OPTIONAL PURE / VECTORS & MATRICES

Eigenvalues, Eigenvectors and Diagonalisation

Worked solutions 15 questions · 169 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR MEI Extra Pure Y435 · AQA 7367 · CIE 9231

Instructions

CIE (9231): eigenvalues are examined only where they are real and distinct, so repeated and non-real eigenvalues are not on the specification. Questions 10, 13 and 14 can be left out.

Questions

Question 1	*6 marks*	2
Question 2	*6 marks*	3
Question 3	*11 marks*	4
Question 4	*6 marks*	6
Question 5	*10 marks*	8
Question 6	*15 marks*	10
Question 7	*13 marks*	12
Question 8	*12 marks*	14
Question 9	*13 marks*	16
Question 10	*8 marks*	19
Question 11	*12 marks*	21
Question 12	*12 marks*	24
Question 13	*13 marks*	26
Question 14	*13 marks*	28
Question 15	*19 marks*	30

1. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & k \\ 2 & -2 \end{pmatrix}$$

where k is a constant.

Given that 4 is an eigenvalue of $\mathbf{A}$, find

- the value of k,
- the other eigenvalue of $\mathbf{A}$,
- an eigenvector associated with each eigenvalue of $\mathbf{A}$.

[6]

Solution

The eigenvalues satisfy the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which is

$$\begin{vmatrix} 3-\lambda & k \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-2-\lambda) - 2k = 0$$

$$\lambda^2 - \lambda - 6 - 2k = 0$$

Since 4 is an eigenvalue, substitute $\lambda = 4$:

$$16 - 4 - 6 - 2k = 0 \implies 6 - 2k = 0 \implies k = 3$$

With $k = 3$ the characteristic equation becomes

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

so the other eigenvalue is $\lambda = -3$.

For $\lambda = 4$, solve $(\mathbf{A} - 4\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both rows give $-x + 3y = 0$, so $x = 3y$. Taking $y = 1$, an eigenvector is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

For $\lambda = -3$, solve $(\mathbf{A} + 3\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$:

$$\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both rows give $2x + y = 0$, so $y = -2x$. Taking $x = 1$, an eigenvector is

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So $k = 3$, the other eigenvalue is -3, and eigenvectors are $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for $\lambda = 4$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for $\lambda = -3$ (or any non-zero multiples of these).

2. The matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} a & 0.6 \\ 0.6 & b \end{pmatrix}$$

where a and b are constants with $a > 0$.

The matrix $\mathbf{M}$ represents a reflection in a line through the origin.

(a) Write down the eigenvalues of $\mathbf{M}$. **[1]**

(b) By considering the characteristic equation of $\mathbf{M}$, use your answer to part (a) to find the values of a and b. **[3]**

(c) Hence find the equation of the mirror line of the reflection represented by $\mathbf{M}$. **[2]**

Solution

(a) A vector along the mirror line is unchanged by the reflection, and a vector perpendicular to the mirror line is reversed. So the eigenvalues are

$$\lambda = 1, \qquad \lambda = -1$$

(b) The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$:

$$(a - \lambda)(b - \lambda) - 0.6 \times 0.6 = 0$$

$$\lambda^2 - (a + b)\lambda + (ab - 0.36) = 0$$

By part (a), its roots are 1 and -1, so it must be

$$(\lambda - 1)(\lambda + 1) = \lambda^2 - 1 = 0$$

Comparing coefficients:

$$a + b = 0, \qquad ab - 0.36 = -1$$

So $b = -a$ and $ab = -0.64$, which gives

$$-a^2 = -0.64 \quad \Longrightarrow \quad a^2 = 0.64$$

Since $a > 0$, we obtain

$$a = 0.8, \qquad b = -0.8$$

(c) The mirror line consists of the invariant points, so it is the line through the origin in the direction of an eigenvector with eigenvalue 1. Solving $(\mathbf{M} - \mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$:

$$\begin{pmatrix} -0.2 & 0.6 \\ 0.6 & -1.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad -0.2x + 0.6y = 0 \quad \Longrightarrow \quad x = 3y$$

So an eigenvector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and the mirror line is

$$y = \frac{1}{3}x$$

3. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 2 & k \\ k & 5 \end{pmatrix}$$

where k is a negative constant.

It is given that 1 is an eigenvalue of $\mathbf{M}$.

(a) Find the value of k. Hence find the other eigenvalue of $\mathbf{M}$, and a corresponding eigenvector for each eigenvalue of $\mathbf{M}$. **[6]**

(b) (i) Describe how the directions of the invariant lines through the origin of the transformation represented by $\mathbf{M}$ are related to each other. Fully justify your answer. **[2]**

(ii) Describe fully the transformation represented by $\mathbf{M}$, stating the equation of any line of invariant points. **[3]**

Solution

(a) The eigenvalues satisfy $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, which is

$$\begin{vmatrix} 2-\lambda & k \\ k & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - k^2 = 0$$

Since $\lambda = 1$ is an eigenvalue:

$$1 \times 4 - k^2 = 0 \quad \Longrightarrow \quad k^2 = 4 \quad \Longrightarrow \quad k = \pm 2$$

Since k is negative, $k = -2$.

The characteristic equation is then

$$\begin{aligned} (2-\lambda)(5-\lambda) - 4 &= 0 \\ \lambda^2 - 7\lambda + 6 &= 0 \\ (\lambda - 1)(\lambda - 6) &= 0 \end{aligned}$$

so the other eigenvalue is $\lambda = 6$.

For $\lambda = 1$, solve $(\mathbf{M} - \mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$:

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad x - 2y = 0$$

so a corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda = 6$, solve $(\mathbf{M} - 6\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$:

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad 2x + y = 0$$

so a corresponding eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(b) (i) The invariant lines through the origin are the lines in the directions of the eigenvectors, namely $y = \frac{1}{2}x$ and $y = -2x$. Their scalar product is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 2 - 2 = 0$$

so the two invariant lines through the origin are perpendicular to each other.

(ii) Using the eigenvectors from part (a):

$$\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \times 2 + (-2) \times 1 \\ (-2) \times 2 + 5 \times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + (-2) \times (-2) \\ (-2) \times 1 + 5 \times (-2) \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Any point can be written with position vector $s\begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and $\mathbf{M}$ maps it to

$$s\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 6t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So every point on $y = \frac{1}{2}x$ (where $t = 0$) is fixed, while the component in the perpendicular direction $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is multiplied by 6.

Hence $\mathbf{M}$ represents a one-way stretch with scale factor 6, parallel to the line $y = -2x$, with line of invariant points $y = \frac{1}{2}x$.

4. A 3×3 matrix $\mathbf{A}$ has eigenvalues 2, -1 and 1, with corresponding eigenvectors

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}$$

respectively.

