OPTIONAL PURE / VECTORS & MATRICES

Cayley-Hamilton Theorem

Worked solutions 10 questions · 72 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR MEI Extra Pure Y435 · CIE 9231

Questions

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1. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

(a) Find the characteristic equation of $\mathbf{A}$, simplifying your answer. **[2]**

(b) Hence, using the Cayley–Hamilton theorem, find $\mathbf{A}^{-2}$ in the form $p\mathbf{A} + q\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix and p and q are rational constants to be found. **[4]**

Solution

(a) The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix}$$

Expanding the determinant:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (4-\lambda)(1-\lambda) - (-1) \times 2 \\ &= 4 - 5\lambda + \lambda^2 + 2 \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 5\lambda + 6 = 0$$

(b) By the Cayley–Hamilton theorem, $\mathbf{A}$ satisfies its own characteristic equation, so

$$\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = \mathbf{0}$$

Since $\det(\mathbf{A}) = 4 \times 1 - (-1) \times 2 = 6 \neq 0$, $\mathbf{A}^{-1}$ exists. Multiplying through by $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A} - 5\mathbf{I} + 6\mathbf{A}^{-1} &= \mathbf{0} \\ \mathbf{A}^{-1} &= \frac{1}{6}(5\mathbf{I} - \mathbf{A}) \end{aligned}$$

Multiplying $\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = \mathbf{0}$ through by $\mathbf{A}^{-2}$ instead gives

$$\begin{aligned} \mathbf{I} - 5\mathbf{A}^{-1} + 6\mathbf{A}^{-2} &= \mathbf{0} \\ \mathbf{A}^{-2} &= \frac{1}{6}\left(5\mathbf{A}^{-1} - \mathbf{I}\right) \end{aligned}$$

Substituting the expression for $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A}^{-2} &= \frac{1}{6}\left(\frac{5}{6}(5\mathbf{I} - \mathbf{A}) - \mathbf{I}\right) \\ &= \frac{1}{36}(25\mathbf{I} - 5\mathbf{A} - 6\mathbf{I}) \\ &= \frac{1}{36}(19\mathbf{I} - 5\mathbf{A}) \\ &= -\frac{5}{36}\mathbf{A} + \frac{19}{36}\mathbf{I} \end{aligned}$$

This is in the form $p\mathbf{A} + q\mathbf{I}$ with $p = -\frac{5}{36}$ and $q = \frac{19}{36}$.

2. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

(a) Find the characteristic equation of $\mathbf{A}$, giving your answer in simplified form. **[2]**

(b) Hence, using the Cayley–Hamilton theorem, show that $\mathbf{A}^{-2} = p\mathbf{I} + q\mathbf{A}$, where $\mathbf{I}$ is the 3×3 identity matrix and p and q are constants to be determined. **[3]**

Solution

(a) The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 0 & -2 & 2-\lambda \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)\big[(-\lambda)(2-\lambda) - (-1)\times(-2)\big] - 0 + 1 \times \big[1\times(-2) - (-\lambda)\times 0\big] \\ &= (1-\lambda)(\lambda^2 - 2\lambda - 2) - 2 \\ &= \lambda^2 - 2\lambda - 2 - \lambda^3 + 2\lambda^2 + 2\lambda - 2 \\ &= -\lambda^3 + 3\lambda^2 - 4 \end{aligned}$$

Setting this equal to zero and multiplying by -1, the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

