OPTIONAL PURE / VECTORS & MATRICES

Orthogonal Diagonalisation

Mark scheme 10 questions · 131 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

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	Total	**131**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$(k-10)(-2-10)-36=0 \Rightarrow k=\ldots$	**M1**	Substitutes $\lambda=10$ into $\det(\mathbf{A}-\lambda\mathbf{I})=0$, or uses another valid eigenvalue condition, and solves for k. Condone sign slips.
	$k=7$	**A1**	
	$\lambda^2-5\lambda-50=0 \Rightarrow \lambda=\ldots$	**M1**	Forms the characteristic equation with their k and solves it by any means, or uses the trace or determinant with their k. Condone sign slips.
	$\lambda=-5$	**A1**	
		[4]	
1(b)	$\begin{pmatrix}-3 & 6\\ 6 & -12\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\mathbf{0} \Rightarrow x=\ldots$	**M1**	Uses $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ or $\mathbf{Ax}=\lambda\mathbf{x}$ with their k and either of their eigenvalues, reaching a relation between x and y.
	$\begin{pmatrix}2\\ 1\end{pmatrix}$ for $\lambda=10$, $\begin{pmatrix}1\\ -2\end{pmatrix}$ for $\lambda=-5$	**A1**	Both correct, or any nonzero multiples, each with its eigenvalue.
	$\sqrt{2^2+1^2}=\sqrt{5}$ $\Rightarrow \frac{1}{\sqrt{5}}\begin{pmatrix}2\\ 1\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ -2\end{pmatrix}$	**M1**	Divides each of their two eigenvectors by its magnitude.
	$\mathbf{P}=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix}$	**A1**	Unit eigenvectors as columns, in either order and of either sign; oe, including decimals awrt 0.894 and 0.447.
	$\mathbf{D}=\begin{pmatrix}10 & 0\\ 0 & -5\end{pmatrix}$	**B1**	Eigenvalues 10 and -5 on the diagonal, in the order of the corresponding eigenvector columns of $\mathbf{P}$.
		[5]	
	Total for Question 1	**9**	

Question	Scheme	Marks	Notes
2(a)	$\begin{pmatrix} 2k-5 \\ 14 \\ k+8 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$	**M1**	Forms $\mathbf{Mv}$ and equates at least two components to those of $\lambda\mathbf{v}$. Condone one arithmetic slip.
	$2\lambda = 14 \Rightarrow \lambda = 7$	**A1**	
	$k + 8 = 14 \Rightarrow k = 6$	**A1**	
		[3]	
2(b)	$\begin{vmatrix} 3-\lambda & -4 & \text{“6”} \\ -4 & 7-\lambda & 2 \\ \text{“6”} & 2 & 2-\lambda \end{vmatrix}$ $= (3-\lambda)\left[(7-\lambda)(2-\lambda)-4\right]$ $+ 4\left[-4(2-\lambda)-12\right] + 6\left[-8-6(7-\lambda)\right]$	**M1**	Forms $\det(\mathbf{M}-\lambda\mathbf{I})$ with their k and expands it to a cubic, or uses the trace and determinant of $\mathbf{M}$ with eigenvalue 7. Condone sign slips.
	$\lambda^3 - 12\lambda^2 - 15\lambda + 350 = 0$	**A1**	Correct cubic, oe, or a correct quadratic factor $\lambda^2 - 5\lambda - 50$; $= 0$ may be implied.
	$(\lambda - 7)(\lambda^2 - 5\lambda - 50) = 0 \Rightarrow \lambda = 10,\ \lambda = -5$	**A1**	
		[3]	
2(c)	$\lambda =$ “10”: $\begin{pmatrix} -7 & -4 & 6 \\ -4 & -3 & 2 \\ 6 & 2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow \ldots$	**M1**	Solves $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for one of their other eigenvalues by any method, such as elimination or a vector product of two rows. Condone sign slips.
	$\lambda = 10$: $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$	**A1**	Any non-zero multiple.
	$\lambda = -5$: $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$	**A1**	Any non-zero multiple.
	$\sqrt{1^2+2^2+2^2} = 3 \Rightarrow \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ldots$	**M1**	Divides each of the three eigenvectors, $\mathbf{v}$ and their two, by its magnitude. Condone a slip in one magnitude.
	$\mathbf{P} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$	**A1**	The three unit eigenvectors as columns, in any order and with either sign. A0 if they are the rows.