(a) By expressing $\mathbf{A}$ in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, find the matrix $\mathbf{A}$. **[4]**

(b) Hence find the matrix $\mathbf{A}^4$. **[2]**

Solution

(a) Form $\mathbf{P}$ with the eigenvectors as its columns, and $\mathbf{D}$ with the corresponding eigenvalues on its leading diagonal in the same order:

$$\mathbf{P} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 1\end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}$$

Then $\mathbf{A} = \mathbf{PDP}^{-1}$.

Expanding along the first row, the determinant of $\mathbf{P}$ is

$$\det(\mathbf{P}) = 1 \times (1-1) - 0 + 1 \times (0-1) = -1$$

The inverse is given by

$$\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\operatorname{adj}(\mathbf{P})$$

where $\operatorname{adj}(\mathbf{P})$ is the transposed matrix of cofactors. The matrix of cofactors of $\mathbf{P}$ is

$$\begin{pmatrix}0 & 1 & -1\\1 & 0 & -1\\-1 & -1 & 1\end{pmatrix}$$

This matrix is symmetric, so it is equal to its transpose, and

$$\mathbf{P}^{-1} = -\begin{pmatrix}0 & 1 & -1\\1 & 0 & -1\\-1 & -1 & 1\end{pmatrix} = \begin{pmatrix}0 & -1 & 1\\-1 & 0 & 1\\1 & 1 & -1\end{pmatrix}$$

Multiplying $\mathbf{P}$ by the diagonal matrix $\mathbf{D}$ scales each column of $\mathbf{P}$ by its eigenvalue:

$$\mathbf{PD} = \begin{pmatrix}2 & 0 & 1\\0 & -1 & 1\\2 & -1 & 1\end{pmatrix}$$

Therefore

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{pmatrix}2 & 0 & 1\\0 & -1 & 1\\2 & -1 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\-1 & 0 & 1\\1 & 1 & -1\end{pmatrix}$$

$$= \begin{pmatrix}1 & -1 & 1\\2 & 1 & -2\\2 & -1 & 0\end{pmatrix}$$

(b) Using part (a):

$$\mathbf{A}^4 = \left(\mathbf{PDP}^{-1}\right)^4 = \mathbf{PD}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{D}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{D}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{DP}^{-1} = \mathbf{PD}^4\mathbf{P}^{-1}$$

where

$$\mathbf{D}^4 = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{PD}^4 = \begin{pmatrix} 16 & 0 & 1 \\ 0 & 1 & 1 \\ 16 & 1 & 1 \end{pmatrix}$$

so

$$\begin{aligned} \mathbf{A}^4 &= \begin{pmatrix} 16 & 0 & 1 \\ 0 & 1 & 1 \\ 16 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -15 & 15 \\ 0 & 1 & 0 \\ 0 & -15 & 16 \end{pmatrix} \end{aligned}$$

5. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 4 & -1 & 1 \\ a & 5 & a \\ -8 & b & -5 \end{pmatrix}$$

where a and b are real numbers.

The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of $\mathbf{M}$, with corresponding eigenvalues λ_1 and λ_2 respectively.

(a) (i) Find the value of λ_1. **[1]**

(ii) Find the value of a and the value of b. **[2]**

It is also given that $\lambda_3 = 3$ is an eigenvalue of $\mathbf{M}$.

(b) Find the value of λ_2, and find an eigenvector $\mathbf{v}_3$ of $\mathbf{M}$ corresponding to the eigenvalue $\lambda_3 = 3$. **[4]**

(c) Find matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$, such that $\mathbf{D}$ is diagonal and $\mathbf{M} = \mathbf{UDU}^{-1}$. **[3]**

Solution

(a) (i) Since $\mathbf{v}_1$ is an eigenvector, $\mathbf{Mv}_1 = \lambda_1 \mathbf{v}_1$:

$$\begin{pmatrix} 4 & -1 & 1 \\ a & 5 & a \\ -8 & b & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ a+10 \\ 2b-8 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Comparing the first components:

$$\lambda_1 = 2$$

(ii) Comparing the second and third components, with $\lambda_1 = 2$:

$$a + 10 = 4 \implies a = -6$$

$$2b - 8 = 0 \implies b = 4$$

(b) With $a = -6$ and $b = 4$, the matrix is

$$\mathbf{M} = \begin{pmatrix} 4 & -1 & 1 \\ -6 & 5 & -6 \\ -8 & 4 & -5 \end{pmatrix}$$

Then

$$\mathbf{Mv}_2 = \begin{pmatrix} 4 & -1 & 1 \\ -6 & 5 & -6 \\ -8 & 4 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

so $\lambda_2 = -1$.

For $\lambda_3 = 3$, let $\mathbf{v}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfy $(\mathbf{M} - 3\mathbf{I})\mathbf{v}_3 = \mathbf{0}$:

$$\begin{pmatrix} 1 & -1 & 1 \\ -6 & 2 & -6 \\ -8 & 4 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two rows give

$$x - y + z = 0$$

$$-6x + 2y - 6z = 0$$

The first gives $y = x + z$; substituting into the second:

$$-6x + 2(x + z) - 6z = 0 \quad \Longrightarrow \quad -4x - 4z = 0 \quad \Longrightarrow \quad z = -x$$

and then $y = 0$. These also satisfy the third row, $-8x + 4y - 8z = 0$. Taking $x = 1$:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(or any non-zero multiple of this).

(c) Take the columns of $\mathbf{U}$ to be the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and the diagonal entries of $\mathbf{D}$ to be the corresponding eigenvalues in the same order:

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The determinant is

$$\det(\mathbf{U}) = 1 \times (-1 - 0) - 0 + 1 \times (2 - 0) = 1$$

which is non-zero, so $\mathbf{U}$ is invertible and

$$\mathbf{U}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

Then $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$. (Any ordering of the eigenvectors in $\mathbf{U}$, with the eigenvalues in $\mathbf{D}$ in the matching order, is equally valid.)

6.

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

(a) Show that -2 is an eigenvalue of $\mathbf{M}$, and find a corresponding eigenvector. **[4]**

(b) Given that $\mathbf{MU} = \mathbf{UD}$, where $\mathbf{D}$ is a diagonal matrix and each column of $\mathbf{U}$ is an eigenvector of $\mathbf{M}$ of unit length, find possible matrices for $\mathbf{D}$ and $\mathbf{U}$. **[11]**

Solution

(a) -2 is an eigenvalue if $\det(\mathbf{M} + 2\mathbf{I}) = 0$. We have

$$\mathbf{M} + 2\mathbf{I} = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

Expanding along the top row:

$$\begin{aligned} \det(\mathbf{M} + 2\mathbf{I}) &= 3 \times (4 \times 2 - 0 \times 0) + 2 \times \big((-2) \times 2 - 0 \times 2\big) + 2 \times \big((-2) \times 0 - 4 \times 2\big) \\ &= 24 - 8 - 16 \\ &= 0 \end{aligned}$$

so -2 is an eigenvalue of $\mathbf{M}$.