(b) By the Cayley–Hamilton theorem, $\mathbf{A}$ satisfies its own characteristic equation, so

$$\mathbf{A}^3 - 3\mathbf{A}^2 + 4\mathbf{I} = \mathbf{0}$$

Putting $\lambda = 0$ in part (a) gives $\det(\mathbf{A}) = -4 \neq 0$, so $\mathbf{A}^{-1}$ exists. Multiplying by $\mathbf{A}^{-1}$:

$$\mathbf{A}^2 - 3\mathbf{A} + 4\mathbf{A}^{-1} = \mathbf{0}$$

Multiplying by $\mathbf{A}^{-1}$ again:

$$\begin{aligned} \mathbf{A} - 3\mathbf{I} + 4\mathbf{A}^{-2} &= \mathbf{0} \\ 4\mathbf{A}^{-2} &= 3\mathbf{I} - \mathbf{A} \\ \mathbf{A}^{-2} &= \tfrac{3}{4}\mathbf{I} - \tfrac{1}{4}\mathbf{A} \end{aligned}$$

This is in the form $\mathbf{A}^{-2} = p\mathbf{I} + q\mathbf{A}$ with $p = \frac{3}{4}$, $q = -\frac{1}{4}$.

3. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & -2 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

Find the characteristic equation of $\mathbf{A}$ and use it to show that

$$\mathbf{A}^5 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$$

where $\mathbf{I}$ is the 3×3 identity matrix and a, b and c are integers to be determined. **[5]**

Solution

The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4-\lambda & 0 & -2 \\ -1 & -1-\lambda & -2 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (4-\lambda)[(-1-\lambda)(1-\lambda) - 0 \times (-2)] - 0 + (-2)[(-1) \times 0 - (-1-\lambda) \times 1] \\ &= -(1+\lambda)(4-\lambda)(1-\lambda) - 2(1+\lambda) \\ &= -(1+\lambda)[(4-\lambda)(1-\lambda) + 2] \\ &= -(1+\lambda)\left(\lambda^2 - 5\lambda + 6\right) \\ &= -\left(\lambda^3 - 4\lambda^2 + \lambda + 6\right) \end{aligned}$$

So the characteristic equation is

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

By the Cayley–Hamilton theorem, $\mathbf{A}$ satisfies its own characteristic equation, so

$$\mathbf{A}^3 - 4\mathbf{A}^2 + \mathbf{A} + 6\mathbf{I} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}^3 = 4\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}$$

Multiplying by $\mathbf{A}$ and substituting for $\mathbf{A}^3$:

$$\begin{aligned} \mathbf{A}^4 &= 4\mathbf{A}^3 - \mathbf{A}^2 - 6\mathbf{A} \\ &= 4\left(4\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}\right) - \mathbf{A}^2 - 6\mathbf{A} \\ &= 15\mathbf{A}^2 - 10\mathbf{A} - 24\mathbf{I} \end{aligned}$$

Multiplying by $\mathbf{A}$ again:

$$\begin{aligned} \mathbf{A}^5 &= 15\mathbf{A}^3 - 10\mathbf{A}^2 - 24\mathbf{A} \\ &= 15\left(4\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}\right) - 10\mathbf{A}^2 - 24\mathbf{A} \\ &= 60\mathbf{A}^2 - 15\mathbf{A} - 90\mathbf{I} - 10\mathbf{A}^2 - 24\mathbf{A} \\ &= 50\mathbf{A}^2 - 39\mathbf{A} - 90\mathbf{I} \end{aligned}$$

This is in the form $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$ with $a = 50$, $b = -39$ and $c = -90$.

4. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ -2 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

Find the characteristic equation of $\mathbf{A}$. Hence, using the Cayley–Hamilton theorem and without calculating any power of $\mathbf{A}$, show that

$$(\mathbf{A} - 2\mathbf{I})^3 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I}$$

where $\mathbf{I}$ is the 3×3 identity matrix and p, q and r are integers to be determined. **[6]**

Solution

The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2-\lambda & 0 & 0 \\ -2 & -2-\lambda & 1 \\ 3 & 1 & -2-\lambda \end{pmatrix}$$

Expanding along the first row, which has two zero entries:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (2-\lambda)\left[(-2-\lambda)^2 - 1 \times 1\right] \\ &= (2-\lambda)\left(\lambda^2 + 4\lambda + 3\right) \\ &= 2\lambda^2 + 8\lambda + 6 - \lambda^3 - 4\lambda^2 - 3\lambda \\ &= -\lambda^3 - 2\lambda^2 + 5\lambda + 6 \end{aligned}$$

