OPTIONAL PURE / VECTORS & MATRICES

Orthogonal Diagonalisation

Worked solutions 10 questions · 131 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

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1. The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} k & 6 \\ 6 & -2 \end{pmatrix}$$

where k is a constant.
It is given that 10 is an eigenvalue of $\mathbf{A}$.

(a) Find the value of k and the other eigenvalue of $\mathbf{A}$. **[4]**

(b) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$$ **[5]**

Solution

(a) The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} k-\lambda & 6 \\ 6 & -2-\lambda \end{vmatrix} = (k-\lambda)(-2-\lambda) - 36$$

Since 10 is an eigenvalue, substituting $\lambda = 10$ gives

$$\begin{aligned} (k-10)(-12) - 36 &= 0 \\ k - 10 &= -3 \\ k &= 7 \end{aligned}$$

With $k = 7$, the characteristic equation becomes

$$\begin{aligned} (7-\lambda)(-2-\lambda) - 36 &= 0 \\ \lambda^2 - 5\lambda - 14 - 36 &= 0 \\ \lambda^2 - 5\lambda - 50 &= 0 \\ (\lambda - 10)(\lambda + 5) &= 0 \end{aligned}$$

So the other eigenvalue of $\mathbf{A}$ is $\lambda = -5$.

(b) For $\lambda = 10$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix} -3 & 6 \\ 6 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -3x + 6y = 0 \implies x = 2y$$

so an eigenvector is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda = -5$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix} 12 & 6 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 12x + 6y = 0 \implies y = -2x$$

so an eigenvector is

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

These eigenvectors are perpendicular, since

$$\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\end{pmatrix}=2\times1+1\times(-2)=0$$

Each has magnitude $\sqrt{2^2+1^2}=\sqrt{5}$, so the unit eigenvectors are

$$\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix},\qquad\frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix}$$

Using these as the columns of $\mathbf{P}$, in the same order as the eigenvalues in $\mathbf{D}$, makes $\mathbf{P}$ orthogonal, so $\mathbf{P}^{\mathrm{T}}=\mathbf{P}^{-1}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$ with

$$\mathbf{P}=\frac{1}{\sqrt{5}}\begin{pmatrix}2&1\\1&-2\end{pmatrix},\qquad\mathbf{D}=\begin{pmatrix}10&0\\0&-5\end{pmatrix}$$

2.

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & k \\ -4 & 7 & 2 \\ k & 2 & 2 \end{pmatrix}$$

where k is a constant.
The vector $\mathbf{v}$ is an eigenvector of $\mathbf{M}$, where

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(a) Find the value of k and the eigenvalue of $\mathbf{M}$ corresponding to $\mathbf{v}$. **[3]**

(b) Determine the other two eigenvalues of $\mathbf{M}$. **[3]**

(c) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$. **[6]**

Solution

(a) Let the eigenvalue be λ, so that $\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$:

$$\begin{pmatrix} 3 & -4 & k \\ -4 & 7 & 2 \\ k & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-8+2k \\ -4+14+4 \\ k+4+4 \end{pmatrix} = \begin{pmatrix} 2k-5 \\ 14 \\ k+8 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

The second row gives $2\lambda = 14$, so $\lambda = 7$. The third row then gives

$$k + 8 = 14 \quad \Longrightarrow \quad k = 6$$

The first row is consistent, since $2 \times 6 - 5 = 7$. So $k = 6$ and the eigenvalue is $\lambda = 7$.

(b) With $k = 6$, the characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & -4 & 6 \\ -4 & 7-\lambda & 2 \\ 6 & 2 & 2-\lambda \end{vmatrix} \\ &= (3-\lambda)[(7-\lambda)(2-\lambda) - 4] + 4[-4(2-\lambda) - 12] + 6[-8 - 6(7-\lambda)] \\ &= (3-\lambda)(\lambda^2 - 9\lambda + 10) + 4(4\lambda - 20) + 6(6\lambda - 50) \\ &= -\lambda^3 + 12\lambda^2 - 37\lambda + 30 + 16\lambda - 80 + 36\lambda - 300 \\ &= -\lambda^3 + 12\lambda^2 + 15\lambda - 350 \end{aligned}$$

So the eigenvalues satisfy

$$\lambda^3 - 12\lambda^2 - 15\lambda + 350 = 0$$

Since $\lambda = 7$ is a root, $\lambda - 7$ is a factor:

$$(\lambda - 7)(\lambda^2 - 5\lambda - 50) = 0$$

$$(\lambda - 7)(\lambda - 10)(\lambda + 5) = 0$$

The other two eigenvalues are $\lambda = 10$ and $\lambda = -5$.

