OPTIONAL PURE / VECTORS & MATRICES

Orthogonal Diagonalisation

Questions by topic 10 questions · 131 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

Question 1	*9 marks*	2
Question 2	*12 marks*	3
Question 3	*15 marks*	4
Question 4	*12 marks*	5
Question 5	*13 marks*	6
Question 6	*10 marks*	7
Question 7	*12 marks*	8
Question 8	*14 marks*	9
Question 9	*14 marks*	10
Question 10	*20 marks*	11

1. The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} k & 6 \\ 6 & -2 \end{pmatrix}$$

where k is a constant.
It is given that 10 is an eigenvalue of $\mathbf{A}$.

(a) Find the value of k and the other eigenvalue of $\mathbf{A}$. **[4]**

(b) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$$

[5]

2.

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & k \\ -4 & 7 & 2 \\ k & 2 & 2 \end{pmatrix}$$

where k is a constant.
The vector $\mathbf{v}$ is an eigenvector of $\mathbf{M}$, where

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(a) Find the value of k and the eigenvalue of $\mathbf{M}$ corresponding to $\mathbf{v}$. **[3]**

(b) Determine the other two eigenvalues of $\mathbf{M}$. **[3]**

(c) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$. **[6]**

3.

$$\mathbf{M} = \begin{pmatrix} 4 & a & 2 \\ a & b & -2 \\ 2 & -2 & 5 \end{pmatrix}$$

where a and b are constants.
It is given that

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is an eigenvector of $\mathbf{M}$.

(a) Find the eigenvalue corresponding to this eigenvector. **[2]**

(b) Hence find the values of a and b. **[3]**

(c) Find the other two eigenvalues of $\mathbf{M}$. **[3]**

(d) Find a normalised eigenvector for each eigenvalue of $\mathbf{M}$. **[5]**

(e) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$, whose diagonal entries are in decreasing order, such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$

[2]

4. The symmetric 3×3 matrix $\mathbf{M}$ has eigenvalues 4, -3 and 6. Eigenvectors of $\mathbf{M}$ corresponding to the eigenvalues 4 and -3 are, respectively,

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) (i) Find an eigenvector of $\mathbf{M}$ corresponding to the eigenvalue 6. **[2]**

(ii) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$ **[3]**

(b) Use the fact that $\mathbf{P}$ is orthogonal to show that $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}$. **[2]**

(c) Hence find the matrix $\mathbf{M}$. **[5]**

5. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & p & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

where p is a constant.
It is given that 1 is an eigenvalue of $\mathbf{A}$.

(a) Show that $p = -3$. **[3]**

(b) Determine

(i) the other two eigenvalues of $\mathbf{A}$, **[3]**

(ii) an eigenvector corresponding to each of the three eigenvalues of $\mathbf{A}$. **[4]**

(c) Hence determine an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[3]**

6. The symmetric matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} -1 & 4 & -4 \\ 4 & 3 & 0 \\ -4 & 0 & a \end{pmatrix}$$

where a is a constant.
Given that $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ is an eigenvector of $\mathbf{M}$,

(a) find the corresponding eigenvalue and the value of a. **[3]**

Given also that 7 is an eigenvalue of $\mathbf{M}$,

(b) find a corresponding eigenvector. **[2]**

(c) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$

[5]

7. The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 9 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

(a) Given that 1 is an eigenvalue of $\mathbf{A}$, find the other eigenvalues of $\mathbf{A}$, showing that one of them is repeated. **[4]**

(b) Find three mutually perpendicular eigenvectors of $\mathbf{A}$. **[5]**

(c) Hence find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[3]**

8.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & k \\ 1 & 4 & 1 \\ k & 1 & 6 \end{pmatrix}$$

where k is a constant.
Given that 2 is an eigenvalue of $\mathbf{M}$,

(a) find the possible values of k. **[4]**

Given further that $k < 0$,

(b) find the other eigenvalues of $\mathbf{M}$. **[3]**

(c) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$. **[7]**

9.

$$\mathbf{M} = \begin{pmatrix} p & q & -3 \\ q & 5 & 3 \\ -3 & 3 & 0 \end{pmatrix}$$

where p and q are constants.
Given that

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is an eigenvector of the matrix $\mathbf{M}$,

(a) find the eigenvalue corresponding to this eigenvector, **[2]**

(b) find the value of q and the value of p, **[3]**

(c) find the other two eigenvalues of $\mathbf{M}$, **[3]**

(d) find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$

[6]

10. The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -4 & 0 \\ -4 & 3 & p \\ 0 & p & 1 \end{pmatrix}$$

where p is a constant.
It is given that

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

is an eigenvector of $\mathbf{A}$.

(a) (i) Determine the eigenvalue corresponding to this eigenvector. **[2]**

(ii) Hence show that $p = -4$. **[2]**

(iii) Determine the remaining eigenvalues of $\mathbf{A}$ and a corresponding eigenvector for each. **[7]**

(b) Write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{\mathrm{T}}$. **[2]**

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant and the dot denotes differentiation with respect to the time t. **[2]**

(ii) With respect to a fixed origin O, a particle moving through space has position (x, y, z) at time t, and its velocity is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

At time $t = 0$ the particle is at the point $(5, 1, 1)$.
By considering

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{\mathrm{T}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

determine the position of the particle at time t. **[5]**