OPTIONAL PURE / VECTORS & MATRICES

Cayley-Hamilton Theorem

Questions by topic 10 questions · 72 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR MEI Extra Pure Y435 · CIE 9231

Questions

Question 1	*6 marks*	2
Question 2	*5 marks*	3
Question 3	*5 marks*	4
Question 4	*6 marks*	5
Question 5	*7 marks*	6
Question 6	*7 marks*	7
Question 7	*10 marks*	8
Question 8	*8 marks*	9
Question 9	*7 marks*	10
Question 10	*11 marks*	11

1. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

(a) Find the characteristic equation of $\mathbf{A}$, simplifying your answer. **[2]**

(b) Hence, using the Cayley–Hamilton theorem, find $\mathbf{A}^{-2}$ in the form $p\mathbf{A} + q\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix and p and q are rational constants to be found. **[4]**

2. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

(a) Find the characteristic equation of $\mathbf{A}$, giving your answer in simplified form. **[2]**

(b) Hence, using the Cayley–Hamilton theorem, show that $\mathbf{A}^{-2} = p\mathbf{I} + q\mathbf{A}$, where $\mathbf{I}$ is the 3×3 identity matrix and p and q are constants to be determined. **[3]**

3. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & -2 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

Find the characteristic equation of $\mathbf{A}$ and use it to show that

$$\mathbf{A}^5 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$$

where $\mathbf{I}$ is the 3×3 identity matrix and a, b and c are integers to be determined. **[5]**

4. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ -2 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

Find the characteristic equation of $\mathbf{A}$. Hence, using the Cayley–Hamilton theorem and without calculating any power of $\mathbf{A}$, show that

$$(\mathbf{A} - 2\mathbf{I})^3 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I}$$

where $\mathbf{I}$ is the 3×3 identity matrix and p, q and r are integers to be determined. **[6]**

5. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are real constants, and $\mathbf{I}$ is the 2×2 identity matrix.

(a) Show that $\mathbf{M}$ satisfies its characteristic equation, that is,

$$\mathbf{M}^2 - (a+d)\mathbf{M} + (ad - bc)\mathbf{I} = \mathbf{0} \quad \textbf{[2]}$$

(b) Deduce that, if $a + d = 0$, then $\mathbf{M}^2 = (a^2 + bc)\mathbf{I}$.
The matrix $\mathbf{N}$ is given by

$$\mathbf{N} = \begin{pmatrix} 5 & -8 \\ 2 & -5 \end{pmatrix}$$

Hence show that

$$(\mathbf{N} + \mathbf{I})^3 = 12\mathbf{N} + 28\mathbf{I} \quad \textbf{[5]}$$

6. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & 3 \\ -4 & 5 \end{pmatrix}$$

where a is a constant.

(a) Find, in expanded form in terms of a, the characteristic equation of $\mathbf{A}$. **[2]**

(b) Hence use the Cayley–Hamilton theorem to find all possible pairs of values of a and b such that

$$\mathbf{A}^3 = 7\mathbf{A} + b\mathbf{I}$$

where $\mathbf{I}$ is the 2×2 identity matrix. **[5]**

7.

$$\mathbf{M} = \begin{pmatrix} 3 & -3 & 1 \\ 2 & -1 & 2 \\ 0 & k & 2 \end{pmatrix}$$

where k is a constant.

(a) Show that a characteristic equation for $\mathbf{M}$ is

$$\lambda^3 - 4\lambda^2 + (7 - 2k)\lambda + 4k - 6 = 0 \quad \textbf{[2]}$$

It is given that 3 is an eigenvalue of $\mathbf{M}$.

(b) (i) Find the value of k and the other two eigenvalues of $\mathbf{M}$. **[3]**

(ii) Use the Cayley–Hamilton theorem to find $\mathbf{M}^{-1}$. **[5]**

8. The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} a & 4 \\ 1 & a-3 \end{pmatrix}$$

where a is a constant.
You are given that $\mathbf{P}$ satisfies the matrix equation $\mathbf{P}^2 + r\mathbf{P} = 6\mathbf{I}$ for some integer r, where $\mathbf{I}$ is the 2×2 identity matrix.

(a) Use the Cayley–Hamilton theorem to find the two possible values of a, and the corresponding value of r in each case. **[4]**

(b) In the case $a > 1$, find integers s and t such that $\mathbf{P}^4 = s\mathbf{P} + t\mathbf{I}$. You should not calculate $\mathbf{P}^4$.
Express $\mathbf{P}^{-1}$ in the form $\alpha\mathbf{P} + \beta\mathbf{I}$, where α and β are rational numbers to be found. **[4]**

9. It is given that k is a positive constant.
The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & k+1 & 2 \\ 0 & k & 0 \\ 4 & 2k & -1 \end{pmatrix}$$

(a) Show that the characteristic equation of $\mathbf{A}$ is

$$\lambda^3 - k\lambda^2 - 9\lambda + 9k = 0$$ **[2]**

(b) Hence use the Cayley–Hamilton theorem to find $\mathbf{A}^{-1}$ in terms of k. **[5]**

10. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} b & 2 & -1 \\ 0 & 0 & a \\ a & 2 & 3 \end{pmatrix}$$

where a and b are integers and $a > 0$.
The characteristic equation of $\mathbf{M}$ is

$$\lambda^3 + \lambda^2 + c\lambda - 24 = 0$$

where c is a constant.

(a) Determine the values of a, b and c. **[6]**

(b) Hence, using the Cayley–Hamilton theorem, find $(\mathbf{M} - 2\mathbf{I})^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. **[5]**