1 \\ 3x + y + 3z &= a \end{aligned}$$

where a is a constant, does not have a unique solution. **[2]**

(b) Given that $a = 4$, show that the system of equations in part (a) is consistent. Interpret this situation geometrically. **[3]**

(c) Given instead that $a \neq 4$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. **[3]**

Solution

(a) Let the coefficient matrix be

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ 3 & 1 & 3 \end{pmatrix}$$

and calculate using, for example, a calculator:

$$\det(\mathbf{A}) = 0$$

Therefore $\mathbf{A}$ is not invertible, so the system cannot have a unique solution for any value of a.

(b) When $a = 4$, the third equation is

$$3x + y + 3z = 4$$

Now add the first two equations:

$$(x + 2y - z) + (2x - y + 4z) = 3 + 1$$

which gives

$$3x + y + 3z = 4$$

So the third equation is exactly the sum of the first two, and adds no new condition. Hence the system is consistent.

To find the solutions, solve the first two equations only. Let

$$z = 5\lambda$$

where $\lambda \in \mathbb{R}$.

From

$$x + 2y - z = 3$$

we have

$$x = 3 - 2y + z$$

Substitute into the second equation:

$$\begin{aligned} 2x - y + 4z &= 1 \\ 2(3 - 2y + z) - y + 4z &= 1 \\ 6 - 4y + 2z - y + 4z &= 1 \\ 6 - 5y + 6z &= 1 \\ -5y + 6z &= -5 \\ 5y - 6z &= 5 \\ y &= 1 + \frac{6}{5}z \end{aligned}$$

Since $z = 5\lambda$, this gives

$$y = 1 + 6\lambda$$

Then

$$\begin{aligned} x &= 3 - 2y + z \\ &= 3 - 2(1 + 6\lambda) + 5\lambda \\ &= 3 - 2 - 12\lambda + 5\lambda \\ &= 1 - 7\lambda \end{aligned}$$

Hence there are infinitely many solutions, forming the line

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 6 \\ 5 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

Geometrically, each equation represents a plane. The first two planes intersect in a line, and the third plane contains that same line, so the planes form a **sheaf**.

(c) If $a \neq 4$, then any point satisfying the first two equations must also satisfy

$$\begin{aligned} 3x + y + 3z &= (x + 2y - z) + (2x - y + 4z) \\ &= 3 + 1 \\ &= 4 \end{aligned}$$

So every common solution of the first two equations lies on the plane

$$3x + y + 3z = 4$$

It cannot also satisfy

$$3x + y + 3z = a$$

when $a \neq 4$.

Therefore the system is inconsistent, so there is no solution.

Geometrically, the normals $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ are not scalar multiples of one another, so no two planes are parallel and each pair of planes meets in a line. Since the three planes have no common point, these three lines are parallel and distinct, so the planes form a **triangular prism**.

6. Three planes have equations

$$\begin{aligned} x - y + 2z &= 3 \\ 2x + y + z &= 4 \\ 3x + (m+1)z &= 7 \end{aligned}$$

where m is a constant.

(a) Investigate the arrangement of the planes:
when $m = 2$;
when $m \neq 2$. **[6]**

(b) A student claims that the position vectors $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{k}$ all lie in a plane through the origin. Determine whether or not the student is correct. **[2]**

Solution

(a) Let the planes be P_1, P_2, P_3, with coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & m+1 \end{pmatrix}$$

Its determinant is

$$\begin{aligned} \det(\mathbf{A}) &= 1\begin{vmatrix} 1 & 1 \\ 0 & m+1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 3 & m+1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \\ &= 1\big(1(m+1) - 0\big) + 1\big(2(m+1) - 3\big) + 2\big(0 - 3\big) \\ &= 3(m-2) \end{aligned}$$

If $m \neq 2$, then $\det(\mathbf{A}) \neq 0$, so the three planes meet in exactly one point.

