OPTIONAL PURE / VECTORS & MATRICES

Scalar Triple Product

Questions by topic 10 questions · 105 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · OCR A Y545

Questions

Question 1	*8 marks*	2
Question 2	*12 marks*	3
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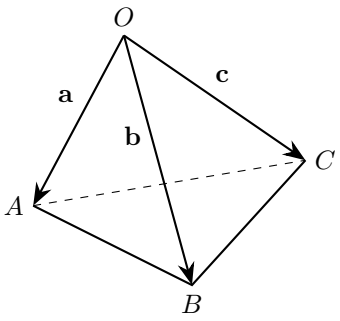
Question 4	*13 marks*	5
Question 5	*11 marks*	6
Question 6	*15 marks*	7
Question 7	*7 marks*	8
Question 8	*13 marks*	9
Question 9	*13 marks*	10
Question 10	*4 marks*	11



1. The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, relative to a fixed origin O, as shown in the diagram above.
It is given that, for a constant λ,

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{pmatrix} \lambda \\ -1 \\ 2 \end{pmatrix}$$

(a) Find $\mathbf{a} \times \mathbf{b}$. **[2]**

(b) Hence find the exact area of triangle OAB. **[2]**

(c) Find $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ in terms of λ. **[2]**

(d) Given that the volume of the tetrahedron $OABC$ is 1 cubic unit, find the two possible values of λ. **[2]**

2. The points A and B have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$ respectively, relative to the origin O. The line ℓ has equation

$$(\mathbf{r} - (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})) \times (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \mathbf{0}$$

The point C lies on ℓ such that OC is perpendicular to ℓ.

(a) Find the position vector of C. **[4]**

(b) Find the volume of the tetrahedron $OABC$. **[4]**

The transformation T is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} k & 1 & 0 \\ 3 & k & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

where k is a positive constant. The transformation T maps the points O, A, B and C to the points O, A', B' and C' respectively. The volume of the tetrahedron $OA'B'C'$ is 36.

(c) Use the determinant of $\mathbf{M}$ to find the value of k. **[4]**

3. The points A and B have position vectors $\mathbf{a} = p\mathbf{i} + 2\mathbf{j} + q\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ respectively, relative to the origin O.

(a) Given that $\mathbf{a} \times \mathbf{b}$ is parallel to $-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and that $\mathbf{a} \cdot \mathbf{b} = -15$, find the value of p and the value of q. **[4]**

(b) The point C has coordinates (d, e, f) and the tetrahedron $OABC$ has volume 4.

(i) Using your values of p and q, find the two possible equations linking d, e and f. **[3]**

(ii) Describe the geometrical significance of these equations, stating the perpendicular distance of C from the plane OAB. **[2]**

4. The points A, B, C and D have coordinates $(2, -1, 1)$, $(3, p, 0)$, $(p, 1, 3)$ and $(4, 0, 2)$ respectively, where p is a constant.

(a) Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of p. **[2]**

(b) Show that

$$\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = (6 - p)(p + 2)$$ **[5]**

(c) Find the values of p for which the points A, B, C and D are coplanar, justifying your answer. **[2]**

(d) Given that the tetrahedron $ABCD$ has volume 8, find the possible values of p. **[4]**

5. The parallelogram P has vertices A, B, C and D, in that order. The vertices A, B and D have coordinates

$$A(3,-2,1) \qquad B(5,-3,0) \qquad D(0,-6,2)$$

(a) Find the coordinates of C, and determine the exact area of P, giving your answer as a simplified surd. **[4]**

The line ℓ passes through C and is perpendicular to the plane containing P.

(b) Determine an equation for ℓ, giving your answer in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. **[2]**

The parallelogram P is one face of a parallelepiped. The opposite face of the parallelepiped lies in the plane Π, and the point $Q(k,k,4)$, where k is a constant, lies in Π.

(c) Given that the volume of the parallelepiped is 36, determine the two possible values of k. **[5]**

6. With respect to a fixed origin O, the points A, B and C have coordinates $(1, 1, 4)$, $(4, 1, -2)$ and $(1, -3, 0)$ respectively.
The plane Π has equation $2x - y + 2z = 7$.
The line segment AB meets Π at the point P, and the line segment AC meets Π at the point Q.

(a) Find the exact area of the quadrilateral $PBCQ$. **[6]**

The point D has coordinates $(-2, 4, 7)$. The line segment AD meets Π at the point R.

(b) Use the scalar triple product to find the volume of the tetrahedron $APQR$. **[6]**

(c) The plane Π divides the tetrahedron $ABCD$ into two pieces. Find the ratio of the volume of the smaller piece to the volume of the larger piece. **[3]**

7. The vectors $\mathbf{b}$ and $\mathbf{d}$ are non-zero and not parallel. The lines ℓ_1 and ℓ_2 have vector equations

$$\ell_1 : \ \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R} \qquad \text{and} \qquad \ell_2 : \ (\mathbf{r} - \mathbf{c}) \times \mathbf{d} = \mathbf{0}$$

The lines ℓ_1 and ℓ_2 intersect.

(a) Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{d})$, giving a reason for each step of your argument. **[4]**

(b) The plane Π contains both ℓ_1 and ℓ_2, and has equation $\mathbf{r} \cdot \mathbf{n} = p$.
Write down a suitable vector $\mathbf{n}$ in terms of $\mathbf{b}$ and $\mathbf{d}$. Give two different expressions for p, one in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ and the other in terms of $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, explaining your reasoning. **[3]**

8. The points P, Q and R have position vectors $p\mathbf{i}$, $q\mathbf{j}$ and $r\mathbf{k}$ respectively, relative to the origin O, where p, q and r are positive constants. The points O, P, Q and R are the vertices of a tetrahedron.

(a) Use the scalar triple product to find the volume of the tetrahedron $OPQR$ in terms of p, q and r. **[3]**

(b) Find $\overrightarrow{RP} \times \overrightarrow{RQ}$ in terms of p, q and r, and hence show that the area of triangle PQR is

$$\frac{1}{2}\sqrt{q^2r^2 + r^2p^2 + p^2q^2}$$ **[4]**

(c) The perpendicular distance from O to the plane PQR is h. By finding the volume of the tetrahedron in a second way, show that

$$\frac{1}{h^2} = \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$$

Hence find the exact value of h when $p = 2$, $q = 3$ and $r = 6$. **[6]**

9. The vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are given by

$$\mathbf{p} = \begin{pmatrix} 2 \\ -1 \\ k \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

where k is a constant.
It is given that $\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = -21$.

(a) Find the value of k, and hence calculate $\mathbf{p} \times (\mathbf{q} \times \mathbf{r})$ and $(\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$. **[7]**

(b) Using your answers to part (a), state whether the vector product is associative for three-dimensional column vectors with real components. Justify your answer. **[1]**

Throughout the rest of this question, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three-dimensional column vectors with real components.

(c) Explain geometrically why the vector $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ must be expressible in the form $\lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are scalar constants. **[2]**

It is given that the following relationship holds for all such $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \qquad (*)$$

(d) Use $(*)$ to show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$ **[3]**

10. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = \begin{pmatrix} p - 2r + 4 \\ q + r - 6 \\ pq - 8 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} p - 2q \\ q^2 - r \\ qr - 2p \end{pmatrix}$$

where p, q and r are real numbers.
It is given that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ for every vector $\mathbf{c}$, and that $\mathbf{a}$ is **not** a multiple of $\mathbf{b}$.
Determine all possible sets of values of p, q and r. **[4]**