Setting this equal to zero and multiplying by -1, the characteristic equation is

$$\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

By the Cayley–Hamilton theorem, $\mathbf{A}$ satisfies its own characteristic equation, so

$$\mathbf{A}^3 + 2\mathbf{A}^2 - 5\mathbf{A} - 6\mathbf{I} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}^3 = -2\mathbf{A}^2 + 5\mathbf{A} + 6\mathbf{I}$$

Since $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$, the matrices $\mathbf{A}$ and $2\mathbf{I}$ commute, so $(\mathbf{A} - 2\mathbf{I})^3$ expands in the same way as $(x-2)^3$:

$$\begin{aligned} (\mathbf{A} - 2\mathbf{I})^3 &= \mathbf{A}^3 - 3 \times 2\mathbf{A}^2 + 3 \times 4\mathbf{A} - 8\mathbf{I} \\ &= \mathbf{A}^3 - 6\mathbf{A}^2 + 12\mathbf{A} - 8\mathbf{I} \end{aligned}$$

Substituting the expression for $\mathbf{A}^3$:

$$\begin{aligned} (\mathbf{A} - 2\mathbf{I})^3 &= \left(-2\mathbf{A}^2 + 5\mathbf{A} + 6\mathbf{I}\right) - 6\mathbf{A}^2 + 12\mathbf{A} - 8\mathbf{I} \\ &= -8\mathbf{A}^2 + 17\mathbf{A} - 2\mathbf{I} \end{aligned}$$

This is in the form $p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I}$ with $p = -8$, $q = 17$ and $r = -2$.

5. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are real constants, and $\mathbf{I}$ is the 2×2 identity matrix.

(a) Show that $\mathbf{M}$ satisfies its characteristic equation, that is,

$$\mathbf{M}^2 - (a+d)\mathbf{M} + (ad - bc)\mathbf{I} = \mathbf{0} \qquad \textbf{[2]}$$

(b) Deduce that, if $a + d = 0$, then $\mathbf{M}^2 = (a^2 + bc)\mathbf{I}$.
The matrix $\mathbf{N}$ is given by

$$\mathbf{N} = \begin{pmatrix} 5 & -8 \\ 2 & -5 \end{pmatrix}$$

Hence show that

$$(\mathbf{N} + \mathbf{I})^3 = 12\mathbf{N} + 28\mathbf{I} \qquad \textbf{[5]}$$

Solution

(a) Squaring the matrix:

$$\mathbf{M}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

Also

$$(a+d)\mathbf{M} = \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix}, \qquad (ad - bc)\mathbf{I} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

so

$$\mathbf{M}^2 - (a+d)\mathbf{M} + (ad - bc)\mathbf{I} = \begin{pmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd \\ ac + cd - ac - cd & bc + d^2 - ad - d^2 + ad - bc \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

as required.

(b) If $a + d = 0$, the result of part (a) becomes

$$\mathbf{M}^2 + (ad - bc)\mathbf{I} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{M}^2 = (bc - ad)\mathbf{I}$$

Since $d = -a$, $bc - ad = bc + a^2$, so

$$\mathbf{M}^2 = (a^2 + bc)\mathbf{I}$$

For $\mathbf{N}$, $a = 5$, $b = -8$, $c = 2$, $d = -5$, so $a + d = 5 + (-5) = 0$. Hence

$$\mathbf{N}^2 = \big(5^2 + (-8) \times 2\big)\mathbf{I} = 9\mathbf{I}$$

and so

$$\mathbf{N}^3 = \mathbf{N}\mathbf{N}^2 = \mathbf{N}(9\mathbf{I}) = 9\mathbf{N}$$

Since $\mathbf{NI} = \mathbf{IN} = \mathbf{N}$, the brackets expand as in ordinary algebra:

$$\begin{aligned}(\mathbf{N}+\mathbf{I})^3 &= \mathbf{N}^3 + 3\mathbf{N}^2 + 3\mathbf{N} + \mathbf{I} \\ &= 9\mathbf{N} + 27\mathbf{I} + 3\mathbf{N} + \mathbf{I} \\ &= 12\mathbf{N} + 28\mathbf{I}\end{aligned}$$

as required.

6. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & 3 \\ -4 & 5 \end{pmatrix}$$

where a is a constant.

(a) Find, in expanded form in terms of a, the characteristic equation of $\mathbf{A}$. **[2]**

(b) Hence use the Cayley–Hamilton theorem to find all possible pairs of values of a and b such that

$$\mathbf{A}^3 = 7\mathbf{A} + b\mathbf{I}$$

where $\mathbf{I}$ is the 2×2 identity matrix. **[5]**

Solution

(a) The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which gives

$$\begin{vmatrix} a-\lambda & 3 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(5-\lambda) + 12 = 0$$

$$\lambda^2 - (a+5)\lambda + 5a + 12 = 0$$

(b) By the Cayley–Hamilton theorem, $\mathbf{A}$ satisfies its own characteristic equation, so using part (a):

$$\mathbf{A}^2 - (a+5)\mathbf{A} + (5a+12)\mathbf{I} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}^2 = (a+5)\mathbf{A} - (5a+12)\mathbf{I}$$

Multiplying by $\mathbf{A}$ and substituting for $\mathbf{A}^2$ again:

$$\begin{aligned} \mathbf{A}^3 &= (a+5)\mathbf{A}^2 - (5a+12)\mathbf{A} \\ &= (a+5)[(a+5)\mathbf{A} - (5a+12)\mathbf{I}] - (5a+12)\mathbf{A} \\ &= \left[(a+5)^2 - (5a+12)\right]\mathbf{A} - (a+5)(5a+12)\mathbf{I} \\ &= (a^2+5a+13)\mathbf{A} - (a+5)(5a+12)\mathbf{I} \end{aligned}$$

Setting this equal to $7\mathbf{A} + b\mathbf{I}$ and rearranging gives

$$(a^2+5a+6)\mathbf{A} = [b + (a+5)(5a+12)]\mathbf{I}$$

The top-right entry of the left-hand side is $3(a^2+5a+6)$, and the top-right entry of the right-hand side is 0, so

$$a^2 + 5a + 6 = 0$$

$$(a+2)(a+3) = 0$$

$$a = -2 \quad \text{or} \quad a = -3$$

The left-hand side is then the zero matrix, so

$$b = -(a+5)(5a+12)$$

When $a = -2$, $b = -3 \times 2 = -6$. When $a = -3$, $b = -2 \times (-3) = 6$.

The possible pairs are $a = -2$, $b = -6$ and $a = -3$, $b = 6$.

7.

$$\mathbf{M} = \begin{pmatrix} 3 & -3 & 1 \\ 2 & -1 & 2 \\ 0 & k & 2 \end{pmatrix}$$

where k is a constant.

(a) Show that a characteristic equation for $\mathbf{M}$ is

$$\lambda^3 - 4\lambda^2 + (7 - 2k)\lambda + 4k - 6 = 0 \quad \textbf{[2]}$$

It is given that 3 is an eigenvalue of $\mathbf{M}$.