To find this point, solve the first two equations:

$$\begin{aligned} x - y + 2z &= 3 \\ 2x + y + z &= 4 \end{aligned}$$

Adding gives

$$\begin{aligned} 3x + 3z &= 7 \\ x + z &= \frac{7}{3} \\ x &= \frac{7}{3} - z \end{aligned}$$

Substitute into $2x + y + z = 4$:

$$2\left(\frac{7}{3} - z\right) + y + z = 4$$

$$\frac{14}{3} - 2z + y + z = 4$$

$$y - z = 4 - \frac{14}{3}$$

$$y - z = -\frac{2}{3}$$

$$y = z - \frac{2}{3}$$

Now substitute into the third plane:

$$3x + (m+1)z = 7$$

$$3\left(\frac{7}{3} - z\right) + (m+1)z = 7$$

$$7 - 3z + (m+1)z = 7$$

$$(m-2)z = 0$$

Since $m \neq 2$, it follows that $z = 0$. Then

$$x = \frac{7}{3}, \qquad y = -\frac{2}{3}$$

So when $m \neq 2$, the three planes intersect at the single point

$$\left(\frac{7}{3}, -\frac{2}{3}, 0\right)$$

If $m = 2$, then the third plane becomes

$$3x + 3z = 7$$

But adding the first two plane equations gives

$$(x - y + 2z) + (2x + y + z) = 3 + 4$$

$$3x + 3z = 7$$

So P_3 is obtained from P_1 and P_2, which means the line of intersection of P_1 and P_2 also lies in P_3. To find that line, let

$$z = \lambda$$

Then from $x + z = \frac{7}{3}$:

$$x = \frac{7}{3} - \lambda$$

And from $y = z - \frac{2}{3}$:

$$y = \lambda - \frac{2}{3}$$

Hence the common line is

$$\mathbf{r} = \begin{pmatrix} \frac{7}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

So the arrangements are:

- if $m \neq 2$, the three planes meet at one point $\left(\frac{7}{3}, -\frac{2}{3}, 0\right)$
- if $m = 2$, the three planes share the line found above, so form a **sheaf**.

(b) The three position vectors are

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

They lie in a plane through the origin if and only if they are coplanar, which is equivalent to their scalar triple product being 0. So we test

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 1(4) + 1(8 - 3) + 2(0 - 3) = 3$$

Since this determinant is not 0, the vectors are not coplanar.

Therefore the student is incorrect.

7.

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & k & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

where k is a constant.

(a) Find the values of k for which the matrix $\mathbf{M}$ has an inverse. **[2]**

(b) Find, in terms of p, the coordinates of the point where the following planes intersect:

$$\begin{aligned} x + y &= 1 \\ 3x + 2y + z &= p \\ 2x - y + z &= 4 \end{aligned}$$ **[5]**

(c) (i) Find the value of q for which the set of simultaneous equations

$$\begin{aligned} x + y &= 1 \\ 3x + z &= q \\ 2x - y + z &= 4 \end{aligned}$$ **[2]**

has at least one solution.

(ii) For this value of q, interpret the solution set geometrically. **[2]**

Solution

(a) A matrix has an inverse if and only if its determinant is non-zero.

For

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & k & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

expand along the first row:

$$\begin{aligned} \det(\mathbf{M}) &= 1\begin{vmatrix} k & 1 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & k \\ 2 & -1 \end{vmatrix} \\ &= 1(k \times 1 - 1 \times (-1)) - 1(3 \times 1 - 1 \times 2) + 0 \\ &= (k + 1) - (3 - 2) \\ &= k \end{aligned}$$

Therefore $\mathbf{M}$ has an inverse when

$$k \neq 0$$

(b) The coefficient matrix is $\mathbf{M}$ with $k = 2$, which has an inverse by part (a). Using a calculator:

$$\mathbf{M}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ -7 & 3 & -1 \end{pmatrix}$$

so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} 1 \\ p \\ 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3-p+4 \\ -1+p-4 \\ -7+3p-4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7-p \\ p-5 \\ 3p-11 \end{pmatrix}$$