Question 2 continued

Question	Scheme	Marks	Notes
	$\mathbf{D} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -5 \end{pmatrix}$	**B1**	Correct eigenvalues, in the order of the eigenvector columns of their **P**.
		[6]	
	Total for Question 2	**12**	

Question	Scheme	Marks	Notes
3(a)	Equates third components: $2+2+10=2\lambda \Rightarrow \lambda = \ldots$	**M1**	Multiplies $\mathbf{M}$ by the eigenvector and equates the third component to 2λ. Condone one arithmetic slip.
	$\lambda = 7$	**A1**	
		[2]	
3(b)	$8-a=$ "7" $a-b-4=-$"7"	**M1**	Equates the first or second component of $\mathbf{M}$ times the eigenvector to their λ times that component. Condone sign slips.
	$a=1$	**A1**	
	$b=4$	**A1**	
		[3]	
3(c)	$\det(\mathbf{M}-\lambda\mathbf{I}) = (4-\lambda)\left(\lambda^2-9\lambda+16\right)$ $-(9-\lambda)+2(2\lambda-10)$	**M1**	Forms $\det(\mathbf{M}-\lambda\mathbf{I})$ with their a and b and expands it to a cubic, or uses the trace and determinant of $\mathbf{M}$ with eigenvalue 7. Condone sign slips.
	$\lambda^3-13\lambda^2+47\lambda-35=0$	**A1**	oe, or a correct quadratic factor $\lambda^2-6\lambda+5$.
	$(\lambda-7)\left(\lambda^2-6\lambda+5\right)=0 \Rightarrow \lambda = 5,\ 1$	**A1**	Both values. Roots of the correct cubic from a calculator accepted.
		[3]	
3(d)	$\lambda =$ "5": $2x-2y=0,\ -x+y+2z=0 \Rightarrow \ldots$	**M1**	Forms $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for one of their other eigenvalues and solves two equations, or takes a vector product of two rows. Condone sign slips.
	$\lambda = 5$: $\begin{pmatrix}1\\1\\0\end{pmatrix}$	**A1**	Any non-zero multiple.
	$\lambda = 1$: $\begin{pmatrix}-1\\1\\1\end{pmatrix}$	**A1**	Any non-zero multiple.
	$\sqrt{1+1+4}=\sqrt{6},\ \sqrt{2},\ \sqrt{3}$	**M1**	Divides at least one of their eigenvectors by its magnitude, the square root of the sum of the squares of its components.

Question 3 continued

Question	Scheme	Marks	Notes
	$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}$	**A1**	All three, each up to sign. Any exact form, or decimals awrt 3 s.f.
		[5]	
3(e)	$\mathbf{D} = \begin{pmatrix}7&0&0\\0&5&0\\0&0&1\end{pmatrix}$	**B1**	
	$\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{2}{\sqrt{6}}&0&\frac{1}{\sqrt{3}}\end{pmatrix}$	**B1ft**	ft their normalised eigenvectors from (d) as columns, in the order of the eigenvalues in **D**. Each column up to sign.
		[2]	
	Total for Question 3	**15**	