(c) For $\lambda = 10$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix} -7 & -4 & 6 \\ -4 & -3 & 2 \\ 6 & 2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two rows give $-7x - 4y + 6z = 0$ and $-4x - 3y + 2z = 0$. Subtracting the first from three times the second gives $-5x - 5y = 0$, so $y = -x$, and then $-4x + 3x + 2z = 0$, so $z = \frac{1}{2}x$. Taking $x = 2$, an eigenvector is

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda = -5$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix} 8 & -4 & 6 \\ -4 & 12 & 2 \\ 6 & 2 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two rows give $4x - 2y + 3z = 0$ and $-2x + 6y + z = 0$. The second gives $z = 2x - 6y$, and substituting into the first gives $10x - 20y = 0$, so $x = 2y$ and $z = -2y$. Taking $y = 1$, an eigenvector is

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$\mathbf{M}$ is symmetric and its eigenvalues are distinct, so these eigenvectors and $\mathbf{v}$ are mutually perpendicular. Each has magnitude $\sqrt{1^2 + 2^2 + 2^2} = 3$, so the unit eigenvectors are

$$\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \qquad \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Using these as the columns of $\mathbf{P}$, with the eigenvalues in the same order down the diagonal of $\mathbf{D}$:

$$\mathbf{P} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

3.

$$\mathbf{M} = \begin{pmatrix} 4 & a & 2 \\ a & b & -2 \\ 2 & -2 & 5 \end{pmatrix}$$

where a and b are constants.
It is given that

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is an eigenvector of $\mathbf{M}$.

(a) Find the eigenvalue corresponding to this eigenvector. **[2]**

(b) Hence find the values of a and b. **[3]**

(c) Find the other two eigenvalues of $\mathbf{M}$. **[3]**

(d) Find a normalised eigenvector for each eigenvalue of $\mathbf{M}$. **[5]**

(e) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$, whose diagonal entries are in decreasing order, such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$ **[2]**

Solution

(a) Let λ be the eigenvalue. Since the given vector is an eigenvector of $\mathbf{M}$, multiplying out gives

$$\begin{pmatrix} 4 & a & 2 \\ a & b & -2 \\ 2 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \times 1 + a \times (-1) + 2 \times 2 \\ a \times 1 + b \times (-1) + (-2) \times 2 \\ 2 \times 1 + (-2) \times (-1) + 5 \times 2 \end{pmatrix} = \begin{pmatrix} 8 - a \\ a - b - 4 \\ 14 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Equating the third components:

$$14 = 2\lambda \quad \Longrightarrow \quad \lambda = 7$$

(b) Using $\lambda = 7$ from part (a), equating the first components gives

$$8 - a = 7 \quad \Longrightarrow \quad a = 1$$

and equating the second components gives

$$a - b - 4 = -7 \quad \Longrightarrow \quad 1 - b - 4 = -7 \quad \Longrightarrow \quad b = 4$$

(c) With $a = 1$ and $b = 4$, the matrix is

$$\mathbf{M} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & -2 \\ 2 & -2 & 5 \end{pmatrix}$$

The eigenvalues satisfy the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. Expanding along the first row:

$$\begin{aligned}\det(\mathbf{M} - \lambda\mathbf{I}) &= (4-\lambda)[(4-\lambda)(5-\lambda) - 4] - 1[(5-\lambda) + 4] + 2[-2 - 2(4-\lambda)] \\ &= (4-\lambda)\big(\lambda^2 - 9\lambda + 16\big) - (9-\lambda) + 2(2\lambda - 10) \\ &= -\lambda^3 + 13\lambda^2 - 52\lambda + 64 - 9 + \lambda + 4\lambda - 20 \\ &= -\lambda^3 + 13\lambda^2 - 47\lambda + 35\end{aligned}$$

So the characteristic equation is

$$\lambda^3 - 13\lambda^2 + 47\lambda - 35 = 0$$

Since $\lambda = 7$ is a root, $\lambda - 7$ is a factor:

$$(\lambda - 7)\big(\lambda^2 - 6\lambda + 5\big) = 0 \quad \Longrightarrow \quad (\lambda - 7)(\lambda - 5)(\lambda - 1) = 0$$

The other two eigenvalues are $\lambda = 5$ and $\lambda = 1$.