Hence the planes intersect at

$$\left(\frac{7-p}{2}, \frac{p-5}{2}, \frac{3p-11}{2} \right)$$

Alternative Solution:

We solve the three plane equations

$$\begin{aligned} x + y &= 1 \\ 3x + 2y + z &= p \\ 2x - y + z &= 4 \end{aligned}$$

From the first equation:

$$y = 1 - x$$

Substitute into the second equation:

$$\begin{aligned} 3x + 2(1-x) + z &= p \\ 3x + 2 - 2x + z &= p \\ x + z &= p - 2 \end{aligned}$$

Substitute into the third equation:

$$\begin{aligned} 2x - (1-x) + z &= 4 \\ 2x - 1 + x + z &= 4 \\ 3x + z &= 5 \end{aligned}$$

So we now have

$$x + z = p - 2 \qquad \text{and} \qquad 3x + z = 5$$

Subtract the first from the second:

$$\begin{aligned} (3x+z) - (x+z) &= 5 - (p-2) \\ 2x &= 7 - p \\ x &= \frac{7-p}{2} \end{aligned}$$

Then

$$\begin{aligned} y &= 1 - x \\ &= 1 - \frac{7-p}{2} \\ &= \frac{2-7+p}{2} \\ &= \frac{p-5}{2} \end{aligned}$$

And using $3x + z = 5$:

$$\begin{aligned} z &= 5 - 3x \\ &= 5 - 3\left(\frac{7-p}{2}\right) \\ &= \frac{10 - 21 + 3p}{2} \\ &= \frac{3p - 11}{2} \end{aligned}$$

which gives the same point.

(c) (i) The equations are

$$\begin{aligned} x + y &= 1 \\ 3x + z &= q \\ 2x - y + z &= 4 \end{aligned}$$

Using the first two equations:

$$(3x + z) - (x + y) = q - 1$$

But

$$(3x + z) - (x + y) = 2x - y + z$$

So the third equation requires

$$q - 1 = 4$$

Hence

$$q = 5$$

(ii) For $q = 5$, the second equation is

$$3x + z = 5$$

Then

$$(3x + z) - (x + y) = 5 - 1 = 4$$

which gives

$$2x - y + z = 4$$

So the third equation is implied by the first two, and does not give any new restriction.

Therefore the three planes have infinitely many common points, forming a line.

Let $x = \lambda$. Then

$$y = 1 - \lambda$$

and from $3x + z = 5$:

$$z = 5 - 3\lambda$$

So the solution set is the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

Geometrically, the three planes intersect in a common line, so form a **sheaf**.

8. The equations

$$\begin{aligned} x - y + 2z &= 4 \\ (k+1)x + ky + (2-k)z &= 4 - k \\ (5-2k)x + (2-2k)y + (k+2)z &= 8 - k \end{aligned}$$

represent three planes, where k is a real constant.
The planes do not intersect in exactly one point.

(a) Find the possible values of k. **[4]**

(b) For each value of k found in part (a), describe the configuration of the planes.
Fully justify your answer, stating in each case whether the corresponding system is consistent or inconsistent. **[6]**

Solution

(a) Let the coefficient matrix be

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ k+1 & k & 2-k \\ 5-2k & 2-2k & k+2 \end{pmatrix}$$

The planes fail to intersect in exactly one point when the system does *not* have a unique solution, so we require