(b) (i) Find the value of k and the other two eigenvalues of $\mathbf{M}$. **[3]**

(ii) Use the Cayley–Hamilton theorem to find $\mathbf{M}^{-1}$. **[5]**

Solution

(a) The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, where

$$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 3 - \lambda & -3 & 1 \\ 2 & -1 - \lambda & 2 \\ 0 & k & 2 - \lambda \end{pmatrix}$$

Expanding along the first column:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= (3 - \lambda)\big[(-1 - \lambda)(2 - \lambda) - 2k\big] - 2\big[(-3)(2 - \lambda) - k\big] \\ &= (3 - \lambda)\big(\lambda^2 - \lambda - 2 - 2k\big) + 12 - 6\lambda + 2k \\ &= -\lambda^3 + 4\lambda^2 + (2k - 1)\lambda - 6 - 6k + 12 - 6\lambda + 2k \\ &= -\lambda^3 + 4\lambda^2 + (2k - 7)\lambda + 6 - 4k \end{aligned}$$

Setting this equal to zero and multiplying by -1 gives

$$\lambda^3 - 4\lambda^2 + (7 - 2k)\lambda + 4k - 6 = 0$$

as required.

(b) (i) Since 3 is an eigenvalue, $\lambda = 3$ satisfies the characteristic equation:

$$\begin{aligned} 27 - 36 + 3(7 - 2k) + 4k - 6 &= 0 \\ 6 - 2k &= 0 \\ k &= 3 \end{aligned}$$

With $k = 3$, the characteristic equation is $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$. $\lambda - 3$ is a factor, so

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda - 3)\big(\lambda^2 - \lambda - 2\big) = (\lambda - 3)(\lambda - 2)(\lambda + 1)$$

so the other two eigenvalues are $\lambda = 2$ and $\lambda = -1$.

(ii) With $k = 3$, by the Cayley–Hamilton theorem, $\mathbf{M}$ satisfies its own characteristic equation, so

$$\mathbf{M}^3 - 4\mathbf{M}^2 + \mathbf{M} + 6\mathbf{I} = \mathbf{0}$$

Since 0 is not an eigenvalue of $\mathbf{M}$, $\det(\mathbf{M}) \neq 0$ and $\mathbf{M}^{-1}$ exists. Multiplying by $\mathbf{M}^{-1}$:

$$\mathbf{M}^2 - 4\mathbf{M} + \mathbf{I} + 6\mathbf{M}^{-1} = \mathbf{0}$$

$$\mathbf{M}^{-1} = \frac{1}{6}(4\mathbf{M} - \mathbf{M}^2 - \mathbf{I})$$

Substituting $k = 3$:

$$\mathbf{M} = \begin{pmatrix} 3 & -3 & 1 \\ 2 & -1 & 2 \\ 0 & 3 & 2 \end{pmatrix}, \qquad \mathbf{M}^2 = \begin{pmatrix} 3 & -3 & -1 \\ 4 & 1 & 4 \\ 6 & 3 & 10 \end{pmatrix}$$

so

$$4\mathbf{M} - \mathbf{M}^2 - \mathbf{I} = \begin{pmatrix} 12 & -12 & 4 \\ 8 & -4 & 8 \\ 0 & 12 & 8 \end{pmatrix} - \begin{pmatrix} 3 & -3 & -1 \\ 4 & 1 & 4 \\ 6 & 3 & 10 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -9 & 5 \\ 4 & -6 & 4 \\ -6 & 9 & -3 \end{pmatrix}$$

Hence

$$\mathbf{M}^{-1} = \frac{1}{6}\begin{pmatrix} 8 & -9 & 5 \\ 4 & -6 & 4 \\ -6 & 9 & -3 \end{pmatrix}$$

8. The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} a & 4 \\ 1 & a-3 \end{pmatrix}$$

where a is a constant.
You are given that $\mathbf{P}$ satisfies the matrix equation $\mathbf{P}^2 + r\mathbf{P} = 6\mathbf{I}$ for some integer r, where $\mathbf{I}$ is the 2×2 identity matrix.