For an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, solve $(\mathbf{M} + 2\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$:

$$\begin{aligned} 3x - 2y + 2z &= 0 \\ -2x + 4y &= 0 \\ 2x + 2z &= 0 \end{aligned}$$

The second equation gives $x = 2y$ and the third gives $z = -x$. Taking $y = 1$ gives $x = 2$, $z = -2$, which also satisfies the first equation since $6 - 2 - 4 = 0$. A corresponding eigenvector is

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

(b) The jth column of $\mathbf{MU}$ is $\mathbf{Mu}_j$, where $\mathbf{u}_j$ is the jth column of $\mathbf{U}$, and the jth column of $\mathbf{UD}$ is $d_j\mathbf{u}_j$, where d_j is the jth diagonal entry of $\mathbf{D}$. So the diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{M}$, in the same order as the corresponding eigenvectors in $\mathbf{U}$.

The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$:

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\big((2-\lambda)(-\lambda) - 0\big) + 2\big((-2)(-\lambda) - 0\big) + 2\big(0 - 2(2-\lambda)\big) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda) + 4\lambda - 8 + 4\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda + 8\lambda - 8 = 0$$

$$\lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$$

From part (a), $\lambda = -2$ is a root, so $(\lambda + 2)$ is a factor:

$$(\lambda+2)(\lambda^2 - 5\lambda + 4) = 0 \quad \Longrightarrow \quad (\lambda+2)(\lambda-1)(\lambda-4) = 0$$

so the eigenvalues are $\lambda = -2,\ 1,\ 4$.

$\lambda = 1$**:** solve $(\mathbf{M} - \mathbf{I})\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$:

$$-2y + 2z = 0$$

$$-2x + y = 0$$

$$2x - z = 0$$

So $y = z$ and $y = 2x$. Taking $x = 1$ gives the eigenvector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

$\lambda = 4$**:** solve $(\mathbf{M} - 4\mathbf{I})\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$:

$$-3x - 2y + 2z = 0$$

$$-2x - 2y = 0$$

$$2x - 4z = 0$$

So $y = -x$ and $x = 2z$. Taking $z = 1$ gives $x = 2$, $y = -2$, which satisfies the first equation since $-6 + 4 + 2 = 0$. The eigenvector is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Each of the three eigenvectors has length 3, for example

$$\left| \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \right| = \sqrt{4 + 1 + 4} = 3$$

and similarly $\sqrt{1 + 4 + 4} = 3$ and $\sqrt{4 + 4 + 1} = 3$. Dividing each by 3 gives unit eigenvectors, so possible matrices are

$$\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad \mathbf{U} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{pmatrix}$$

The columns of $\mathbf{U}$ may be listed in any order, provided the diagonal entries of $\mathbf{D}$ are reordered to match, and any column may be multiplied by -1.

7. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{8} & \frac{5}{16} \\ \frac{5}{4} & \frac{3}{8} \end{pmatrix}$$

(a) Find the eigenvalues of $\mathbf{A}$ and a corresponding eigenvector for each eigenvalue. **[5]**

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. **[1]**

(c) Find $\mathbf{A}^n$, where n is a positive integer, as a single matrix whose entries are expressed in terms of n. Hence find the matrix $\mathbf{L}$ such that $\mathbf{A}^n \to \mathbf{L}$ as $n \to \infty$, explaining briefly why this limit exists. **[5]**

(d) The transformation represented by $\mathbf{L}$ maps every point of the plane onto a single line through the origin.
Find the equation of this line. **[2]**

Solution

(a) The eigenvalues satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$:

$$\left(\frac{3}{8} - \lambda\right)^2 - \frac{5}{16} \times \frac{5}{4} = 0$$

$$\lambda^2 - \frac{3}{4}\lambda + \frac{9}{64} - \frac{25}{64} = 0$$

$$4\lambda^2 - 3\lambda - 1 = 0$$

$$(4\lambda + 1)(\lambda - 1) = 0$$

so the eigenvalues are $\lambda = 1$ and $\lambda = -\frac{1}{4}$.

For $\lambda = 1$, $(\mathbf{A} - \mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$ gives

$$-\frac{5}{8}x + \frac{5}{16}y = 0 \quad \Longrightarrow \quad y = 2x$$

so an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda = -\frac{1}{4}$, $\left(\mathbf{A} + \frac{1}{4}\mathbf{I}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$ gives

$$\frac{5}{8}x + \frac{5}{16}y = 0 \quad \Longrightarrow \quad y = -2x$$

so an eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(b) Using the eigenvectors as columns, in the same order as the eigenvalues on the diagonal:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$$

(c) Since $\mathbf{A} = \mathbf{PDP}^{-1}$, in the product $\mathbf{A}^n = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1})$ each inner $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, so

$$\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}, \qquad \mathbf{D}^n = \begin{pmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{4}\right)^n \end{pmatrix}$$

Using

$$\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\,\mathrm{adj}(\mathbf{P})$$

where $\mathrm{adj}(\mathbf{P})$ is the transposed matrix of cofactors, with $\det(\mathbf{P}) = -2 - 2 = -4$:

$$\mathbf{P}^{-1} = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

Then

$$\mathbf{PD}^n = \begin{pmatrix} 1 & \left(-\frac{1}{4}\right)^n \\ 2 & -2\left(-\frac{1}{4}\right)^n \end{pmatrix}$$

so

$$\mathbf{A}^n = \begin{pmatrix} 1 & \left(-\frac{1}{4}\right)^n \\ 2 & -2\left(-\frac{1}{4}\right)^n \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{4}\right)^n & \frac{1}{4} - \frac{1}{4}\left(-\frac{1}{4}\right)^n \\ 1 - \left(-\frac{1}{4}\right)^n & \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{4}\right)^n \end{pmatrix}$$

Since $\left|-\frac{1}{4}\right| < 1$, $\left(-\frac{1}{4}\right)^n \to 0$ as $n \to \infty$, so every entry of $\mathbf{A}^n$ tends to a finite limit and the limit exists:

$$\mathbf{L} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} \end{pmatrix}$$

(d) The image of the point (x, y) under $\mathbf{L}$ is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{4}y \\ x + \frac{1}{2}y \end{pmatrix} = \left(\frac{1}{2}x + \frac{1}{4}y\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The second coordinate of every image point is twice the first, so the line is

$$y = 2x$$

8. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -4 & -13 & 8 \\ -6 & -7 & 6 \\ -12 & -22 & 16 \end{pmatrix}$$

(a) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$ and hence find the eigenvalues of $\mathbf{A}$. **[4]**

(b) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. **[5]**

(c) Hence show that, for all positive integers n, the element in the first row and first column of $\mathbf{A}^n$ is $2(-1)^n - 2^n$. **[3]**

Solution

(a) The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -4-\lambda & -13 & 8 \\ -6 & -7-\lambda & 6 \\ -12 & -22 & 16-\lambda \end{vmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (-4-\lambda)\big[(-7-\lambda)(16-\lambda) + 132\big] + 13\big[-6(16-\lambda) + 72\big] + 8\big[132 - 12(7+\lambda)\big] \\ &= (-4-\lambda)(\lambda^2 - 9\lambda + 20) + 13(6\lambda - 24) + 8(48 - 12\lambda) \\ &= (-\lambda^3 + 5\lambda^2 + 16\lambda - 80) + (78\lambda - 312) + (384 - 96\lambda) \\ &= -\lambda^3 + 5\lambda^2 - 2\lambda - 8 \end{aligned}$$

Setting this equal to zero and multiplying by -1, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$$

as required.