(d) For $\lambda = 7$, the given eigenvector has magnitude $\sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$, so a normalised eigenvector is

$$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

For $\lambda = 5$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix}-1 & 1 & 2\\ 1 & -1 & -2\\ 2 & -2 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

The third row gives $2x - 2y = 0$, so $y = x$, and then the first row gives $2z = 0$, so $z = 0$. Taking $x = 1$ gives the eigenvector

$$\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Its magnitude is $\sqrt{2}$, so a normalised eigenvector is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

For $\lambda = 1$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix}3 & 1 & 2\\ 1 & 3 & -2\\ 2 & -2 & 4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Adding the first two rows gives $4x + 4y = 0$, so $y = -x$. The first row then gives $3x - x + 2z = 0$, so $z = -x$. Taking $x = -1$ gives the eigenvector

$$\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Its magnitude is $\sqrt{3}$, so a normalised eigenvector is

$$\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

(e) $\mathbf{M}$ is symmetric and its eigenvalues are distinct, so its eigenvectors are mutually perpendicular. The matrix $\mathbf{P}$ whose columns are the normalised eigenvectors from part (d) is therefore orthogonal, with $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$. Ordering the columns to match the eigenvalues 7, 5, 1 in decreasing order:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. The symmetric 3×3 matrix $\mathbf{M}$ has eigenvalues 4, -3 and 6. Eigenvectors of $\mathbf{M}$ corresponding to the eigenvalues 4 and -3 are, respectively,

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) (i) Find an eigenvector of $\mathbf{M}$ corresponding to the eigenvalue 6. **[2]**

(ii) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$ **[3]**

(b) Use the fact that $\mathbf{P}$ is orthogonal to show that $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}$. **[2]**

(c) Hence find the matrix $\mathbf{M}$. **[5]**

Solution

(a) (i) Since $\mathbf{M}$ is symmetric, eigenvectors corresponding to distinct eigenvalues are perpendicular. So an eigenvector for the eigenvalue 6 is perpendicular to both given eigenvectors, and their vector product gives one:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

(ii) The three eigenvectors are mutually perpendicular, so dividing each by its magnitude gives mutually perpendicular unit vectors. The magnitudes are

$$\left| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right| = \sqrt{2}, \qquad \left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{3}, \qquad \left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| = \sqrt{6}$$

so the unit eigenvectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

Taking these as the columns of $\mathbf{P}$, with the eigenvalues on the diagonal of $\mathbf{D}$ in the same order:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

(b) Since $\mathbf{P}$ is orthogonal, $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{P}\mathbf{P}^{\mathrm{T}} = \mathbf{I}$. Premultiplying $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$ by $\mathbf{P}$:

$$\begin{aligned}\mathbf{P}\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} &= \mathbf{P}\mathbf{D}\\ \mathbf{I}\mathbf{M}\mathbf{P} &= \mathbf{P}\mathbf{D}\\ \mathbf{M}\mathbf{P} &= \mathbf{P}\mathbf{D}\end{aligned}$$

Postmultiplying by $\mathbf{P}^{\mathrm{T}}$:

$$\begin{aligned}\mathbf{M}\mathbf{P}\mathbf{P}^{\mathrm{T}} &= \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}\\ \mathbf{M}\mathbf{I} &= \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}\\ \mathbf{M} &= \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}\end{aligned}$$

as required.

(c) Using part (b), $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}$. First, multiplying each column of $\mathbf{P}$ by the corresponding eigenvalue gives

$$\mathbf{P}\mathbf{D} = \begin{pmatrix} \frac{4}{\sqrt{2}} & -\frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{4}{\sqrt{2}} & \frac{3}{\sqrt{3}} & -\frac{6}{\sqrt{6}} \\ 0 & -\frac{3}{\sqrt{3}} & -\frac{12}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & -\sqrt{3} & \sqrt{6} \\ 2\sqrt{2} & \sqrt{3} & -\sqrt{6} \\ 0 & -\sqrt{3} & -2\sqrt{6} \end{pmatrix}$$

The transpose of $\mathbf{P}$ is

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

so

$$\begin{aligned}\mathbf{M} &= \begin{pmatrix} 2\sqrt{2} & -\sqrt{3} & \sqrt{6} \\ 2\sqrt{2} & \sqrt{3} & -\sqrt{6} \\ 0 & -\sqrt{3} & -2\sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}\\ &= \begin{pmatrix} 2-1+1 & 2+1-1 & 0-1-2 \\ 2+1-1 & 2-1+1 & 0+1+2 \\ 0-1-2 & 0+1+2 & 0-1+4 \end{pmatrix}\\ &= \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\end{aligned}$$

5. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & p & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

where p is a constant.
It is given that 1 is an eigenvalue of $\mathbf{A}$.