$$\det(\mathbf{A}) = 0$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A}) &= 1\begin{vmatrix} k & 2-k \\ 2-2k & k+2 \end{vmatrix} - (-1)\begin{vmatrix} k+1 & 2-k \\ 5-2k & k+2 \end{vmatrix} + 2\begin{vmatrix} k+1 & k \\ 5-2k & 2-2k \end{vmatrix} \\ &= k(k+2) - (2-k)(2-2k) \\ &\quad + (k+1)(k+2) - (2-k)(5-2k) \\ &\quad + 2\big((k+1)(2-2k) - k(5-2k)\big) \\ &= \big(k^2 + 2k - (4 - 6k + 2k^2)\big) \\ &\quad + \big(k^2 + 3k + 2 - (10 - 9k + 2k^2)\big) \\ &\quad + 2\big((2 - 2k^2) - (5k - 2k^2)\big) \\ &= (-k^2 + 8k - 4) + (-k^2 + 12k - 8) + (4 - 10k) \\ &= -2k^2 + 10k - 8 \\ &= -2(k^2 - 5k + 4) \\ &= -2(k-1)(k-4) \end{aligned}$$

So

$$-2(k-1)(k-4) = 0$$

Hence the possible values are

$$k = 1 \quad \text{or} \quad k = 4$$

(b) We consider each value of k separately.

For $k = 1$, the planes are

$$x - y + 2z = 4$$
$$2x + y + z = 3$$
$$3x + 3z = 7$$

Adding the first two equations gives

$$(x - y + 2z) + (2x + y + z) = 4 + 3$$

so

$$3x + 3z = 7$$

which is exactly the third equation.

Therefore any point that lies on the first two planes automatically lies on the third plane as well.

Now the normals of the first two planes are

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

These are not scalar multiples, so the first two planes are not parallel. Hence they intersect in a line, and the third plane contains this same line.

Therefore the system is consistent, and the three planes form a **sheaf**: they share a common line, so there are infinitely many solutions.

For $k = 4$, the planes are

$$x - y + 2z = 4$$
$$5x + 4y - 2z = 0$$
$$-3x - 6y + 6z = 4$$

Suppose a point lies on the first two planes. Then

$$2(x - y + 2z) - (5x + 4y - 2z) = 2 \times 4 - 0$$

so

$$-3x - 6y + 6z = 8$$

But the third plane is

$$-3x - 6y + 6z = 4$$

This is impossible, so there is no point common to all three planes. Therefore the system is inconsistent.

Now check how the planes are arranged. Their normals are

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{n}_2 = \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}, \qquad \mathbf{n}_3 = \begin{pmatrix} -3 \\ -6 \\ 6 \end{pmatrix}$$

No two of these are scalar multiples, so no two planes are parallel. Hence each pair of planes meets in a line. Since the three planes have no common point, these three lines are parallel and distinct, so when $k = 4$ the three planes form a **triangular prism**.

9. Three planes have equations

$$\begin{aligned} x + ay &= -1 \\ 2x + y + z &= 3 \\ 3x + by + 2z &= c \end{aligned}$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a. **[2]**

(ii) find the value of c for which the planes form a sheaf. **[3]**

(b) In the case where $b = 1 - a$ and $c = 5$, find the coordinates of the point of intersection of the planes in terms of a. **[5]**

Solution

(a) Let the planes be

$$P_1 : \ x + ay = -1, \qquad P_2 : \ 2x + y + z = 3, \qquad P_3 : \ 3x + by + 2z = c$$

(i) For the three planes to intersect at a unique point, the coefficient matrix must have non-zero determinant:

$$\mathbf{A} = \begin{pmatrix} 1 & a & 0 \\ 2 & 1 & 1 \\ 3 & b & 2 \end{pmatrix}$$

So

$$\begin{aligned} \det(\mathbf{A}) &= \begin{vmatrix} 1 & a & 0 \\ 2 & 1 & 1 \\ 3 & b & 2 \end{vmatrix} \\ &= 1\begin{vmatrix} 1 & 1 \\ b & 2 \end{vmatrix} - a\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 3 & b \end{vmatrix} \\ &= 1(2 - b) - a(4 - 3) \\ &= 2 - b - a \end{aligned}$$

In the case where the planes do **not** intersect at a unique point, we need

$$\det(\mathbf{A}) = 0$$

so

$$\begin{aligned} 2 - b - a &= 0 \\ b &= 2 - a \end{aligned}$$

(ii) A sheaf of planes means that all three planes contain a common line.