Substituting $\lambda = -1$ gives $-1 - 5 - 2 + 8 = 0$, so $(\lambda + 1)$ is a factor:

$$(\lambda + 1)(\lambda^2 - 6\lambda + 8) = (\lambda + 1)(\lambda - 2)(\lambda - 4) = 0$$

The eigenvalues are therefore

$$\lambda = -1, \qquad \lambda = 2, \qquad \lambda = 4$$

(b) For each eigenvalue, solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ with $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$\lambda = -1$:

$$\begin{pmatrix} -3 & -13 & 8 \\ -6 & -6 & 6 \\ -12 & -22 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

The second row gives $-6x - 6y + 6z = 0$, so $z = x + y$. Substituting into the first row, $-3x - 13y + 8(x + y) = 5x - 5y = 0$, so $y = x$ and $z = 2x$. Taking $x = 1$, an eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

$\lambda = 2$:

$$\begin{pmatrix} -6 & -13 & 8 \\ -6 & -9 & 6 \\ -12 & -22 & 14 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

Subtracting the second equation from the first gives $-4y + 2z = 0$, so $z = 2y$. The second row then gives $-6x - 9y + 12y = 0$, so $y = 2x$ and $z = 4x$. Taking $x = 1$, an eigenvector is $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

$\lambda = 4$:

$$\begin{pmatrix} -8 & -13 & 8 \\ -6 & -11 & 6 \\ -12 & -22 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

Taking $3\times$ the first equation minus $4\times$ the second gives $-39y + 44y = 5y = 0$, so $y = 0$. The second row then gives $-6x + 6z = 0$, so $z = x$. Taking $x = 1$, an eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Using the eigenvectors as the columns of $\mathbf{P}$, in the same order as the eigenvalues on the diagonal of $\mathbf{D}$:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(c) Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, the inner factors cancel in

$$\mathbf{A}^n = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) \cdots (\mathbf{PDP}^{-1}) = \mathbf{PD}^n\mathbf{P}^{-1}$$

where $\mathbf{D}^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$. Here $\det(\mathbf{P}) = 1$ and

$$\mathbf{P}^{-1} = \begin{pmatrix} 2 & 3 & -2 \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

The first row of $\mathbf{PD}^n$ is $\begin{pmatrix} (-1)^n & 2^n & 4^n \end{pmatrix}$, and the first column of $\mathbf{P}^{-1}$ is $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$. So the element in the first row and first column of $\mathbf{A}^n$ is

$$(-1)^n \times 2 + 2^n \times (-1) + 4^n \times 0 = 2(-1)^n - 2^n$$

as required.

9. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & -4 \\ 1 & 3 & -1 \\ 1 & 4 & -2 \end{pmatrix}$$

(a) Show that the characteristic equation of $\mathbf{M}$ is

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \qquad \textbf{[3]}$$

(b) Hence find the three eigenvalues of $\mathbf{M}$. **[2]**

(c) For each eigenvalue of $\mathbf{M}$ find an associated eigenvector. **[4]**

(d) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$. Hence find $\mathbf{M}^n$ as a single matrix, where n is a positive integer, giving each entry in terms of n. **[4]**

Solution

(a) The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, where

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 & -4 \\ 1 & 3-\lambda & -1 \\ 1 & 4 & -2-\lambda \end{vmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= (3-\lambda)\big[(3-\lambda)(-2-\lambda) + 4\big] - 4\big[(-2-\lambda) + 1\big] - 4\big[4 - (3-\lambda)\big] \\ &= (3-\lambda)(\lambda^2 - \lambda - 2) - 4(-1-\lambda) - 4(1+\lambda) \\ &= (3-\lambda)(\lambda^2 - \lambda - 2) + 4 + 4\lambda - 4 - 4\lambda \\ &= 3\lambda^2 - 3\lambda - 6 - \lambda^3 + \lambda^2 + 2\lambda \\ &= -\lambda^3 + 4\lambda^2 - \lambda - 6 \end{aligned}$$

Setting this equal to zero and multiplying by -1 gives

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

as required.

(b) Let $f(\lambda) = \lambda^3 - 4\lambda^2 + \lambda + 6$. Then

$$f(-1) = -1 - 4 - 1 + 6 = 0$$

so, by the factor theorem, $(\lambda + 1)$ is a factor. Factorising:

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda+1)(\lambda^2 - 5\lambda + 6) = (\lambda+1)(\lambda-2)(\lambda-3)$$

Hence the eigenvalues are $\lambda = -1$, $\lambda = 2$ and $\lambda = 3$.

(c) Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and solve $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ for each eigenvalue.

$\lambda = -1$:

$$\begin{pmatrix} 4 & 4 & -4 \\ 1 & 4 & -1 \\ 1 & 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x + y - z = 0, \qquad x + 4y - z = 0$$

Subtracting gives $3y = 0$, so $y = 0$ and $x = z$. An eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda = 2$:

$$\begin{pmatrix} 1 & 4 & -4 \\ 1 & 1 & -1 \\ 1 & 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x + 4y - 4z = 0, \qquad x + y - z = 0$$

Subtracting gives $3y - 3z = 0$, so $y = z$ and then $x = 0$. An eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$\lambda = 3$:

$$\begin{pmatrix} 0 & 4 & -4 \\ 1 & 0 & -1 \\ 1 & 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad 4y - 4z = 0, \qquad x - z = 0, \qquad x + 4y - 5z = 0$$

So $y = z$ and $x = z$, which also satisfy the third equation. An eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Any non-zero multiple of each of these vectors is also an eigenvector.