(a) Show that $p = -3$. **[3]**

(b) Determine

(i) the other two eigenvalues of $\mathbf{A}$, **[3]**

(ii) an eigenvector corresponding to each of the three eigenvalues of $\mathbf{A}$. **[4]**

(c) Hence determine an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$. **[3]**

Solution

(a) Since 1 is an eigenvalue, $\det(\mathbf{A} - \mathbf{I}) = 0$, where

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & -2 & 2 \\ -2 & p-1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A} - \mathbf{I}) &= 0 - (-2)\begin{vmatrix} -2 & 2 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} -2 & p-1 \\ 2 & 2 \end{vmatrix} \\ &= 2(0-4) + 2(-4-2(p-1)) \\ &= -8 + 2(-2-2p) \\ &= -12 - 4p \end{aligned}$$

Setting this equal to zero gives

$$-12 - 4p = 0 \quad \Longrightarrow \quad p = -3$$

as required.

(b) (i) Substituting $p = -3$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & -3-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)[(-3-\lambda)(1-\lambda) - 4] + 2[-2(1-\lambda) - 4] + 2[-4 - 2(-3-\lambda)] \\ &= (1-\lambda)(\lambda^2 + 2\lambda - 7) + (4\lambda - 12) + (4 + 4\lambda) \\ &= (1-\lambda)(\lambda^2 + 2\lambda - 7) - 8(1-\lambda) \\ &= (1-\lambda)(\lambda^2 + 2\lambda - 15) \\ &= (1-\lambda)(\lambda - 3)(\lambda + 5) \end{aligned}$$

The characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ therefore gives $\lambda = 1$, $\lambda = 3$ and $\lambda = -5$, so the other two eigenvalues are

$$\lambda = 3 \quad \text{and} \quad \lambda = -5$$

(ii) For each eigenvalue λ, an eigenvector with components x, y, z satisfies the equations given by the rows of $\mathbf{A} - \lambda\mathbf{I}$.

$\lambda = 1$: the first and third rows of $\mathbf{A} - \mathbf{I}$ give

$$-2y + 2z = 0, \qquad 2x + 2y = 0$$

so $z = y$ and $x = -y$, and the second row, $-2x - 4y + 2z = 0$, also holds. Taking $y = 1$:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 3$: with

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & -2 & 2 \\ -2 & -6 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

the first two rows give

$$-2x - 2y + 2z = 0, \qquad -2x - 6y + 2z = 0$$

Subtracting gives $4y = 0$, so $y = 0$ and $z = x$; the third row also holds. Taking $x = 1$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = -5$: with

$$\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 6 \end{pmatrix}$$

the first two rows give

$$6x - 2y + 2z = 0, \qquad -2x + 2y + 2z = 0$$

Adding gives $4x + 4z = 0$, so $z = -x$, and then $y = x - z = 2x$; the third row, $2x + 2y + 6z = 0$, also holds. Taking $x = 1$:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Any non-zero multiples of these vectors are also eigenvectors.

(c) $\mathbf{A}$ is symmetric with three distinct eigenvalues, so its eigenvectors are mutually perpendicular. The magnitudes are $|\mathbf{v}_1| = \sqrt{3}$, $|\mathbf{v}_2| = \sqrt{2}$ and $|\mathbf{v}_3| = \sqrt{6}$. Using the unit eigenvectors as the columns of $\mathbf{P}$,

in the same order as the eigenvalues on the leading diagonal of $\mathbf{D}$, gives an orthogonal matrix, so $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$ with

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

6. The symmetric matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} -1 & 4 & -4 \\ 4 & 3 & 0 \\ -4 & 0 & a \end{pmatrix}$$

where a is a constant.
Given that $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ is an eigenvector of $\mathbf{M}$,

(a) find the corresponding eigenvalue and the value of a. **[3]**

Given also that 7 is an eigenvalue of $\mathbf{M}$,

(b) find a corresponding eigenvector. **[2]**

(c) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$ **[5]**

Solution

(a) Let λ be the corresponding eigenvalue. Then

$$\begin{pmatrix} -1 & 4 & -4 \\ 4 & 3 & 0 \\ -4 & 0 & a \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \times 2 + 4 \times (-1) - 4 \times 1 \\ 4 \times 2 + 3 \times (-1) + 0 \times 1 \\ -4 \times 2 + 0 \times (-1) + a \times 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ a - 8 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

The first two components give $\lambda = -5$, and then the third gives

$$a - 8 = -5 \quad \Longrightarrow \quad a = 3$$

So the eigenvalue is -5 and $a = 3$.

