OPTIONAL PURE / VECTORS & MATRICES

Plane Intersections

Mark scheme 10 questions · 83 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 CIE 9231

Questions

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	Total	**83**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$\det(\mathbf{M}) = 1(2-3) - 1(-4-1) + 1(6+1)$ $= \ldots$	**M1**	Expands the determinant of the coefficient matrix along any row or column, three 2×2 minors attempted, or by Sarrus' rule. Condone sign slips. Not for a calculator value alone.
	$\det(\mathbf{M}) = 11 \neq 0$ Non-singular, so the planes meet at a single point	**A1 AG**	Correct expansion leading without error to 11, with $11 \neq 0$ and a conclusion in words that the planes meet at a single point.
		[2]	
1(b)	$\mathbf{M}^{-1} = \frac{1}{11}\begin{pmatrix} -1 & 5 & 2 \\ 5 & -3 & 1 \\ 7 & -2 & -3 \end{pmatrix}$	**B1**	Correct inverse, from a calculator or otherwise.
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{“}\mathbf{M}^{-1}\text{”} \begin{pmatrix} 4 \\ 8 \\ -7 \end{pmatrix} = \ldots$	**M1**	Multiplies their 3×3 inverse by the column of constants, on its right, attempting all three components.
	$(2, -1, 3)$	**A1**	Accept $x = 2$, $y = -1$, $z = 3$ or the column vector.
		[3]	
	Total for Question 1	**5**	

Question	Scheme	Marks	Notes
2(a)	$\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}2\\-1\\3\end{pmatrix} = \ldots$	**M1**	Attempts $\mathbf{n}_1 \times \mathbf{n}_2$, all three components, condoning sign slips; a product written without working needs two components correct. Or any valid method for the direction.
	$\begin{pmatrix}7\\-1\\-5\end{pmatrix}$	**A1**	Or any nonzero multiple.
	Sets $z = 0$: $x + 2y = 4,\ 2x - y = 3 \Rightarrow x = \ldots,\ y = \ldots$	**M1**	Finds a point on both planes, by fixing one coordinate and solving the resulting pair or by any valid method. Condone arithmetic slips.
	$(2, 1, 0)$ $\mathbf{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}7\\-1\\-5\end{pmatrix},\ \lambda \in \mathbb{R}$	**A1**	Any correct point on L with any nonzero multiple of the direction, in a vector equation of the form $\mathbf{r} = \ldots$ oe.
		[4]	
2(b)	$3(2 + 7\lambda) - 4(1 - \lambda) + 5(-5\lambda)$	**M1**	Substitutes their general point $\mathbf{a} + \lambda\mathbf{d}$ into $3x - 4y + 5z$; or checks a point and $\mathbf{d} \cdot \mathbf{n}_3$, two points of L, or Π_3 as a combination of Π_1 and Π_2.
	$= 6 + 21\lambda - 4 + 4\lambda - 25\lambda = 2$ for all λ, so L lies in Π_3	**A1 AG**	Correct substitution simplified to 2 without error, with a conclusion that it holds for all λ; or that route's checks all correct, with a conclusion.
		[2]	
2(c)	The three planes form a sheaf, meeting in the common line L	**E1**	States that the planes form a sheaf, or that all three meet in the common line L.
		[1]	
	Total for Question 2	**7**	

Question	Scheme	Marks	Notes
3(a)	$\begin{pmatrix}4\\-2\\6\end{pmatrix} = 2\begin{pmatrix}2\\-1\\3\end{pmatrix}$, and Π_2: $2x - y + 3z = \frac{5}{2} \neq 4$	**B1**	Both needed, normals shown to be multiples and the equations shown inconsistent, oe $8 \neq 5$.
		[1]	
3(b)	No point lies on both Π_1 and Π_2, so no common solution	**E1**	States that no point lies on both parallel distinct planes, so the three equations have no common solution.
	$\begin{pmatrix}1\\1\\-1\end{pmatrix}$ not a multiple of $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ Two parallel planes cut by the third in two parallel lines	**E1**	States two parallel planes cut by a third, non-parallel plane, oe; the lines of intersection need not be called parallel.
		[2]	
3(c)	$2x - y + 3z = \frac{k}{2}$, $\frac{k}{2} = 4 \Rightarrow k = 8$	**B1**	
	Π_1 and Π_2 coincide; Π_3 meets them in a line	**E1**	States the first two planes coincide (are the same plane) and the third meets them in a line. Not for sheaf or a common line alone.
		[2]	
	Total for Question 3	**5**	