(d) Taking the eigenvectors as the columns of $\mathbf{P}$, with the eigenvalues in the same order on the diagonal of $\mathbf{D}$:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $\det(\mathbf{P}) = 1 \times (1 - 1) - 0 + 1 \times (0 - 1) = -1 \neq 0$, and

$$\mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, we have

$$\mathbf{M}^n = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) \cdots (\mathbf{PDP}^{-1}) = \mathbf{PD}^n\mathbf{P}^{-1}, \qquad \mathbf{D}^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

First calculate:

$$\mathbf{PD}^n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} = \begin{pmatrix} (-1)^n & 0 & 3^n \\ 0 & 2^n & 3^n \\ (-1)^n & 2^n & 3^n \end{pmatrix}$$

Then

$$\begin{aligned}\mathbf{M}^n &= \begin{pmatrix}(-1)^n & 0 & 3^n\\ 0 & 2^n & 3^n\\ (-1)^n & 2^n & 3^n\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ -1 & 0 & 1\\ 1 & 1 & -1\end{pmatrix}\\ &= \begin{pmatrix}3^n & 3^n-(-1)^n & (-1)^n-3^n\\ 3^n-2^n & 3^n & 2^n-3^n\\ 3^n-2^n & 3^n-(-1)^n & (-1)^n+2^n-3^n\end{pmatrix}\end{aligned}$$

10. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} -1 & -1 & 2 \\ -3 & -1 & 6 \\ k & 1 & 0 \end{pmatrix}$$

where k is a constant.

(a) Find the characteristic equation of $\mathbf{M}$ in terms of k, and show that 2 is an eigenvalue of $\mathbf{M}$ for every value of k. **[3]**

(b) Given that $\mathbf{M}$ has a repeated eigenvalue, find the possible values of k and, for each value of k, all the eigenvalues of $\mathbf{M}$. **[5]**

Solution

(a) The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, where

$$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} -1-\lambda & -1 & 2 \\ -3 & -1-\lambda & 6 \\ k & 1 & -\lambda \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= (-1-\lambda)\big[(-1-\lambda)(-\lambda) - 6\big] + 1 \times \big[(-3)(-\lambda) - 6k\big] + 2\big[(-3) \times 1 - k(-1-\lambda)\big] \\ &= -(1+\lambda)(\lambda^2 + \lambda - 6) + 3\lambda - 6k - 6 + 2k + 2k\lambda \\ &= -\lambda^3 - 2\lambda^2 + 5\lambda + 6 + 3\lambda - 6k - 6 + 2k + 2k\lambda \\ &= -\lambda^3 - 2\lambda^2 + (2k+8)\lambda - 4k \end{aligned}$$

Setting this equal to zero and multiplying by -1, the characteristic equation is

$$\lambda^3 + 2\lambda^2 - (2k+8)\lambda + 4k = 0$$

Substituting $\lambda = 2$:

$$2^3 + 2 \times 2^2 - (2k+8) \times 2 + 4k = 8 + 8 - 4k - 16 + 4k = 0$$

This holds for every value of k, so 2 is an eigenvalue of $\mathbf{M}$ for every value of k.

(b) Since $\lambda = 2$ is a root, $\lambda - 2$ is a factor of the characteristic equation, and equating coefficients gives

$$(\lambda - 2)(\lambda^2 + 4\lambda - 2k) = 0$$

so the other two eigenvalues are the roots of $\lambda^2 + 4\lambda - 2k = 0$. An eigenvalue is repeated either when this quadratic has a repeated root or when 2 is also a root of the quadratic.

Case 1: the quadratic has a repeated root, so its discriminant is zero:

$$b^2 - 4ac = 4^2 - 4 \times 1 \times (-2k) = 16 + 8k = 0 \quad \Longrightarrow \quad k = -2$$

Then the quadratic is

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \quad \Longrightarrow \quad \lambda = -2$$

so when $k = -2$ the eigenvalues are 2, -2 and -2.

Case 2: $\lambda = 2$ is a root of the quadratic:

$$2^2 + 4 \times 2 - 2k = 0 \quad \Longrightarrow \quad 12 - 2k = 0 \quad \Longrightarrow \quad k = 6$$

Then the quadratic is

$$\lambda^2 + 4\lambda - 12 = (\lambda - 2)(\lambda + 6) = 0 \quad \Longrightarrow \quad \lambda = 2 \text{ or } \lambda = -6$$

so when $k = 6$ the eigenvalues are 2, 2 and -6.

11. The square matrix $\mathbf{A}$ has an eigenvalue λ with corresponding eigenvector $\mathbf{e}$. The non-singular matrix $\mathbf{M}$ is of the same order as $\mathbf{A}$, and the matrix $\mathbf{B}$ is defined by $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$.

(a) Show that $\mathbf{M}^{-1}\mathbf{e}$ is an eigenvector of $\mathbf{B}$, and that λ is the corresponding eigenvalue. **[3]**

Let

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 5 & 1 \end{pmatrix}$$

(b) Write down the eigenvalues of $\mathbf{A}$ and obtain corresponding eigenvectors. **[4]**

It is given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

(c) Find $\mathbf{M}^{-1}$ and hence, without finding $\mathbf{B}$ explicitly, find the eigenvalues and corresponding eigenvectors of $\mathbf{B}$. **[5]**

Solution

(a) Since $\mathbf{e}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ with $\mathbf{e} \neq \mathbf{0}$. Then

$$\begin{aligned} \mathbf{B}\left(\mathbf{M}^{-1}\mathbf{e}\right) &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{M}^{-1}\mathbf{e} \\ &= \mathbf{M}^{-1}\mathbf{A}\mathbf{I}\mathbf{e} \\ &= \mathbf{M}^{-1}(\lambda\mathbf{e}) \\ &= \lambda\left(\mathbf{M}^{-1}\mathbf{e}\right) \end{aligned}$$

Also $\mathbf{M}^{-1}\mathbf{e} \neq \mathbf{0}$: if $\mathbf{M}^{-1}\mathbf{e} = \mathbf{0}$, then multiplying on the left by $\mathbf{M}$ would give $\mathbf{e} = \mathbf{M}\mathbf{0} = \mathbf{0}$, which is false. Hence $\mathbf{M}^{-1}\mathbf{e}$ is an eigenvector of $\mathbf{B}$ with corresponding eigenvalue λ, as required.

(b) $\mathbf{A}$ is lower triangular, so

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (3 - \lambda)(-2 - \lambda)(1 - \lambda)$$

and the eigenvalues are the diagonal entries, $\lambda = 3,\ -2,\ 1$.

For an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, solve $(\mathbf{A} - \lambda\mathbf{I})\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$.

$\lambda = 3$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -5 & 0 \\ 2 & 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x = 5y, \qquad 2x + 5y - 2z = 0$$

Taking $y = 2$ gives $x = 10$ and $2z = 20 + 10 = 30$, so $z = 15$. An eigenvector is $\begin{pmatrix} 10 \\ 2 \\ 15 \end{pmatrix}$.