Question	Scheme	Marks	Notes
4(a)(i)	$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = \ldots$	**M1**	Forms the vector product of the two given eigenvectors, or sets both scalar products to zero and solves. Condone sign slips; unworked, needs two components correct.
	$\begin{pmatrix}1\\-1\\-2\end{pmatrix}$	**A1**	Any non-zero multiple.
		[2]	
4(a)(ii)	Divides each eigenvector by its magnitude: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$	**M1**	Divides each of their three eigenvectors by its magnitude. Condone a slip in one magnitude.
	$\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\end{pmatrix}$	**A1**	Any correct orthogonal $\mathbf{P}$ of unit eigenvectors, columns in any order and any column negated; surds need not be rationalised.
	$\mathbf{D} = \begin{pmatrix}4 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 6\end{pmatrix}$	**B1**	Diagonal entries 4, -3, 6 in the order of the corresponding eigenvectors in the columns of $\mathbf{P}$.
		[3]	
4(b)	$\mathbf{PP^{T}MPP^{T}} = \mathbf{PDP^{T}}$	**M1**	Multiplies $\mathbf{P^{T}MP} = \mathbf{D}$ by $\mathbf{P}$ on the left and $\mathbf{P^{T}}$ on the right, in either order, or inverts using $\mathbf{P}^{-1} = \mathbf{P^{T}}$.
	$\mathbf{PP^{T}} = \mathbf{I} \Rightarrow \mathbf{IMI} = \mathbf{PDP^{T}}$ $\Rightarrow \mathbf{M} = \mathbf{PDP^{T}}$	**A1 AG**	States $\mathbf{PP^{T}} = \mathbf{I}$, or $\mathbf{P}^{-1} = \mathbf{P^{T}}$, and uses it on each side of $\mathbf{M}$, reaching the printed result without error; combined steps accepted.
		[2]	
4(c)	$\mathbf{PD} =$ "$\mathbf{P}$" $\begin{pmatrix}4 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 6\end{pmatrix} = \ldots$	**M1**	Multiplies their $\mathbf{P}$ by their $\mathbf{D}$, or their $\mathbf{D}$ by their $\mathbf{P^{T}}$, in that order, reaching a 3×3 matrix. Condone slips.
	$\mathbf{PD} = \begin{pmatrix}2\sqrt{2} & -\sqrt{3} & \sqrt{6}\\ 2\sqrt{2} & \sqrt{3} & -\sqrt{6}\\ 0 & -\sqrt{3} & -2\sqrt{6}\end{pmatrix}$	**A1**	Correct product for a correct $\mathbf{P}$ and $\mathbf{D}$, any equivalent form.

Question 4 continued

Question	Scheme	Marks	Notes
	$\mathbf{M} =$ "$\mathbf{PD}$" "$\mathbf{P}^{\mathrm{T}}$" $= \ldots$	**dM1**	Multiplies their **PD** by their $\mathbf{P}^{\mathrm{T}}$, or their **P** by their $\mathbf{DP}^{\mathrm{T}}$, in that order, reaching a 3×3 matrix.
	$\mathbf{M} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}$	**A1 A1**	**A1** At least five correct entries. **A1** cao, all nine entries correct.
		[5]	
	Total for Question 4	**12**	

Question	Scheme	Marks	Notes
5(a)	$\det(\mathbf{A}-\mathbf{I}) = 0 + 2(0-4) + 2\left(-4-2(p-1)\right)$	**M1**	Forms $\mathbf{A}-\mathbf{I}$, or $\mathbf{A}-\lambda\mathbf{I}$ with $\lambda = 1$, and expands its determinant with three terms. Condone sign slips.
	$= -12 - 4p$	**A1**	oe
	$-12 - 4p = 0 \Rightarrow p = -3$	**A1 AG**	Correct determinant set equal to zero, leading without error to $p = -3$.
		[3]	
5(b)(i)	$\det(\mathbf{A}-\lambda\mathbf{I}) = (1-\lambda)\left[(-3-\lambda)(1-\lambda)-4\right]$ $+ 2\left[-2(1-\lambda)-4\right] + 2\left[-4-2(-3-\lambda)\right]$	**M1**	Forms $\det(\mathbf{A}-\lambda\mathbf{I})$ with $p = -3$ and expands with three terms, or uses both the trace and the determinant of $\mathbf{A}$ with eigenvalue 1. Condone sign slips.
	$(1-\lambda)(\lambda-3)(\lambda+5) = 0$	**A1**	Correct cubic, oe, such as $\lambda^3 + \lambda^2 - 17\lambda + 15 = 0$, or a correct quadratic factor $\lambda^2 + 2\lambda - 15$.
	$\lambda = 3,\ \lambda = -5$	**A1**	
		[3]	
5(b)(ii)	$\lambda = 1$: $-2y + 2z = 0,\ 2x + 2y = 0 \Rightarrow z = y,\ x = -y$	**M1**	Uses $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for one of their eigenvalues, solving two equations or taking a vector product of two rows, to reach a non-zero vector. Condone sign slips.
	$\lambda = 1$: $\begin{pmatrix}-1\\1\\1\end{pmatrix}$	**A1**	Or any non-zero multiple.
	$\lambda = 3$: $\begin{pmatrix}1\\0\\1\end{pmatrix}$	**A1**	Or any non-zero multiple.
	$\lambda = -5$: $\begin{pmatrix}1\\2\\-1\end{pmatrix}$	**A1**	Or any non-zero multiple.
		[4]	
5(c)	$\frac{1}{\sqrt{3}}\text{“}\begin{pmatrix}-1\\1\\1\end{pmatrix}\text{”},\ \frac{1}{\sqrt{2}}\text{“}\begin{pmatrix}1\\0\\1\end{pmatrix}\text{”},\ \frac{1}{\sqrt{6}}\text{“}\begin{pmatrix}1\\2\\-1\end{pmatrix}\text{”}$	**M1**	Divides each of their three eigenvectors from (b)(ii) by its magnitude. Condone a slip in one magnitude.

