OPTIONAL PURE / VECTORS & MATRICES

Eigenvalues, Eigenvectors and Diagonalisation

Questions by topic 15 questions · 169 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR MEI Extra Pure Y435 · AQA 7367 · CIE 9231

Instructions

CIE (9231): eigenvalues are examined only where they are real and distinct, so repeated and non-real eigenvalues are not on the specification. Questions 10, 13 and 14 can be left out.

Questions

Question 1	*6 marks*	2
Question 2	*6 marks*	3
Question 3	*11 marks*	4
Question 4	*6 marks*	5
Question 5	*10 marks*	6
Question 6	*15 marks*	7
Question 7	*13 marks*	8
Question 8	*12 marks*	9
Question 9	*13 marks*	10
Question 10	*8 marks*	11
Question 11	*12 marks*	12
Question 12	*12 marks*	13
Question 13	*13 marks*	14
Question 14	*13 marks*	15
Question 15	*19 marks*	16

1. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & k \\ 2 & -2 \end{pmatrix}$$

where k is a constant.

Given that 4 is an eigenvalue of $\mathbf{A}$, find

- the value of k,
- the other eigenvalue of $\mathbf{A}$,
- an eigenvector associated with each eigenvalue of $\mathbf{A}$.

[6]

2. The matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} a & 0.6 \\ 0.6 & b \end{pmatrix}$$

where a and b are constants with $a > 0$.

The matrix $\mathbf{M}$ represents a reflection in a line through the origin.

(a) Write down the eigenvalues of $\mathbf{M}$. **[1]**

(b) By considering the characteristic equation of $\mathbf{M}$, use your answer to part (a) to find the values of a and b. **[3]**

(c) Hence find the equation of the mirror line of the reflection represented by $\mathbf{M}$. **[2]**

3. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 2 & k \\ k & 5 \end{pmatrix}$$

where k is a negative constant.

It is given that 1 is an eigenvalue of $\mathbf{M}$.

(a) Find the value of k. Hence find the other eigenvalue of $\mathbf{M}$, and a corresponding eigenvector for each eigenvalue of $\mathbf{M}$. **[6]**

(b) (i) Describe how the directions of the invariant lines through the origin of the transformation represented by $\mathbf{M}$ are related to each other. Fully justify your answer. **[2]**

(ii) Describe fully the transformation represented by $\mathbf{M}$, stating the equation of any line of invariant points. **[3]**

4. A 3×3 matrix $\mathbf{A}$ has eigenvalues 2, -1 and 1, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

respectively.

(a) By expressing $\mathbf{A}$ in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, find the matrix $\mathbf{A}$. **[4]**

(b) Hence find the matrix $\mathbf{A}^4$. **[2]**

5. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 4 & -1 & 1 \\ a & 5 & a \\ -8 & b & -5 \end{pmatrix}$$

where a and b are real numbers.

The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of $\mathbf{M}$, with corresponding eigenvalues λ_1 and λ_2 respectively.

(a) (i) Find the value of λ_1. **[1]**

(ii) Find the value of a and the value of b. **[2]**

It is also given that $\lambda_3 = 3$ is an eigenvalue of $\mathbf{M}$.

(b) Find the value of λ_2, and find an eigenvector $\mathbf{v}_3$ of $\mathbf{M}$ corresponding to the eigenvalue $\lambda_3 = 3$. **[4]**

(c) Find matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$, such that $\mathbf{D}$ is diagonal and $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$. **[3]**

6.

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

(a) Show that -2 is an eigenvalue of $\mathbf{M}$, and find a corresponding eigenvector. **[4]**

(b) Given that $\mathbf{MU} = \mathbf{UD}$, where $\mathbf{D}$ is a diagonal matrix and each column of $\mathbf{U}$ is an eigenvector of $\mathbf{M}$ of unit length, find possible matrices for $\mathbf{D}$ and $\mathbf{U}$. **[11]**

7. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{8} & \frac{5}{16} \\ \frac{5}{4} & \frac{3}{8} \end{pmatrix}$$

(a) Find the eigenvalues of $\mathbf{A}$ and a corresponding eigenvector for each eigenvalue. **[5]**

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. **[1]**

(c) Find $\mathbf{A}^n$, where n is a positive integer, as a single matrix whose entries are expressed in terms of n. Hence find the matrix $\mathbf{L}$ such that $\mathbf{A}^n \to \mathbf{L}$ as $n \to \infty$, explaining briefly why this limit exists. **[5]**

(d) The transformation represented by $\mathbf{L}$ maps every point of the plane onto a single line through the origin.
Find the equation of this line. **[2]**

8. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -4 & -13 & 8 \\ -6 & -7 & 6 \\ -12 & -22 & 16 \end{pmatrix}$$

(a) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$ and hence find the eigenvalues of $\mathbf{A}$. **[4]**

(b) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. **[5]**

(c) Hence show that, for all positive integers n, the element in the first row and first column of $\mathbf{A}^n$ is $2(-1)^n - 2^n$. **[3]**

9. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & -4 \\ 1 & 3 & -1 \\ 1 & 4 & -2 \end{pmatrix}$$

(a) Show that the characteristic equation of $\mathbf{M}$ is

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$ **[3]**

(b) Hence find the three eigenvalues of $\mathbf{M}$. **[2]**

(c) For each eigenvalue of $\mathbf{M}$ find an associated eigenvector. **[4]**

(d) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$. Hence find $\mathbf{M}^n$ as a single matrix, where n is a positive integer, giving each entry in terms of n. **[4]**

10. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} -1 & -1 & 2 \\ -3 & -1 & 6 \\ k & 1 & 0 \end{pmatrix}$$

where k is a constant.