$\lambda = -2$:

$$\begin{pmatrix} 5 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x = 0, \qquad 5y + 3z = 0$$

Taking $y = 3$ gives $z = -5$. An eigenvector is $\begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$.

$\lambda = 1$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 2 & 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x = 0, \qquad y = 0$$

with z free. An eigenvector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Any non-zero multiples of these vectors are also eigenvectors.

(c) Expanding along the first column:

$$\det(\mathbf{M}) = 1 \times (1 \times 1 - 0 \times 2) = 1$$

Using

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \operatorname{adj}(\mathbf{M})$$

where $\operatorname{adj}(\mathbf{M})$ is the transposed matrix of cofactors. The matrix of cofactors is

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing:

$$\mathbf{M}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

By part (a), each eigenvalue λ of $\mathbf{A}$ with eigenvector $\mathbf{e}$ is an eigenvalue of $\mathbf{B}$ with eigenvector $\mathbf{M}^{-1}\mathbf{e}$. Since $\mathbf{B}$ is 3×3 it has at most three eigenvalues, so the eigenvalues of $\mathbf{B}$ are $3,\ -2,\ 1$.

$\lambda = 3$:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \\ 15 \end{pmatrix} = \begin{pmatrix} 10 - 2 \\ 2 \\ -4 + 15 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 11 \end{pmatrix}$$

$\lambda = -2$:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -6 - 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -11 \end{pmatrix}$$

$\lambda = 1$:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So $\mathbf{B}$ has eigenvalue 3 with eigenvector $\begin{pmatrix} 8 \\ 2 \\ 11 \end{pmatrix}$, eigenvalue -2 with eigenvector $\begin{pmatrix} -3 \\ 3 \\ -11 \end{pmatrix}$, and eigenvalue 1 with eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, or any non-zero multiples of these.

12. It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

(a) Show that λ^2 is an eigenvalue of $\mathbf{A}^2$ and that λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$, and that $\mathbf{e}$ is a corresponding eigenvector in each case. **[3]**

(b) Deduce that $\lambda^2 + 2\lambda^{-1}$ is an eigenvalue of $\mathbf{A}^2 + 2\mathbf{A}^{-1}$. **[1]**

It is given that -1 is an eigenvalue of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix}$$

(c) Find a corresponding eigenvector. **[2]**

(d) It is also given that

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of the matrix $\mathbf{A}$. Find the corresponding eigenvalues. **[2]**

(e) Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\left(\mathbf{A}^2 + 2\mathbf{A}^{-1}\right)^2 = \mathbf{PDP}^{-1}$$ **[4]**

Solution

(a) Since $\mathbf{e}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, we have $\mathbf{Ae} = \lambda\mathbf{e}$ with $\mathbf{e} \neq \mathbf{0}$. Then

$$\mathbf{A}^2\mathbf{e} = \mathbf{A}(\mathbf{Ae}) = \mathbf{A}(\lambda\mathbf{e}) = \lambda\mathbf{Ae} = \lambda^2\mathbf{e}$$

so λ^2 is an eigenvalue of $\mathbf{A}^2$ with corresponding eigenvector $\mathbf{e}$.

If $\lambda = 0$, then $\mathbf{Ae} = \mathbf{0}$, so $\mathbf{e} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$, a contradiction. Hence $\lambda \neq 0$. Premultiplying $\mathbf{Ae} = \lambda\mathbf{e}$ by $\mathbf{A}^{-1}$:

$$\mathbf{e} = \lambda\mathbf{A}^{-1}\mathbf{e} \quad \Longrightarrow \quad \mathbf{A}^{-1}\mathbf{e} = \lambda^{-1}\mathbf{e}$$

so λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ with corresponding eigenvector $\mathbf{e}$.

(b) Using part (a):

$$\left(\mathbf{A}^2 + 2\mathbf{A}^{-1}\right)\mathbf{e} = \mathbf{A}^2\mathbf{e} + 2\mathbf{A}^{-1}\mathbf{e} = \lambda^2\mathbf{e} + 2\lambda^{-1}\mathbf{e} = \left(\lambda^2 + 2\lambda^{-1}\right)\mathbf{e}$$

so $\lambda^2 + 2\lambda^{-1}$ is an eigenvalue of $\mathbf{A}^2 + 2\mathbf{A}^{-1}$, as required.

(c) Let the eigenvector be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. It satisfies $(\mathbf{A} + \mathbf{I})\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$, that is

$$\begin{pmatrix} 3 & 1 & -1 \\ 3 & 3 & -3 \\ 3 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives $3x + y - z = 0$ and $3x + 3y - 3z = 0$, i.e. $x + y - z = 0$. Subtracting gives $2x = 0$, so $x = 0$ and $y = z$. A corresponding eigenvector is

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(d) Multiplying each eigenvector by $\mathbf{A}$:

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The corresponding eigenvalues are 1 and 2 respectively.

(e) Let $\mathbf{B} = \mathbf{A}^2 + 2\mathbf{A}^{-1}$. By part (b), each eigenvector of $\mathbf{A}$ with eigenvalue λ is an eigenvector of $\mathbf{B}$ with eigenvalue $\mu = \lambda^2 + 2\lambda^{-1}$:

$$\lambda = -1: \quad \mu = 1 - 2 = -1$$

$$\lambda = 1: \quad \mu = 1 + 2 = 3$$

$$\lambda = 2: \quad \mu = 4 + 1 = 5$$

If $\mathbf{Be} = \mu\mathbf{e}$, then, exactly as in part (a), $\mathbf{B}^2\mathbf{e} = \mathbf{B}(\mu\mathbf{e}) = \mu\mathbf{Be} = \mu^2\mathbf{e}$. So $\mathbf{B}^2$ has the same eigenvectors with eigenvalues $(-1)^2 = 1$, $3^2 = 9$ and $5^2 = 25$.

Taking the eigenvectors as the columns of $\mathbf{P}$ and the corresponding eigenvalues, in the same order, on the diagonal of $\mathbf{D}$:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

Since $\det(\mathbf{P}) = 0\times(0-1) - 1\times(1-1) + 1\times(1-0) = 1 \neq 0$, $\mathbf{P}$ is non-singular, so $\left(\mathbf{A}^2 + 2\mathbf{A}^{-1}\right)^2 = \mathbf{PDP}^{-1}$ with these matrices.

13. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

(a) (i) Find the characteristic equation of $\mathbf{M}$, giving your answer in the form $\lambda^3 + p\lambda^2 + q\lambda + r = 0$, where p, q and r are integers. Hence show that 1 is an eigenvalue of $\mathbf{M}$. **[3]**

(ii) Find the other two eigenvalues of $\mathbf{M}$, giving each in the form $x + yi$ where x and y are real. Deduce that 1 is the only real eigenvalue of $\mathbf{M}$. **[3]**

(b) A student claims that $\mathbf{M}$ can be written in the form $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and **both** $\mathbf{P}$ and $\mathbf{D}$ have only real entries. Using your answer to part (a)(ii), explain why the student is wrong. **[2]**

You are now given the following information.