Question 5 continued

Question	Scheme	Marks	Notes
	$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$	**A1**	Correct unit eigenvectors as the columns, in any order, each up to sign, oe. Not for the vectors as rows.
	$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$	**B1**	Correct eigenvalues on the leading diagonal, in the order of the corresponding columns of $\mathbf{P}$.
		[3]	
Total for Question 5		**13**	

Question	Scheme	Marks	Notes
6(a)	$\mathbf{M}\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}-10\\5\\a-8\end{pmatrix} = \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$	**M1**	Multiplies $\mathbf{M}$ by the vector and equates at least one component to λ times the vector's. Condone one arithmetic slip.
	$\lambda = -5$	**A1**	
	$a - 8 = -5 \Rightarrow a = 3$	**A1**	
		[3]	
6(b)	$\begin{pmatrix}-8 & 4 & -4\\4 & -4 & 0\\-4 & 0 & -4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \mathbf{0}$ $\Rightarrow y = x,\ z = -x$	**M1**	Forms $(\mathbf{M} - 7\mathbf{I})\mathbf{x} = \mathbf{0}$ or $\mathbf{Mx} = 7\mathbf{x}$ with their numerical a, and solves for the ratio $x : y : z$, or takes the vector product of two rows. Condone one slip.
	$\begin{pmatrix}1\\1\\-1\end{pmatrix}$	**A1**	Any non-zero multiple.
		[2]	
6(c)	$\begin{pmatrix}2\\-1\\1\end{pmatrix} \times \text{“}\begin{pmatrix}1\\1\\-1\end{pmatrix}\text{”} = \ldots$	**M1**	Takes the vector product of the two eigenvectors, all three components attempted, or finds the third eigenvalue and solves $(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. Condone sign slips; unworked, needs two components correct.
	$\begin{pmatrix}0\\3\\3\end{pmatrix}$, or $\begin{pmatrix}0\\1\\1\end{pmatrix}$	**A1**	Any non-zero multiple of $\mathbf{j} + \mathbf{k}$.
	$\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\1\end{pmatrix},\ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$	**M1**	Divides each of their three eigenvectors by its magnitude. Condone a slip in one magnitude.
	$\mathbf{P} = \begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0\\-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{pmatrix}$	**A1**	Correct unit eigenvectors as columns, in any order, any column multiplied by -1; oe exact or awrt 3 s.f. entries.

Question 6 continued

Question	Scheme	Marks	Notes
	$\mathbf{D} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	**B1**	Eigenvalues -5, 7, 3 on the leading diagonal, in the order of the corresponding columns of $\mathbf{P}$, zeros elsewhere.
		[5]	
	Total for Question 6	**10**	