(b) With $a = 3$, the eigenvectors for the eigenvalue 7 are the non-zero solutions of

$$\begin{pmatrix} -8 & 4 & -4 \\ 4 & -4 & 0 \\ -4 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The second row gives $4x - 4y = 0$, so $y = x$, and the third row gives $-4x - 4z = 0$, so $z = -x$. These also satisfy the first row, since $-8x + 4x + 4x = 0$. Taking $x = 1$, an eigenvector is

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(c) Since $\mathbf{M}$ is symmetric, its eigenvectors can be chosen to be mutually perpendicular. The eigenvectors found so far are perpendicular, since

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

A third eigenvector is perpendicular to both, so take their vector product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (1 - 1)\mathbf{i} - (-2 - 1)\mathbf{j} + (2 + 1)\mathbf{k} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

Dividing by 3 gives the eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then

$$\begin{pmatrix} -1 & 4 & -4 \\ 4 & 3 & 0 \\ -4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

so this is an eigenvector with eigenvalue 3.

The eigenvectors have magnitudes

$$\left| \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{6}, \qquad \left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right| = \sqrt{3}, \qquad \left| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right| = \sqrt{2}$$

so the unit eigenvectors are

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The orthogonal matrix $\mathbf{P}$ has these unit eigenvectors as its columns, and $\mathbf{D}$ has the corresponding eigenvalues on its leading diagonal in the same order:

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

7. The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 9 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

(a) Given that 1 is an eigenvalue of $\mathbf{A}$, find the other eigenvalues of $\mathbf{A}$, showing that one of them is repeated. **[4]**

(b) Find three mutually perpendicular eigenvectors of $\mathbf{A}$. **[5]**

(c) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[3]**

Solution

(a) The eigenvalues satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Expanding along the first row:

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 9-\lambda & -2 & 2 \\ -2 & 6-\lambda & 4 \\ 2 & 4 & 6-\lambda \end{vmatrix} \\
&= (9-\lambda)\left[(6-\lambda)^2 - 16\right] + 2[-2(6-\lambda) - 8] + 2[-8 - 2(6-\lambda)] \\
&= (9-\lambda)\left(\lambda^2 - 12\lambda + 20\right) + 2(2\lambda - 20) + 2(2\lambda - 20) \\
&= -\lambda^3 + 21\lambda^2 - 128\lambda + 180 + 8\lambda - 80 \\
&= -\lambda^3 + 21\lambda^2 - 120\lambda + 100
\end{aligned}$$

So the characteristic equation is

$$\lambda^3 - 21\lambda^2 + 120\lambda - 100 = 0$$

Since 1 is an eigenvalue, $\lambda - 1$ is a factor. Factorising:

$$\begin{aligned}
(\lambda - 1)\left(\lambda^2 - 20\lambda + 100\right) &= 0 \\
(\lambda - 1)(\lambda - 10)^2 &= 0
\end{aligned}$$

The other eigenvalues are $\lambda = 10$ and $\lambda = 10$: the eigenvalue 10 is repeated.

(b) For $\lambda = 1$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The first row gives $8x - 2y + 2z = 0$, so $z = y - 4x$. Substituting into the second row:

$$-2x + 5y + 4(y - 4x) = 0 \quad \Longrightarrow \quad 9y = 18x \quad \Longrightarrow \quad y = 2x$$

Then $z = 2x - 4x = -2x$, and the third row, $2x + 8x - 10x = 0$, is satisfied. Taking $x = 1$, an eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

For $\lambda = 10$, the eigenvectors are the non-zero solutions of

$$\begin{pmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Every row is a multiple of the first, so the only condition is

$$x + 2y - 2z = 0$$

Any non-zero vector satisfying this is an eigenvector with eigenvalue 10. Taking $x = 0$, $y = 1$ gives $z = 1$:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

For a third eigenvector $\mathbf{v}_3$ with eigenvalue 10, perpendicular to $\mathbf{v}_2$, it must also satisfy

$$\mathbf{v}_3 \cdot \mathbf{v}_2 = 0$$

so $y + z = 0$. Taking $y = -1$, $z = 1$, the condition $x + 2y - 2z = 0$ gives $x = 4$:

$$\mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

Checking the scalar products:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 + 2 - 2 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = 4 - 2 - 2 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = 0 - 1 + 1 = 0$$

So three mutually perpendicular eigenvectors of $\mathbf{A}$ are

$$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

(c) Using part (b), divide each eigenvector by its magnitude to make it a unit vector:

$$|\mathbf{v}_1| = \sqrt{1 + 4 + 4} = 3, \qquad |\mathbf{v}_2| = \sqrt{0 + 1 + 1} = \sqrt{2}, \qquad |\mathbf{v}_3| = \sqrt{16 + 1 + 1} = 3\sqrt{2}$$

The columns of $\mathbf{P}$ are these unit eigenvectors, which are mutually perpendicular, so $\mathbf{P}$ is orthogonal. The diagonal entries of $\mathbf{D}$ are the corresponding eigenvalues, in the same order as the columns of $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