- $\mathbf{M}$ represents a rotation in 3-D space by an acute angle about an axis through the origin.
- $\mathbf{e}$ is a real eigenvector of $\mathbf{M}$ associated with the eigenvalue 1.

(c) (i) Find a possible vector $\mathbf{e}$. **[2]**

(ii) Explain the significance of $\mathbf{e}$, and the fact that its associated eigenvalue is 1, in relation to the rotation that $\mathbf{M}$ represents. Hence write down a vector equation for the axis of this rotation. **[2]**

(iii) Explain **geometrically** why $\mathbf{M}$ has no real eigenvectors other than non-zero multiples of $\mathbf{e}$. **[1]**

Solution

(a) (i) Since $\mathbf{M} - \lambda\mathbf{I} = \frac{1}{3}(\mathbf{A} - 3\lambda\mathbf{I})$, where $\mathbf{A} = \begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$, and the matrices are 3×3, we have

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \frac{1}{27}\det\begin{pmatrix} 2-3\lambda & -1 & 2 \\ 2 & 2-3\lambda & -1 \\ -1 & 2 & 2-3\lambda \end{pmatrix}$$

Expanding the determinant along the first row:

$$\begin{aligned} &(2-3\lambda)\left[(2-3\lambda)^2 + 2\right] + [2(2-3\lambda) - 1] + 2[4 + (2-3\lambda)] \\ &= (2-3\lambda)^3 + 6(2-3\lambda) + 7 \\ &= 8 - 36\lambda + 54\lambda^2 - 27\lambda^3 + 12 - 18\lambda + 7 \\ &= -27\lambda^3 + 54\lambda^2 - 54\lambda + 27 \end{aligned}$$

So $\det(\mathbf{M} - \lambda\mathbf{I}) = -\lambda^3 + 2\lambda^2 - 2\lambda + 1$, and the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ is

$$\lambda^3 - 2\lambda^2 + 2\lambda - 1 = 0$$

with $p = -2$, $q = 2$, $r = -1$.

Substituting $\lambda = 1$:

$$1 - 2 + 2 - 1 = 0$$

so $\lambda = 1$ satisfies the characteristic equation and 1 is an eigenvalue of $\mathbf{M}$.

(ii) Since $\lambda - 1$ is a factor:

$$\lambda^3 - 2\lambda^2 + 2\lambda - 1 = (\lambda - 1)(\lambda^2 - \lambda + 1)$$

The other eigenvalues satisfy $\lambda^2 - \lambda + 1 = 0$, so

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

The discriminant of $\lambda^2 - \lambda + 1$ is $b^2 - 4ac = 1 - 4 = -3 < 0$, so this quadratic has no real roots. Hence the only real root of the characteristic equation, and so the only real eigenvalue of $\mathbf{M}$, is 1.

(b) If $\mathbf{M} = \mathbf{PDP}^{-1}$, then

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \det\big(\mathbf{P}(\mathbf{D} - \lambda\mathbf{I})\mathbf{P}^{-1}\big) = \det(\mathbf{P})\det(\mathbf{D} - \lambda\mathbf{I})\det(\mathbf{P}^{-1}) = \det(\mathbf{D} - \lambda\mathbf{I})$$

so $\mathbf{M}$ and $\mathbf{D}$ have the same eigenvalues. The eigenvalues of the diagonal matrix $\mathbf{D}$ are its diagonal entries, so these would have to be 1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. By part (a)(ii), two of these are not real, so $\mathbf{D}$ cannot have only real entries, and the student is wrong.

(c) (i) Let $\mathbf{e} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfy $(\mathbf{M} - \mathbf{I})\mathbf{e} = \mathbf{0}$, that is

$$\frac{1}{3}\begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two rows give $-x - y + 2z = 0$ and $2x - y - z = 0$. From the second, $y = 2x - z$; substituting into the first:

$$-x - (2x - z) + 2z = 0 \quad \Longrightarrow \quad x = z \quad \Longrightarrow \quad y = x$$

These also satisfy the third row, $-x + 2y - z = 0$. Taking $x = 1$:

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) Since $\mathbf{Me} = 1\,\mathbf{e} = \mathbf{e}$, the vector $\mathbf{e}$, and every multiple $t\mathbf{e}$, is mapped to itself: it is neither turned nor stretched nor reversed. So every point on the line through the origin in the direction of $\mathbf{e}$ is invariant under the rotation. The points left fixed by a rotation are exactly those on its axis, so this line is the axis of rotation:

$$\mathbf{r} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(iii) A real eigenvector $\mathbf{v}$ satisfies $\mathbf{Mv} = k\mathbf{v}$ for some real k, so it is mapped to a vector parallel to itself. A rotation through an acute angle (so not 0° or 180°) turns every vector not parallel to the axis into a vector in a different direction, not parallel to the original. So the only vectors mapped to multiples of themselves lie along the axis, which means the only real eigenvectors are non-zero multiples of $\mathbf{e}$.

14. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

(a) Obtain the characteristic equation of $\mathbf{M}$. Hence, or otherwise, find the value of $\det(\mathbf{M})$. **[3]**

(b) Find the real eigenvalue of $\mathbf{M}$, and show that the other two eigenvalues of $\mathbf{M}$ are not real.
Find an eigenvector of $\mathbf{M}$ corresponding to its real eigenvalue.
Hence, or otherwise, write down the solution of the following system of equations, and explain why the system has no other solution.

$$\begin{aligned} x \quad - \quad z &= 0.4 \\ 3x + y \quad &= -0.4 \\ x + y - 2z &= 1.2 \end{aligned}$$

[10]

Solution

(a) The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, where

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= (1-\lambda)\big[(1-\lambda)(-2-\lambda) - 0\big] - 0 + (-1)\big[3 \times 1 - (1-\lambda) \times 1\big] \\ &= -(1-\lambda)^2(2+\lambda) - (2+\lambda) \\ &= -(1 - 2\lambda + \lambda^2)(2+\lambda) - 2 - \lambda \\ &= -\big(2 - 3\lambda + \lambda^3\big) - 2 - \lambda \\ &= -\lambda^3 + 2\lambda - 4 \end{aligned}$$

So the characteristic equation is

$$\lambda^3 - 2\lambda + 4 = 0$$

Setting $\lambda = 0$ in $\det(\mathbf{M} - \lambda\mathbf{I}) = -\lambda^3 + 2\lambda - 4$ gives

$$\det(\mathbf{M}) = -4$$

(b) Testing $\lambda = -2$ in the characteristic equation:

$$(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0$$

so $\lambda + 2$ is a factor, and

$$\lambda^3 - 2\lambda + 4 = (\lambda + 2)\big(\lambda^2 - 2\lambda + 2\big)$$

The quadratic factor has discriminant

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 2 = -4 < 0$$

so $\lambda^2 - 2\lambda + 2 = 0$ has no real roots, and the other two eigenvalues of $\mathbf{M}$ are not real. The real eigenvalue is $\lambda = -2$.