Question	Scheme	Marks	Notes
7(a)	$\begin{vmatrix} 9-\lambda & -2 & 2 \\ -2 & 6-\lambda & 4 \\ 2 & 4 & 6-\lambda \end{vmatrix}$ $= (9-\lambda)\left[(6-\lambda)^2-16\right]$ $\quad + 2\left[-2(6-\lambda)-8\right] + 2\left[-8-2(6-\lambda)\right]$	**M1**	Expands $\det(\mathbf{A}-\lambda\mathbf{I})$, or $\det(\lambda\mathbf{I}-\mathbf{A})$, towards a cubic, or uses the trace and determinant of $\mathbf{A}$ with eigenvalue 1. Condone slips.
	$\lambda^3 - 21\lambda^2 + 120\lambda - 100 = 0$	**A1**	Correct cubic oe, such as $-\lambda^3 + 21\lambda^2 - 120\lambda + 100$, or a correct quadratic factor $\lambda^2 - 20\lambda + 100$; condone $= 0$ missing.
	$(\lambda-1)\left(\lambda^2-20\lambda+100\right) = 0 \Rightarrow \lambda = \ldots$	**dM1**	Takes out the factor $\lambda - 1$ from their cubic, by any method, to a quadratic factor, and solves it.
	$(\lambda-1)(\lambda-10)^2 = 0$ $\lambda = 10$, repeated	**A1**	Needs $(\lambda-10)^2$, both roots of the quadratic 10, or a statement that 10 is repeated.
		[4]	
7(b)	$8x - 2y + 2z = 0, \quad -2x + 5y + 4z = 0$ $\Rightarrow x : y : z = \ldots$	**M1**	Solves two equations from $(\mathbf{A}-\mathbf{I})\mathbf{x} = \mathbf{0}$ to a ratio, or takes the vector product of two rows of $\mathbf{A}-\mathbf{I}$. Condone arithmetic slips.
	$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$	**A1**	Any non-zero multiple.
	$-x - 2y + 2z = 0 \Rightarrow x + 2y - 2z = 0$	**M1**	Obtains a single condition for eigenvectors with eigenvalue 10, from $(\mathbf{A}-10\mathbf{I})\mathbf{x} = \mathbf{0}$ or perpendicularity to their eigenvector for 1.
	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	**A1**	Any non-zero vector satisfying $x + 2y - 2z = 0$.
	$y + z = 0$ and $x + 2y - 2z = 0 \Rightarrow \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$	**A1**	Any non-zero vector satisfying $x + 2y - 2z = 0$ and perpendicular to their other eigenvector for 10, by any method such as a vector product.
		[5]	
7(c)	$\sqrt{1+4+4} = 3,$ $\sqrt{0+1+1} = \sqrt{2}, \quad \sqrt{16+1+1} = 3\sqrt{2}$	**M1**	Divides each of their three eigenvectors from (b) by its magnitude. Condone a slip in one magnitude.

Question 7 continued

Question	Scheme	Marks	Notes
	$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}$	**A1**	Any orthogonal $\mathbf{P}$ whose columns are correct unit eigenvectors, in any order and with any signs; decimals awrt 3 s.f.
	$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$	**B1**	Correct eigenvalues on the diagonal, in the order of the corresponding columns of $\mathbf{P}$.
		[3]	
	Total for Question 7	**12**	

Question	Scheme	Marks	Notes
8(a)	$\det(\mathbf{M}-2\mathbf{I}) = 4(8-1)-1(4-k)+k(1-2k)$	**M1**	Forms $\mathbf{M}-2\mathbf{I}$, or $\mathbf{M}-\lambda\mathbf{I}$ with $\lambda=2$, and expands its determinant with three terms. Condone sign slips.
	$24+2k-2k^2=0$	**A1**	Correct quadratic in k, oe.
	$k^2-k-12=0 \Rightarrow (k-4)(k+3)=0 \Rightarrow k=\ldots$	**dM1**	Solves their three-term quadratic by factorising, formula or completing the square, leading to $k=\ldots$; a calculator may be used.
	$k=4$ or $k=-3$	**A1**	
		[4]	
8(b)	$\det(\mathbf{M}-\lambda\mathbf{I}) = (6-\lambda)[(4-\lambda)(6-\lambda)-1]$ $-[(6-\lambda)+3]-3[1+3(4-\lambda)]$	**M1**	Expands $\det(\mathbf{M}-\lambda\mathbf{I})$ with their negative k to a cubic, three terms attempted, or uses the trace and determinant of $\mathbf{M}$ with eigenvalue 2. Condone sign slips.
	$\lambda^3-16\lambda^2+73\lambda-90=0$	**A1**	Correct cubic, oe such as multiplied by -1, or a correct quadratic factor $\lambda^2-14\lambda+45$.
	$(\lambda-2)(\lambda^2-14\lambda+45)=0 \Rightarrow \lambda=5,\ 9$	**A1**	Both values.
		[3]	
8(c)	$\lambda=2$: $4x+y-3z=0,\ x+2y+z=0$ $\Rightarrow \begin{pmatrix}1\\-1\\1\end{pmatrix}$	**M1 A1**	**M1** Solves $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for one of their eigenvalues, by equations from two rows or a vector product of two rows, reaching a non-zero vector. Condone slips. **A1** Any non-zero multiple, here and for each eigenvector.
	$\lambda=5$: $\begin{pmatrix}1\\2\\1\end{pmatrix}$	**A1**	
	$\lambda=9$: $\begin{pmatrix}1\\0\\-1\end{pmatrix}$	**A1**	
	$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$	**M1**	Divides each of their eigenvectors by its magnitude. Condone one wrong magnitude.
	$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$	**A1**	The three unit eigenvectors as columns, in any order and each of either sign; rationalised denominators accepted.