Question	Scheme	Marks	Notes
4(a)	$\det \mathbf{A} = 3(-7-0) + 2(-14+28) + 1(0-7)$	**M1**	Expands along any row or column, with three 2×2 minors and their signs. Condone one slip in a minor.
	$= -21 + 28 - 7 = 0$ so $\mathbf{A}$ is singular and there is no unique solution	**A1 AG**	Correct expansion leading without error to 0, then both conclusions, singular and no unique solution.
		[2]	
4(b)	$(3x - 2y + z) + 2(2x + y - 4z) = 5 + 2(1)$	**M1**	Combines two equations to compare with the third, or solves two and substitutes into the third. Condone arithmetic slips.
	$\Rightarrow 7x - 7z = 7$, the third equation, so the system is consistent	**A1**	Correct combination matching the third equation, constants included, and consistent stated.
	The three planes meet in a common line (a sheaf) $\left[\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right]$	**E1**	States the planes meet in a common line, or form a sheaf. Not for infinitely many solutions alone; the line's equation is not needed.
		[3]	
Total for Question 4		**5**	

Question	Scheme	Marks	Notes
5(a)	$\det \mathbf{A} = 1(-7) - 2(-6) - 1(5) = 0$	**B1**	$\det \mathbf{A} = 0$ for the coefficient matrix; a calculator value is accepted.
	A is singular, so no unique solution	**E1**	States that **A** is singular or not invertible, so there is no unique solution. Needs $\det \mathbf{A} = 0$.
		[2]	
5(b)	$(x + 2y - z) + (2x - y + 4z) = 3 + 1$	**M1**	Adds the first two equations, or eliminates one variable from the system by any method, and compares with the third.
	$\Rightarrow 3x + y + 3z = 4$, the third equation, so consistent	**A1**	Correct sum identified with the third equation, or $0 = 0$ by elimination, and a conclusion of consistency.
	The three planes meet in a common line (a sheaf)	**E1**	States the planes meet in a single common line, or form a sheaf. Not for infinitely many solutions alone.
		[3]	
5(c)	First two equations $\Rightarrow 3x + y + 3z = 4$	**M1**	Combines the first two equations to give $3x + y + 3z = 4$, or eliminates by any method. May be seen in (b).
	$3x + y + 3z = a \neq 4$: contradiction, so inconsistent	**A1**	Contradiction with the third equation for $a \neq 4$ and a conclusion of inconsistency.
	Normals $(1, 2, -1)$, $(2, -1, 4)$, $(3, 1, 3)$: no two parallel $\Rightarrow$ triangular prism	**E1**	States triangular prism, with a reason no two planes are parallel, such as no two normals being multiples.
		[3]	
	Total for Question 5	**8**	

Question	Scheme	Marks	Notes
6(a)	$\begin{vmatrix}1 & -1 & 2\\2 & 1 & 1\\3 & 0 & m+1\end{vmatrix}$ $= 1(m+1)+1\big(2(m+1)-3\big)+2(0-3)$	**M1**	Expands the determinant of the coefficient matrix in terms of m by any row or column. Condone sign slips.
	$= 3(m-2)$	**A1**	$3(m-2)$ oe, such as $3m-6$.
	$m \neq 2 \Rightarrow \det \neq 0$: the planes meet in a single point	**E1**	Relates the non-zero determinant to a unique point of intersection. The point need not be found.
	$m = 2$: first + second gives $3x+3z=7$ third: $3x+3z=7$	**M1**	Substitutes $m=2$ and tests consistency, by combining the first two equations to compare with the third, or by elimination.
	Third equation is the sum of the first two, so consistent	**A1**	Third equation shown to be the sum of the first two, or elimination reaching $0=0$, with consistency stated or implied.
	The planes meet in a common line (a sheaf)	**E1**	States the planes share one common line, or form a sheaf, from a correct consistency result. The line need not be found.
		[6]	
6(b)	$\begin{vmatrix}1 & -1 & 2\\2 & 1 & 1\\3 & 0 & 4\end{vmatrix} = 1(4)+1(8-3)+2(0-3)$	**M1**	Forms the determinant or scalar triple product of the three vectors and attempts to evaluate it, or uses another valid coplanarity test. Condone sign slips.
	$= 3 \neq 0$, so not coplanar: the student is incorrect	**A1**	Value ± 3, depending on order, and concludes the student is incorrect.
		[2]	
	Total for Question 6	**8**	