First find the line of intersection of P_1 and P_2.

From P_1:

$$x = -1 - ay$$

Substitute into P_2:

$$\begin{aligned} 2x + y + z &= 3 \\ 2(-1 - ay) + y + z &= 3 \\ -2 - 2ay + y + z &= 3 \\ z &= 5 + (2a - 1)y \end{aligned}$$

So the points on the line of intersection of P_1 and P_2 satisfy

$$x = -1 - ay, \qquad z = 5 + (2a - 1)y$$

For P_3 to contain this whole line, substitute these expressions into P_3, using $b = 2 - a$:

$$\begin{aligned} 3x + (2 - a)y + 2z &= 3(-1 - ay) + (2 - a)y + 2\big(5 + (2a - 1)y\big) \\ &= -3 - 3ay + (2 - a)y + 10 + (4a - 2)y \\ &= 7 + \big(-3a + 2 - a + 4a - 2\big)y \\ &= 7 \end{aligned}$$

This is satisfied for every point on the line precisely when

$$c = 7$$

(b) Now take $b = 1 - a$ and $c = 5$.

From the first plane:

$$x = -1 - ay$$

Substitute into the second plane:

$$\begin{aligned} 2x + y + z &= 3 \\ 2(-1 - ay) + y + z &= 3 \\ -2 - 2ay + y + z &= 3 \\ z &= 5 + (2a - 1)y \end{aligned}$$

Now substitute $x = -1 - ay$ and $z = 5 + (2a - 1)y$ into the third plane:

$$3x + (1 - a)y + 2z = 5$$

Then

$$\begin{aligned} 3(-1 - ay) + (1 - a)y + 2\big(5 + (2a - 1)y\big) &= 5 \\ -3 - 3ay + (1 - a)y + 10 + (4a - 2)y &= 5 \\ 7 + \big(-3a + 1 - a + 4a - 2\big)y &= 5 \\ 7 - y &= 5 \\ y &= 2 \end{aligned}$$

Now find x and z:

$$\begin{aligned} x &= -1 - a(2) = -1 - 2a \\ z &= 5 + (2a - 1)(2) = 5 + 4a - 2 = 4a + 3 \end{aligned}$$

Therefore the point of intersection is

$$(-1-2a,\ 2,\ 4a+3)$$

10. You are given that a is a parameter which can take only real values.
The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & a \\ 1 & 3 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

(a) Find an expression for the determinant of $\mathbf{A}$ in terms of a. **[2]**

You are given the following system of equations in x, y and z.

$$\begin{aligned} 3x + 2y + az &= 8 \\ x + 3y &= 5 \\ 2x - y + z &= 3 \end{aligned}$$

The system can be written in the form

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 3 \end{pmatrix}$$

(b) (i) In the case where $\mathbf{A}$ is not singular, solve the given system by using $\mathbf{A}^{-1}$. **[6]**

(ii) In the case where $\mathbf{A}$ is singular, describe the configuration of the planes whose equations are the three equations of the system. **[3]**

The transformation represented by $\mathbf{A}$ is denoted by T.
A 3-D solid of volume $a^2 + 3$ is transformed by T to an image.

(c) (i) Determine the range of values of a for which the orientation of the image is the reverse of the orientation of the object. **[1]**

(ii) Determine the range of values of a for which the volume of the image is less than the volume of the object. **[2]**

Solution

(a) Expanding $\det(\mathbf{A})$ along the first row:

$$\begin{aligned} \det(\mathbf{A}) &= 3\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + a\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\ &= 3(3 \times 1 - 0 \times (-1)) - 2(1 \times 1 - 0 \times 2) + a(1 \times (-1) - 3 \times 2) \\ &= 3(3) - 2(1) + a(-1 - 6) \\ &= 9 - 2 - 7a \\ &= 7(1 - a) \end{aligned}$$