8.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & k \\ 1 & 4 & 1 \\ k & 1 & 6 \end{pmatrix}$$

where k is a constant.
Given that 2 is an eigenvalue of $\mathbf{M}$,

(a) find the possible values of k. **[4]**

Given further that $k < 0$,

(b) find the other eigenvalues of $\mathbf{M}$. **[3]**

(c) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$. **[7]**

Solution

(a) Since 2 is an eigenvalue of $\mathbf{M}$, $\det(\mathbf{M} - 2\mathbf{I}) = 0$, where

$$\mathbf{M} - 2\mathbf{I} = \begin{pmatrix} 4 & 1 & k \\ 1 & 2 & 1 \\ k & 1 & 4 \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{M} - 2\mathbf{I}) &= 4(2 \times 4 - 1 \times 1) - 1(1 \times 4 - 1 \times k) + k(1 \times 1 - 2k) \\ &= 28 - 4 + k + k - 2k^2 \\ &= 24 + 2k - 2k^2 \end{aligned}$$

Setting this equal to zero:

$$\begin{aligned} k^2 - k - 12 &= 0 \\ (k - 4)(k + 3) &= 0 \end{aligned}$$

so $k = 4$ or $k = -3$.

(b) Since $k < 0$, $k = -3$, and

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -3 \\ 1 & 4 & 1 \\ -3 & 1 & 6 \end{pmatrix}$$

The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. Expanding along the first row:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= (6 - \lambda)\big[(4 - \lambda)(6 - \lambda) - 1\big] - 1\big[(6 - \lambda) + 3\big] - 3\big[1 + 3(4 - \lambda)\big] \\ &= (6 - \lambda)(\lambda^2 - 10\lambda + 23) - (9 - \lambda) - 3(13 - 3\lambda) \\ &= -\lambda^3 + 16\lambda^2 - 83\lambda + 138 - 9 + \lambda - 39 + 9\lambda \\ &= -\lambda^3 + 16\lambda^2 - 73\lambda + 90 \end{aligned}$$

So the characteristic equation is $\lambda^3 - 16\lambda^2 + 73\lambda - 90 = 0$. Since $\lambda = 2$ is a root, $\lambda - 2$ is a factor:

$$\begin{aligned} (\lambda - 2)(\lambda^2 - 14\lambda + 45) &= 0 \\ (\lambda - 2)(\lambda - 5)(\lambda - 9) &= 0 \end{aligned}$$

The other eigenvalues of $\mathbf{M}$ are 5 and 9.

(c) For each eigenvalue λ, an eigenvector with components x, y, z satisfies the equations given by the rows of $\mathbf{M} - \lambda\mathbf{I}$.

$\lambda = 2$: the first two rows of $\mathbf{M} - 2\mathbf{I}$ give

$$4x + y - 3z = 0, \qquad x + 2y + z = 0$$

From the second, $x = -2y - z$. Substituting into the first gives $-7y - 7z = 0$, so $y = -z$ and $x = z$. An eigenvector is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 5$: the first two rows of $\mathbf{M} - 5\mathbf{I}$ give

$$x + y - 3z = 0, \qquad x - y + z = 0$$

Adding gives $2x - 2z = 0$, so $x = z$ and then $y = 2z$. An eigenvector is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda = 9$: the first two rows of $\mathbf{M} - 9\mathbf{I}$ give

$$-3x + y - 3z = 0, \qquad x - 5y + z = 0$$

From the first, $y = 3x + 3z$. Substituting into the second gives $-14x - 14z = 0$, so $z = -x$ and then $y = 0$. An eigenvector is

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\mathbf{M}$ is symmetric and its eigenvalues are distinct, so these eigenvectors are mutually perpendicular. Their magnitudes are $\sqrt{3}$, $\sqrt{6}$ and $\sqrt{2}$. Dividing each by its magnitude gives unit eigenvectors, which form the columns of an orthogonal matrix $\mathbf{P}$, with $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$. Then $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

9.

$$\mathbf{M} = \begin{pmatrix} p & q & -3 \\ q & 5 & 3 \\ -3 & 3 & 0 \end{pmatrix}$$

where p and q are constants.
Given that

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is an eigenvector of the matrix $\mathbf{M}$,

(a) find the eigenvalue corresponding to this eigenvector, **[2]**

(b) find the value of q and the value of p, **[3]**

(c) find the other two eigenvalues of $\mathbf{M}$, **[3]**

(d) find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$ **[6]**

Solution

(a) Let λ be the corresponding eigenvalue. Then

$$\begin{pmatrix} p & q & -3 \\ q & 5 & 3 \\ -3 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Multiplying out the left-hand side gives

$$\begin{pmatrix} p-q-6 \\ q+1 \\ -6 \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \\ 2\lambda \end{pmatrix}$$