Question	Scheme	Marks	Notes
7(a)	$\det \mathbf{M} = 1(k+1) - 1(3-2) + 0$	**M1**	Expands the 3×3 determinant along any row or column, with every nonzero 2×2 minor attempted. Condone sign slips.
	$\det \mathbf{M} = k$, so an inverse exists for $k \neq 0$	**A1**	
		[2]	
7(b)	$\mathbf{M}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ -7 & 3 & -1 \end{pmatrix}$	**B1**	Correct inverse with $k = 2$, from a calculator or otherwise.
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$ "$\mathbf{M}^{-1}$" $\begin{pmatrix} 1 \\ p \\ 4 \end{pmatrix} = \ldots$	**M1**	Multiplies their 3×3 inverse by $\begin{pmatrix} 1 \\ p \\ 4 \end{pmatrix}$, in that order, for at least one component.
	$x = \frac{7-p}{2}$	**A1**	oe. Coordinates may be given as a column vector.
	$y = \frac{p-5}{2}$	**A1**	
	$z = \frac{3p-11}{2}$	**A1**	
	Alternative: Elimination		
	$y = 1 - x \Rightarrow 3x + 2(1-x) + z = p$ $2x - (1-x) + z = 4$	**M1**	Eliminates one variable from two pairs of the equations, giving two equations in the other two. Condone sign slips.
	$x + z = p - 2, \quad 3x + z = 5$	**A1**	oe, or the correct pair for another eliminated variable.
	$2x = 5 - (p-2) \Rightarrow x = \ldots$ $y = 1 - x = \ldots, \quad z = 5 - 3x = \ldots$	**dM1**	Solves their pair for one variable and substitutes back to find the other two.
	$x = \frac{7-p}{2}$	**A1**	oe. Coordinates may be given as a column vector.
	$y = \frac{p-5}{2}, \quad z = \frac{3p-11}{2}$	**A1**	Both correct, oe.
		[5]	
7(c)(i)	$(3x + z) - (x + y) = 2x - y + z \Rightarrow q - 1 = 4$	**M1**	Combines two of the equations to match the variable terms of the third, or eliminates variables to compare with it, reaching an equation in q. Condone sign slips.
	$q = 5$	**A1**	
		[2]	

Question 7 continued

Question	Scheme	Marks	Notes
7(c)(ii)	Third equation follows from the first two, so infinitely many solutions	**E1**	States that one equation is a combination of the other two, or that the system is consistent with no unique solution, and so has infinitely many solutions.
	The three planes meet in a common line (a sheaf)	**E1**	States that the planes meet in a single common line; "sheaf" alone accepted. The line's equation is not needed.
		[2]	
	Total for Question 7	**11**	

Question	Scheme	Marks	Notes
8(a)	$\begin{vmatrix} 1 & -1 & 2 \\ k+1 & k & 2-k \\ 5-2k & 2-2k & k+2 \end{vmatrix} = 0$ $\Rightarrow k(k+2) - (2-k)(2-2k) + \ldots = 0$	**M1**	Sets the determinant of the coefficient matrix to zero and expands it, with at least two of the three 2×2 terms correct in form. Condone sign slips.
	$-2k^2 + 10k - 8 = 0$	**A1**	Correct quadratic, oe such as $k^2 - 5k + 4 = 0$.
	$-2(k-1)(k-4) = 0 \Rightarrow k = \ldots$	**dM1**	Solves their three-term quadratic by any means, a calculator allowed.
	$k = 1$ or $k = 4$	**A1**	Both values and no others.
		[4]	
8(b)	$k = 1$: $(x - y + 2z) + (2x + y + z) = 4 + 3$ $\Rightarrow 3x + 3z = 7$	**M1**	Substitutes $k = 1$ and combines two equations to compare with the third, or eliminates a variable from two pairs and compares.
	Same as the third equation, so consistent	**A1**	Shows the third equation is the sum of the first two and states consistent.
	$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ not parallel $\Rightarrow$ the planes meet in a common line: a sheaf	**E1**	States sheaf, or a common line, with at least two normals shown not parallel.
	$k = 4$: $2(x - y + 2z) - (5x + 4y - 2z) = 2 \times 4 - 0$ $\Rightarrow -3x - 6y + 6z = 8$	**M1**	Substitutes $k = 4$ and combines two equations to compare with the third, or eliminates a variable from two pairs and compares. Condone arithmetic slips.
	But $-3x - 6y + 6z = 4$: $8 \neq 4$, so inconsistent	**A1**	Correct contradiction and states inconsistent.
	$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ 6 \end{pmatrix}$: no two parallel $\Rightarrow$ triangular prism	**E1**	States triangular prism, with all three pairs of normals shown or stated not parallel. E0 without the normals.
		[6]	
	Total for Question 8	**10**	