(a) Use the Cayley–Hamilton theorem to find the two possible values of a, and the corresponding value of r in each case. **[4]**

(b) In the case $a > 1$, find integers s and t such that $\mathbf{P}^4 = s\mathbf{P} + t\mathbf{I}$. You should not calculate $\mathbf{P}^4$. Express $\mathbf{P}^{-1}$ in the form $\alpha\mathbf{P} + \beta\mathbf{I}$, where α and β are rational numbers to be found. **[4]**

Solution

(a) The characteristic equation of $\mathbf{P}$ is $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$:

$$(a-\lambda)(a-3-\lambda) - 4 \times 1 = 0$$

$$\lambda^2 - (2a-3)\lambda + a^2 - 3a - 4 = 0$$

By the Cayley–Hamilton theorem, $\mathbf{P}$ satisfies its own characteristic equation, so

$$\mathbf{P}^2 - (2a-3)\mathbf{P} + \big(a^2 - 3a - 4\big)\mathbf{I} = \mathbf{0}$$

The given equation is $\mathbf{P}^2 + r\mathbf{P} - 6\mathbf{I} = \mathbf{0}$. Subtracting the Cayley–Hamilton equation from it gives

$$(r + 2a - 3)\mathbf{P} = \big(a^2 - 3a + 2\big)\mathbf{I}$$

The top-right entry of $\mathbf{I}$ is 0, and the top-right entry of $\mathbf{P}$ is 4, so equating top-right entries:

$$4(r + 2a - 3) = 0 \quad \Longrightarrow \quad r = 3 - 2a$$

The left-hand side is then the zero matrix, so equating top-left entries:

$$a^2 - 3a + 2 = 0$$

$$(a-1)(a-2) = 0$$

so $a = 1$ or $a = 2$. Using $r = 3 - 2a$, when $a = 1$, $r = 1$, and when $a = 2$, $r = -1$.

(b) Since $a > 1$, $a = 2$ and $r = -1$, so

$$\mathbf{P}^2 - \mathbf{P} = 6\mathbf{I} \quad \Longrightarrow \quad \mathbf{P}^2 = \mathbf{P} + 6\mathbf{I}$$

Multiplying by $\mathbf{P}$ and replacing each $\mathbf{P}^2$ by $\mathbf{P} + 6\mathbf{I}$:

$$\begin{aligned} \mathbf{P}^3 &= \mathbf{P}^2 + 6\mathbf{P} \\ &= \mathbf{P} + 6\mathbf{I} + 6\mathbf{P} \\ &= 7\mathbf{P} + 6\mathbf{I} \\ \mathbf{P}^4 &= 7\mathbf{P}^2 + 6\mathbf{P} \\ &= 7(\mathbf{P} + 6\mathbf{I}) + 6\mathbf{P} \\ &= 13\mathbf{P} + 42\mathbf{I} \end{aligned}$$

This is in the form $s\mathbf{P} + t\mathbf{I}$ with $s = 13$ and $t = 42$.

With $a = 2$, $\det(\mathbf{P}) = 2 \times (-1) - 4 \times 1 = -6 \neq 0$, so $\mathbf{P}^{-1}$ exists. Multiplying $\mathbf{P}^2 - \mathbf{P} = 6\mathbf{I}$ by $\mathbf{P}^{-1}$:

$$\mathbf{P} - \mathbf{I} = 6\mathbf{P}^{-1}$$

$$\mathbf{P}^{-1} = \frac{1}{6}\mathbf{P} - \frac{1}{6}\mathbf{I}$$

This is in the form $\alpha\mathbf{P} + \beta\mathbf{I}$ with $\alpha = \frac{1}{6}$ and $\beta = -\frac{1}{6}$.