The third row contains neither p nor q, so

$$2\lambda = -6 \implies \lambda = -3$$

(b) Using $\lambda = -3$ in the second row:

$$q + 1 = 3 \implies q = 2$$

Then the first row gives

$$p - 2 - 6 = -3 \implies p = 5$$

(c) Substituting $p = 5$ and $q = 2$:

$$\mathbf{M} = \begin{pmatrix} 5 & 2 & -3 \\ 2 & 5 & 3 \\ -3 & 3 & 0 \end{pmatrix}$$

The eigenvalues satisfy $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. Expanding along the first row:

$$\begin{aligned}\det(\mathbf{M} - \lambda\mathbf{I}) &= (5-\lambda)[(5-\lambda)(-\lambda) - 9] - 2[-2\lambda + 9] - 3[6 + 3(5-\lambda)] \\ &= (5-\lambda)(\lambda^2 - 5\lambda - 9) + 4\lambda - 18 - 63 + 9\lambda \\ &= -\lambda^3 + 10\lambda^2 - 16\lambda - 45 + 13\lambda - 81 \\ &= -\lambda^3 + 10\lambda^2 - 3\lambda - 126\end{aligned}$$

So the characteristic equation is

$$\lambda^3 - 10\lambda^2 + 3\lambda + 126 = 0$$

Since $\lambda = -3$ is a root, $\lambda + 3$ is a factor:

$$\begin{aligned}(\lambda+3)(\lambda^2 - 13\lambda + 42) &= 0 \\ (\lambda+3)(\lambda-6)(\lambda-7) &= 0\end{aligned}$$

The other two eigenvalues are 6 and 7.

(d) For $\lambda = 6$, the first two rows of $\mathbf{M} - 6\mathbf{I}$ give

$$\begin{aligned}-x + 2y - 3z &= 0 \\ 2x - y + 3z &= 0\end{aligned}$$

Adding gives $x + y = 0$, so $y = -x$, and then $-x - 2x - 3z = 0$ gives $z = -x$. Taking $x = -1$, an eigenvector is

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 7$, the first and third rows of $\mathbf{M} - 7\mathbf{I}$ give

$$\begin{aligned}-2x + 2y - 3z &= 0 \\ -3x + 3y - 7z &= 0\end{aligned}$$

Three times the first equation minus twice the second gives $5z = 0$, so $z = 0$ and $y = x$. An eigenvector is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Since $\mathbf{M}$ is symmetric and its eigenvalues are distinct, these three eigenvectors are mutually perpendicular. Dividing each by its magnitude, $\sqrt{6}$, $\sqrt{3}$ and $\sqrt{2}$ respectively, gives unit eigenvectors, which form the columns of $\mathbf{P}$, with the eigenvalues in the same order down the diagonal of $\mathbf{D}$:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

10. The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -4 & 0 \\ -4 & 3 & p \\ 0 & p & 1 \end{pmatrix}$$

where p is a constant.
It is given that

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

is an eigenvector of $\mathbf{A}$.

(a) (i) Determine the eigenvalue corresponding to this eigenvector. **[2]**

(ii) Hence show that $p = -4$. **[2]**

(iii) Determine the remaining eigenvalues of $\mathbf{A}$ and a corresponding eigenvector for each. **[7]**

(b) Write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{\mathrm{T}}$. **[2]**

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant and the dot denotes differentiation with respect to the time t. **[2]**

(ii) With respect to a fixed origin O, a particle moving through space has position (x, y, z) at time t, and its velocity is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

At time $t = 0$ the particle is at the point $(5, 1, 1)$.
By considering

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{\mathrm{T}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

determine the position of the particle at time t. **[5]**

Solution

(a) (i) Multiplying the eigenvector by $\mathbf{A}$ gives

$$\begin{pmatrix} 5 & -4 & 0 \\ -4 & 3 & p \\ 0 & p & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \times 2 + (-4) \times (-2) + 0 \times 1 \\ (-4) \times 2 + 3 \times (-2) + p \times 1 \\ 0 \times 2 + p \times (-2) + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 18 \\ p - 14 \\ 1 - 2p \end{pmatrix}$$

This is λ times the eigenvector, where λ is the eigenvalue. Equating first components, $18 = 2\lambda$, so

$$\lambda = 9$$

(ii) By part (a)(i), the product above equals

$$9 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ -18 \\ 9 \end{pmatrix}$$

Equating second and third components:

$$p - 14 = -18$$

$$1 - 2p = 9$$

Both equations give

$$p = -4$$

as required.