(a) Find the characteristic equation of $\mathbf{M}$ in terms of k, and show that 2 is an eigenvalue of $\mathbf{M}$ for every value of k. **[3]**

(b) Given that $\mathbf{M}$ has a repeated eigenvalue, find the possible values of k and, for each value of k, all the eigenvalues of $\mathbf{M}$. **[5]**

11. The square matrix $\mathbf{A}$ has an eigenvalue λ with corresponding eigenvector $\mathbf{e}$. The non-singular matrix $\mathbf{M}$ is of the same order as $\mathbf{A}$, and the matrix $\mathbf{B}$ is defined by $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$.

(a) Show that $\mathbf{M}^{-1}\mathbf{e}$ is an eigenvector of $\mathbf{B}$, and that λ is the corresponding eigenvalue. **[3]**

Let

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 5 & 1 \end{pmatrix}$$

(b) Write down the eigenvalues of $\mathbf{A}$ and obtain corresponding eigenvectors. **[4]**

It is given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

(c) Find $\mathbf{M}^{-1}$ and hence, without finding $\mathbf{B}$ explicitly, find the eigenvalues and corresponding eigenvectors of $\mathbf{B}$. **[5]**

12. It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

(a) Show that λ^2 is an eigenvalue of $\mathbf{A}^2$ and that λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$, and that $\mathbf{e}$ is a corresponding eigenvector in each case. **[3]**

(b) Deduce that $\lambda^2 + 2\lambda^{-1}$ is an eigenvalue of $\mathbf{A}^2 + 2\mathbf{A}^{-1}$. **[1]**

It is given that -1 is an eigenvalue of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix}$$

(c) Find a corresponding eigenvector. **[2]**

(d) It is also given that

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of the matrix $\mathbf{A}$. Find the corresponding eigenvalues. **[2]**

(e) Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\left(\mathbf{A}^2 + 2\mathbf{A}^{-1}\right)^2 = \mathbf{PDP}^{-1}$$

[4]

13. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

(a) (i) Find the characteristic equation of $\mathbf{M}$, giving your answer in the form $\lambda^3 + p\lambda^2 + q\lambda + r = 0$, where p, q and r are integers. Hence show that 1 is an eigenvalue of $\mathbf{M}$. **[3]**

(ii) Find the other two eigenvalues of $\mathbf{M}$, giving each in the form $x + yi$ where x and y are real. Deduce that 1 is the only real eigenvalue of $\mathbf{M}$. **[3]**

(b) A student claims that $\mathbf{M}$ can be written in the form $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and **both** $\mathbf{P}$ and $\mathbf{D}$ have only real entries. Using your answer to part (a)(ii), explain why the student is wrong. **[2]**

You are now given the following information.

- $\mathbf{M}$ represents a rotation in 3-D space by an acute angle about an axis through the origin.
- $\mathbf{e}$ is a real eigenvector of $\mathbf{M}$ associated with the eigenvalue 1.

(c) (i) Find a possible vector $\mathbf{e}$. **[2]**

(ii) Explain the significance of $\mathbf{e}$, and the fact that its associated eigenvalue is 1, in relation to the rotation that $\mathbf{M}$ represents. Hence write down a vector equation for the axis of this rotation. **[2]**

(iii) Explain **geometrically** why $\mathbf{M}$ has no real eigenvectors other than non-zero multiples of $\mathbf{e}$. **[1]**

14. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

(a) Obtain the characteristic equation of $\mathbf{M}$. Hence, or otherwise, find the value of $\det(\mathbf{M})$. **[3]**

(b) Find the real eigenvalue of $\mathbf{M}$, and show that the other two eigenvalues of $\mathbf{M}$ are not real.
Find an eigenvector of $\mathbf{M}$ corresponding to its real eigenvalue.
Hence, or otherwise, write down the solution of the following system of equations, and explain why the system has no other solution.

$$\begin{aligned} x \quad - \quad z &= 0.4 \\ 3x + y \quad &= -0.4 \\ x + y - 2z &= 1.2 \end{aligned}$$

[10]

15.

$$\mathbf{A} = \begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & p \\ 10 & -5 & 3 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$,

(a) (i) determine the eigenvalue corresponding to this eigenvector, **[1]**

(ii) hence show that $p = -4$, **[2]**

(iii) determine the remaining eigenvalues and corresponding eigenvectors of $\mathbf{A}$. **[7]**

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[1]**

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant. **[2]**
With respect to a fixed origin O, the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

By considering

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

(ii) determine a general solution for the displacement of the particle. **[4]**

At time $t = 0$ the particle is at the point with position vector $\begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$.

(iii) Find expressions for x, y and z in terms of t for this particle. **[2]**