(b) (i) For $\mathbf{A}$ to be non-singular, we need

$$\det(\mathbf{A}) \neq 0$$

So

$$7(1 - a) \neq 0 \quad \Longrightarrow \quad a \neq 1$$

The matrix of cofactors of $\mathbf{A}$ is

$$\begin{pmatrix} 3 & -1 & -7 \\ -(2+a) & 3-2a & 7 \\ -3a & a & 7 \end{pmatrix}$$

Transposing this gives the adjugate, and dividing by $\det(\mathbf{A})$, for $a \neq 1$, gives

$$\mathbf{A}^{-1} = \frac{1}{7(1-a)} \begin{pmatrix} 3 & -a-2 & -3a \\ -1 & 3-2a & a \\ -7 & 7 & 7 \end{pmatrix}$$

Now

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 8 \\ 5 \\ 3 \end{pmatrix}$$

So

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{7(1-a)} \begin{pmatrix} 3 & -a-2 & -3a \\ -1 & 3-2a & a \\ -7 & 7 & 7 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \\ 3 \end{pmatrix} \\ &= \frac{1}{7(1-a)} \begin{pmatrix} 3(8)+(-a-2)(5)+(-3a)(3) \\ (-1)(8)+(3-2a)(5)+a(3) \\ (-7)(8)+7(5)+7(3) \end{pmatrix} \\ &= \frac{1}{7(1-a)} \begin{pmatrix} 24-5a-10-9a \\ -8+15-10a+3a \\ -56+35+21 \end{pmatrix} \\ &= \frac{1}{7(1-a)} \begin{pmatrix} 14-14a \\ 7-7a \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Hence, for $a \neq 1$, the solution is

$$x = 2, \quad y = 1, \quad z = 0$$

(ii) $\mathbf{A}$ is singular when

$$7(1-a) = 0 \quad \Longrightarrow \quad a = 1$$

In this case the equations are

$$\begin{aligned} 3x + 2y + z &= 8 \\ x + 3y &= 5 \\ 2x - y + z &= 3 \end{aligned}$$

Now add the second and third equations:

$$\begin{aligned} (x+3y) + (2x - y + z) &= 5 + 3 \\ 3x + 2y + z &= 8 \end{aligned}$$

This is exactly the first equation.

So the first plane contains the line of intersection of the second and third planes.

Also, the second and third planes are not parallel, since their normal vectors

$$\begin{pmatrix}1\\3\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}2\\-1\\1\end{pmatrix}$$

are not scalar multiples of each other.

Therefore the second and third planes intersect in a line, and the first plane also contains this same line.

Hence the configuration is: the three distinct planes all share a common line, so form a **sheaf**.

(c) (i) Orientation is reversed when

$$\det(\mathbf{A}) < 0$$

Since

$$\det(\mathbf{A}) = 7(1-a)$$

we need

$$7(1-a) < 0 \quad \Longrightarrow \quad 1-a < 0 \quad \Longrightarrow \quad a > 1$$

Hence, the orientation is reversed for

$$a > 1$$

(ii) Volume scale factor under the transformation T is

$$|\det(\mathbf{A})| = |7(1-a)|$$

So the image volume is

$$|7(1-a)|(a^2+3)$$

We want this to be less than the original volume a^2+3:

$$|7(1-a)|(a^2+3) < a^2+3$$

Since

$$a^2+3 > 0$$

for all real a, divide through by a^2+3 to get

$$|7(1-a)| < 1$$

So

$$7|1-a| < 1 \quad \Longrightarrow \quad |1-a| < \frac{1}{7}$$

Hence

$$-\frac{1}{7} < 1-a < \frac{1}{7}$$

Subtracting 1 throughout gives

$$-\frac{8}{7} < -a < -\frac{6}{7}$$

Multiplying through by -1 and reversing the inequalities:

$$\frac{6}{7} < a < \frac{8}{7}$$

Hence, the image volume is less than the object volume for

$$\frac{6}{7} < a < \frac{8}{7}$$