9. It is given that k is a positive constant.
The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & k+1 & 2 \\ 0 & k & 0 \\ 4 & 2k & -1 \end{pmatrix}$$

(a) Show that the characteristic equation of $\mathbf{A}$ is

$$\lambda^3 - k\lambda^2 - 9\lambda + 9k = 0 \qquad \textbf{[2]}$$

(b) Hence use the Cayley–Hamilton theorem to find $\mathbf{A}^{-1}$ in terms of k. **[5]**

Solution

(a) The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1-\lambda & k+1 & 2 \\ 0 & k-\lambda & 0 \\ 4 & 2k & -1-\lambda \end{pmatrix}$$

The second row has only one non-zero entry, so expanding along the second row gives

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (k-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} \\ &= (k-\lambda)[(1-\lambda)(-1-\lambda) - 8] \\ &= (k-\lambda)\left(\lambda^2 - 1 - 8\right) \\ &= (k-\lambda)\left(\lambda^2 - 9\right) \\ &= k\lambda^2 - 9k - \lambda^3 + 9\lambda \end{aligned}$$

Setting this equal to zero and multiplying by -1, the characteristic equation is

$$\lambda^3 - k\lambda^2 - 9\lambda + 9k = 0$$

as required.

(b) By the Cayley–Hamilton theorem, $\mathbf{A}$ satisfies its own characteristic equation, so

$$\mathbf{A}^3 - k\mathbf{A}^2 - 9\mathbf{A} + 9k\mathbf{I} = \mathbf{0}$$

Since $k > 0$, $\det(\mathbf{A}) = -9k \neq 0$, so $\mathbf{A}^{-1}$ exists. Multiplying through by $\mathbf{A}^{-1}$:

$$\mathbf{A}^2 - k\mathbf{A} - 9\mathbf{I} + 9k\mathbf{A}^{-1} = \mathbf{0}$$

$$\mathbf{A}^{-1} = \frac{1}{9k}\left(9\mathbf{I} + k\mathbf{A} - \mathbf{A}^2\right)$$

Now find $\mathbf{A}^2$:

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix}1 & k+1 & 2\\ 0 & k & 0\\ 4 & 2k & -1\end{pmatrix}\begin{pmatrix}1 & k+1 & 2\\ 0 & k & 0\\ 4 & 2k & -1\end{pmatrix}\\ &= \begin{pmatrix}1+0+8 & (k+1)+k(k+1)+4k & 2+0-2\\ 0 & k^2 & 0\\ 4+0-4 & 4(k+1)+2k^2-2k & 8+0+1\end{pmatrix}\\ &= \begin{pmatrix}9 & k^2+6k+1 & 0\\ 0 & k^2 & 0\\ 0 & 2k^2+2k+4 & 9\end{pmatrix}\end{aligned}$$

Then

$$\begin{aligned}9\mathbf{I}+k\mathbf{A}-\mathbf{A}^2 &= \begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}+\begin{pmatrix}k & k^2+k & 2k\\ 0 & k^2 & 0\\ 4k & 2k^2 & -k\end{pmatrix}-\begin{pmatrix}9 & k^2+6k+1 & 0\\ 0 & k^2 & 0\\ 0 & 2k^2+2k+4 & 9\end{pmatrix}\\ &= \begin{pmatrix}k & -5k-1 & 2k\\ 0 & 9 & 0\\ 4k & -2k-4 & -k\end{pmatrix}\end{aligned}$$

so

$$\mathbf{A}^{-1} = \frac{1}{9k}\begin{pmatrix}k & -5k-1 & 2k\\ 0 & 9 & 0\\ 4k & -2k-4 & -k\end{pmatrix}$$

10. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} b & 2 & -1 \\ 0 & 0 & a \\ a & 2 & 3 \end{pmatrix}$$

where a and b are integers and $a > 0$.
The characteristic equation of $\mathbf{M}$ is

$$\lambda^3 + \lambda^2 + c\lambda - 24 = 0$$

where c is a constant.