(iii) With $p = -4$, the eigenvalues satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 5-\lambda & -4 & 0 \\ -4 & 3-\lambda & -4 \\ 0 & -4 & 1-\lambda \end{vmatrix} \\ &= (5-\lambda)[(3-\lambda)(1-\lambda) - (-4)\times(-4)] - (-4)[(-4)(1-\lambda) - (-4)\times 0] + 0 \\ &= (5-\lambda)(\lambda^2 - 4\lambda - 13) - 16(1-\lambda) \\ &= -\lambda^3 + 9\lambda^2 - 7\lambda - 65 - 16 + 16\lambda \\ &= -\lambda^3 + 9\lambda^2 + 9\lambda - 81\end{aligned}$$

So the characteristic equation is

$$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$$

By part (a)(i), $\lambda = 9$ is a root, so $\lambda - 9$ is a factor:

$$\lambda^2(\lambda - 9) - 9(\lambda - 9) = (\lambda - 9)(\lambda^2 - 9) = (\lambda - 9)(\lambda - 3)(\lambda + 3) = 0$$

The remaining eigenvalues are $\lambda = 3$ and $\lambda = -3$.

For $\lambda = 3$, an eigenvector with components α, β, γ satisfies

$$(\mathbf{A} - 3\mathbf{I})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & -4 \\ 0 & -4 & -2 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

which gives

$$2\alpha - 4\beta = 0$$

$$-4\alpha - 4\gamma = 0$$

$$-4\beta - 2\gamma = 0$$

The first equation gives $\alpha = 2\beta$ and the third gives $\gamma = -2\beta$, and these satisfy the second. Taking $\beta = 1$, an eigenvector for $\lambda = 3$ is

$$\begin{pmatrix}2\\ 1\\ -2\end{pmatrix}$$

For $\lambda = -3$:

$$(\mathbf{A} + 3\mathbf{I})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix} 8 & -4 & 0 \\ -4 & 6 & -4 \\ 0 & -4 & 4 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

which gives

$$\begin{aligned} 8\alpha - 4\beta &= 0 \\ -4\alpha + 6\beta - 4\gamma &= 0 \\ -4\beta + 4\gamma &= 0 \end{aligned}$$

The first equation gives $\beta = 2\alpha$ and the third gives $\gamma = \beta = 2\alpha$, and these satisfy the second, since $-4\alpha + 12\alpha - 8\alpha = 0$. Taking $\alpha = 1$, an eigenvector for $\lambda = -3$ is

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(b) Since $\mathbf{A}$ is symmetric and its eigenvalues are distinct, the three eigenvectors are mutually perpendicular. Each has magnitude 3, for example $\sqrt{2^2 + (-2)^2 + 1^2} = 3$, so dividing each by 3 gives unit eigenvectors, which form the columns of $\mathbf{P}$ in the same order as the eigenvalues in $\mathbf{D}$:

$$\mathbf{P} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

(c) (i) Separating the variables:

$$\int \frac{1}{u}\,\mathrm{d}u = \int k\,\mathrm{d}t$$

Integrating gives

$$\begin{aligned} \ln|u| &= kt + c \\ |u| &= e^c e^{kt} \end{aligned}$$

Writing $K = \pm e^c$, the general solution is

$$u = Ke^{kt}$$

where K is a constant.

(ii) Since $\mathbf{P}$ is orthogonal, $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{P}\mathbf{P}^{\mathrm{T}} = \mathbf{I}$. Using $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}$ from part (b):

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} &= \mathbf{P}^{\mathrm{T}}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \\ &= \mathbf{P}^{\mathrm{T}}\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \mathbf{P}^{\mathrm{T}}\mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \mathbf{D}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \end{aligned}$$

So

$$\dot{u} = 9u, \qquad \dot{v} = 3v, \qquad \dot{w} = -3w$$

At $t = 0$, $x = 5$, $y = 1$ and $z = 1$, so

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

By part (c)(i), with $k = 9$, 3 and -3 in turn, and these values at $t = 0$ as the constants:

$$u = 3e^{9t}, \qquad v = 3e^{3t}, \qquad w = 3e^{-3t}$$

Multiplying both sides of

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{\mathrm{T}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

on the left by $\mathbf{P}$, and using $\mathbf{P}\mathbf{P}^{\mathrm{T}} = \mathbf{I}$:

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{P}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 3e^{9t} \\ 3e^{3t} \\ 3e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{9t} + 2e^{3t} + e^{-3t} \\ -2e^{9t} + e^{3t} + 2e^{-3t} \\ e^{9t} - 2e^{3t} + 2e^{-3t} \end{pmatrix} \end{aligned}$$