For an eigenvector with components a, b, c and $\lambda = -2$, solve

$$\begin{pmatrix} 3 & 0 & -1 \\ 3 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$3a - c = 0, \qquad a + b = 0$$

Taking $a = 1$ gives $b = -1$ and $c = 3$, so an eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

The system of equations is

$$\mathbf{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.4 \\ -0.4 \\ 1.2 \end{pmatrix} = 0.4 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0.4\,\mathbf{v}$$

Since $\mathbf{M}\mathbf{v} = -2\mathbf{v}$, multiplying by -0.2 gives

$$\mathbf{M}(-0.2\,\mathbf{v}) = -0.2 \times (-2)\,\mathbf{v} = 0.4\,\mathbf{v}$$

so a solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -0.2 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.2 \\ -0.6 \end{pmatrix}$$

that is, $x = -0.2$, $y = 0.2$, $z = -0.6$.

From part (a), $\det(\mathbf{M}) = -4 \neq 0$, so $\mathbf{M}$ is non-singular and $\mathbf{M}^{-1}$ exists. Multiplying the system by $\mathbf{M}^{-1}$ gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} 0.4 \\ -0.4 \\ 1.2 \end{pmatrix}$$

This is a single vector, so the system has no other solution.

15.

$$\mathbf{A} = \begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & p \\ 10 & -5 & 3 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$,

(a) (i) determine the eigenvalue corresponding to this eigenvector, **[1]**

(ii) hence show that $p = -4$, **[2]**

(iii) determine the remaining eigenvalues and corresponding eigenvectors of $\mathbf{A}$. **[7]**

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[1]**

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant. **[2]**
With respect to a fixed origin O, the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

By considering

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

(ii) determine a general solution for the displacement of the particle. **[4]**

At time $t = 0$ the particle is at the point with position vector $\begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$.

(iii) Find expressions for x, y and z in terms of t for this particle. **[2]**

Solution

(a) (i)

$$\mathbf{A}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3+4+2 \\ -2+4-p \\ 10-10-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2-p \\ -3 \end{pmatrix}$$

The first and third components are 3×1 and $3 \times (-1)$, so the eigenvalue is 3.

(ii) Since the vector is an eigenvector with eigenvalue 3, the second component must equal 3×2:

$$2 - p = 6 \quad \Longrightarrow \quad p = -4$$

as required.

(iii) With $p = -4$, the matrix is

$$\mathbf{A} = \begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & -4 \\ 10 & -5 & 3 \end{pmatrix}$$

The eigenvalues satisfy $\det(\mathbf{A}-\lambda\mathbf{I})=0$. Expanding along the first row:

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= (-3-\lambda)\big[(2-\lambda)(3-\lambda)-20\big]-2\big[-2(3-\lambda)+40\big]-2\big[10-10(2-\lambda)\big]\\ &= -(\lambda+3)(\lambda^2-5\lambda-14)-2(2\lambda+34)-2(10\lambda-10)\\ &= -\lambda^3+2\lambda^2+29\lambda+42-4\lambda-68-20\lambda+20\\ &= -\lambda^3+2\lambda^2+5\lambda-6\end{aligned}$$

So the characteristic equation is

$$\lambda^3-2\lambda^2-5\lambda+6=0$$

Since $\lambda=3$ is a root, $(\lambda-3)$ is a factor:

$$(\lambda-3)(\lambda^2+\lambda-2)=0 \quad\Longrightarrow\quad (\lambda-3)(\lambda-1)(\lambda+2)=0$$

so the remaining eigenvalues are $\lambda=1$ and $\lambda=-2$.

$\lambda=1$**:** solve $(\mathbf{A}-\mathbf{I})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\mathbf{0}$, that is

$$\begin{pmatrix}-4 & 2 & -2\\ -2 & 1 & -4\\ 10 & -5 & 2\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

The first two rows give

$$-2\alpha+\beta-\gamma=0$$

$$-2\alpha+\beta-4\gamma=0$$

Subtracting gives $3\gamma=0$, so $\gamma=0$ and $\beta=2\alpha$. The third row, $10\alpha-5\beta+2\gamma=10\alpha-10\alpha=0$, is then satisfied. Taking $\alpha=1$, an eigenvector is

$$\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$$

$\lambda=-2$**:** solve $(\mathbf{A}+2\mathbf{I})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\mathbf{0}$, that is

$$\begin{pmatrix}-1 & 2 & -2\\ -2 & 4 & -4\\ 10 & -5 & 5\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

The first and third rows give

$$-\alpha+2\beta-2\gamma=0$$

$$2\alpha-\beta+\gamma=0$$

From the first, $\alpha=2\beta-2\gamma$. Substituting into the second gives $4\beta-4\gamma-\beta+\gamma=3\beta-3\gamma=0$, so $\beta=\gamma$ and $\alpha=0$. The second row is a multiple of the first, so is also satisfied. Taking $\beta=1$, an eigenvector is

$$\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$$

(b) Using the eigenvectors as columns, in the same order as their eigenvalues on the diagonal:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(c) (i) Separating the variables:

$$\int \frac{1}{u}\,\mathrm{d}u = \int k\,\mathrm{d}t$$
$$\ln|u| = kt + c$$
$$u = Ke^{kt}$$

where K is an arbitrary constant.

(ii) Using the given relations and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{P}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{P}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{D}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

So the equations separate as

$$\dot{u} = 3u, \qquad \dot{v} = v, \qquad \dot{w} = -2w$$

By part (c)(i), we have

$$u = C_1e^{3t}, \qquad v = C_2e^{t}, \qquad w = C_3e^{-2t}$$

where C_1, C_2, C_3 are arbitrary constants. Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_1e^{3t} \\ C_2e^{t} \\ C_3e^{-2t} \end{pmatrix}$$
$$= C_1e^{3t}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + C_2e^{t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_3e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(iii) When $t = 0$:

$$C_1 + C_2 = 3$$
$$2C_1 + 2C_2 + C_3 = 7$$
$$-C_1 + C_3 = -1$$

The first two equations give $C_3 = 7 - 6 = 1$, then the third gives $C_1 = 2$, and so $C_2 = 1$. Hence

$$x = 2e^{3t} + e^{t}, \qquad y = 4e^{3t} + 2e^{t} + e^{-2t}, \qquad z = -2e^{3t} + e^{-2t}$$