Question	Scheme	Marks	Notes
9(a)(i)	$\det \mathbf{A} = 1(2-b) - a(4-3) = 0$	**M1**	Sets an expansion of the determinant of the coefficient matrix equal to zero. Condone one sign slip.
	$b = 2 - a$	**A1**	oe
		[2]	
9(a)(ii)	$x = -1 - ay, \quad z = 5 + (2a-1)y$	**M1**	Finds the line of intersection of two planes, or a point on it, or reduces to two equations in two variables, or writes one plane as a combination of the others. Condone sign slips.
	$3(-1-ay) + (2-a)y + 2\big(5 + (2a-1)y\big) = c$	**dM1**	Uses their b in terms of a with their line, point, reduced equations or combination to find c.
	$c = 7$	**A1**	
		[3]	
9(b)	$x = -1 - ay \Rightarrow 2(-1-ay) + y + z = 3$	**M1**	Eliminates a variable between two of the equations, giving one variable in terms of the others. Condone sign slips.
	$z = 5 + (2a-1)y$	**A1**	oe, or the correct result of another elimination.
	$3(-1-ay) + (1-a)y + 2\big(5 + (2a-1)y\big) = 5$ $\Rightarrow 7 - y = 5 \Rightarrow y = \ldots$	**dM1**	Substitutes their expressions into the third equation, with $b = 1 - a$ and $c = 5$, and solves for the remaining variable.
	$y = 2$	**A1**	
	$x = -1 - 2a, \quad z = 4a + 3$	**A1**	Both, oe, as coordinates or separate values.
		[5]	
	Total for Question 9	**10**	

Question	Scheme	Marks	Notes
10(a)	$\det \mathbf{A} = 3(3-0) - 2(1-0) + a(-1-6)$	**M1**	Expands along any row or column, three terms each an element times its 2×2 determinant, or uses Sarrus' rule. Condone sign slips.
	$\det \mathbf{A} = 7(1-a)$	**A1**	oe, such as $7 - 7a$.
		[2]	
10(b)(i)	Minors: $\begin{pmatrix} 3 & 1 & -7 \\ 2+a & 3-2a & -7 \\ -3a & -a & 7 \end{pmatrix}$	**M1**	Attempts all nine 2×2 minors; implied by a matrix of cofactors.
	$\text{adj}\,\mathbf{A} = \begin{pmatrix} 3 & -a-2 & -3a \\ -1 & 3-2a & a \\ -7 & 7 & 7 \end{pmatrix}$	**M1**	Applies the alternating signs and transposes their matrix of minors, in either order.
	$\mathbf{A}^{-1} = \frac{1}{7(1-a)} \begin{pmatrix} 3 & -a-2 & -3a \\ -1 & 3-2a & a \\ -7 & 7 & 7 \end{pmatrix}$	**M1 A1**	**M1** Divides their adjugate by their determinant from (a), in terms of a. **A1** Correct inverse, oe; the factor may be multiplied in.
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{“}\mathbf{A}^{-1}\text{”} \begin{pmatrix} 8 \\ 5 \\ 3 \end{pmatrix}$ $= \frac{1}{7(1-a)} \begin{pmatrix} 14-14a \\ 7-7a \\ 0 \end{pmatrix}$	**M1**	Multiplies their $\mathbf{A}^{-1}$, in terms of a, by $(8,5,3)^T$ in that order, at least one element attempted.
	$x = 2,\ y = 1,\ z = 0$	**A1**	cso; all three values.
		[6]	
10(b)(ii)	With $a = 1$, adds the second and third equations: $(x + 3y) + (2x - y + z) = 5 + 3$	**M1**	Sets $a = 1$ and compares a combination of two equations with the third.
	$\Rightarrow 3x + 2y + z = 8$, the first equation, so consistent	**A1**	Shows the first equation is the sum of the others, oe, and concludes the equations are consistent or have solutions.
	The three planes meet in a common line: a sheaf	**E1**	States the planes share a single common line, or form a sheaf. No check that the normals are non-parallel is needed.
		[3]	
10(c)(i)	$a > 1$	**B1**	
		[1]	

Question 10 continued

Question	Scheme	Marks	Notes
10(c)(ii)	$\lvert 7(1-a)\rvert(a^2+3) < a^2+3 \Rightarrow \lvert 7(1-a)\rvert < 1$	**M1**	Uses the modulus of their determinant from (a) as the volume scale factor, reaching $\lvert\det \mathbf{A}\rvert < 1$ or both cases. Not for $7(1-a) < 1$ alone.
	$\frac{6}{7} < a < \frac{8}{7}$	**A1**	oe, such as $\left(\frac{6}{7}, \frac{8}{7}\right)$.
		[2]	
	Total for Question 10	**14**	