(a) Determine the values of a, b and c. **[6]**

(b) Hence, using the Cayley–Hamilton theorem, find $(\mathbf{M} - 2\mathbf{I})^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. **[5]**

Solution

(a) The characteristic equation is $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, where

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} b-\lambda & 2 & -1 \\ 0 & -\lambda & a \\ a & 2 & 3-\lambda \end{vmatrix}$$

Expanding along the first row:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= (b-\lambda)\big(-\lambda(3-\lambda) - 2a\big) - 2\big(0 - a^2\big) - 1(0 + a\lambda) \\ &= (b-\lambda)\big(\lambda^2 - 3\lambda - 2a\big) + 2a^2 - a\lambda \\ &= b\lambda^2 - 3b\lambda - 2ab - \lambda^3 + 3\lambda^2 + 2a\lambda + 2a^2 - a\lambda \\ &= -\lambda^3 + (b+3)\lambda^2 + (a-3b)\lambda + 2a^2 - 2ab \end{aligned}$$

Setting this equal to zero and multiplying by -1, the characteristic equation is

$$\lambda^3 - (b+3)\lambda^2 + (3b-a)\lambda + 2ab - 2a^2 = 0$$

Equating coefficients of λ^2 with $\lambda^3 + \lambda^2 + c\lambda - 24 = 0$:

$$-(b+3) = 1 \quad \Longrightarrow \quad b = -4$$

Equating constant terms, with $b = -4$:

$$\begin{aligned} 2a(-4) - 2a^2 &= -24 \\ a^2 + 4a - 12 &= 0 \\ (a+6)(a-2) &= 0 \end{aligned}$$

Since $a > 0$, $a = 2$.

Equating coefficients of λ:

$$c = 3b - a = 3 \times (-4) - 2 = -14$$

So $a = 2$, $b = -4$ and $c = -14$.

(b) By the Cayley–Hamilton theorem, $\mathbf{M}$ satisfies its own characteristic equation, so

$$\mathbf{M}^3 + \mathbf{M}^2 - 14\mathbf{M} - 24\mathbf{I} = \mathbf{0}$$

Adding $40\mathbf{I}$ to both sides:

$$\mathbf{M}^3 + \mathbf{M}^2 - 14\mathbf{M} + 16\mathbf{I} = 40\mathbf{I}$$

The left-hand side factorises, since

$$(\mathbf{M} - 2\mathbf{I})\big(\mathbf{M}^2 + 3\mathbf{M} - 8\mathbf{I}\big) = \mathbf{M}^3 + 3\mathbf{M}^2 - 8\mathbf{M} - 2\mathbf{M}^2 - 6\mathbf{M} + 16\mathbf{I} = \mathbf{M}^3 + \mathbf{M}^2 - 14\mathbf{M} + 16\mathbf{I}$$

so

$$(\mathbf{M} - 2\mathbf{I})\big(\mathbf{M}^2 + 3\mathbf{M} - 8\mathbf{I}\big) = 40\mathbf{I} \quad \Longrightarrow \quad (\mathbf{M} - 2\mathbf{I})^{-1} = \frac{1}{40}\big(\mathbf{M}^2 + 3\mathbf{M} - 8\mathbf{I}\big)$$

With $a = 2$ and $b = -4$:

$$\mathbf{M} = \begin{pmatrix} -4 & 2 & -1 \\ 0 & 0 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \qquad \mathbf{M}^2 = \begin{pmatrix} 14 & -10 & 5 \\ 4 & 4 & 6 \\ -2 & 10 & 11 \end{pmatrix}$$

so

$$\mathbf{M}^2 + 3\mathbf{M} - 8\mathbf{I} = \begin{pmatrix} 14 - 12 - 8 & -10 + 6 & 5 - 3 \\ 4 & 4 - 8 & 6 + 6 \\ -2 + 6 & 10 + 6 & 11 + 9 - 8 \end{pmatrix} = \begin{pmatrix} -6 & -4 & 2 \\ 4 & -4 & 12 \\ 4 & 16 & 12 \end{pmatrix}$$

Therefore

$$(\mathbf{M} - 2\mathbf{I})^{-1} = \frac{1}{40}\begin{pmatrix} -6 & -4 & 2 \\ 4 & -4 & 12 \\ 4 & 16 & 12 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} -3 & -2 & 1 \\ 2 & -2 & 6 \\ 2 & 8 & 6 \end{pmatrix}$$