Question 8 continued

Question	Scheme	Marks	Notes
	$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$	**B1**	2, 5, 9 on the diagonal in the order of the columns of $\mathbf{P}$, zeros elsewhere.
		[7]	
	Total for Question 8	**14**	

Question	Scheme	Marks	Notes
9(a)	$\mathbf{M}\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}p-q-6\\q+1\\-6\end{pmatrix}$ $-6 = 2\lambda \Rightarrow \lambda = \ldots$	**M1**	Multiplies $\mathbf{M}$ by the eigenvector and equates the third component to λ times 2. Condone a slip in the other components.
	$\lambda = -3$	**A1**	
		[2]	
9(b)	$q+1 = \text{“}-3\text{”} \times (-1) \Rightarrow q = \ldots$ $p-q-6 = \text{“}-3\text{”} \Rightarrow p = \ldots$	**M1**	Equates the first or second component of the product to their λ times the corresponding component of the eigenvector.
	$q = 2$	**A1**	
	$p = 5$	**A1**	
		[3]	
9(c)	$\det(\mathbf{M}-\lambda\mathbf{I}) = (5-\lambda)\left[(5-\lambda)(-\lambda)-9\right]$ $-2(-2\lambda+9)-3\left[6+3(5-\lambda)\right]$	**M1**	Expands $\det(\mathbf{M}-\lambda\mathbf{I})$ with their p and q to a cubic, three terms attempted, or uses the trace and determinant of $\mathbf{M}$ with eigenvalue -3. Condone sign slips.
	$\lambda^3 - 10\lambda^2 + 3\lambda + 126 = 0$	**A1**	Correct cubic oe, such as $-\lambda^3 + 10\lambda^2 - 3\lambda - 126 = 0$, or a correct quadratic factor $\lambda^2 - 13\lambda + 42$.
	$(\lambda+3)(\lambda-6)(\lambda-7) = 0 \Rightarrow \lambda = 6,\ 7$	**A1**	Both values, by factorising or by calculator.
		[3]	
9(d)	$\lambda = 6$: $-x+2y-3z = 0,\ 2x-y+3z = 0 \Rightarrow \ldots$	**M1**	Forms $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for one of their other eigenvalues and solves, or takes a vector product of two rows. Condone slips.
	$\begin{pmatrix}-1\\1\\1\end{pmatrix}$	**A1**	Any nonzero multiple.
	$\lambda = 7$: $-2x+2y-3z = 0,\ -3x+3y-7z = 0$ $\Rightarrow \begin{pmatrix}1\\1\\0\end{pmatrix}$	**A1**	Any nonzero multiple.
	$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$	**M1**	Divides each of their three eigenvectors by its magnitude. Condone a slip in one magnitude.

Question 9 continued

Question	Scheme	Marks	Notes
	$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$	**A1**	Unit eigenvectors as columns, in any order and with either sign, exact or awrt 3 s.f.
	$\mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$	**B1**	Eigenvalues -3, 6, 7 on the diagonal in the order of their eigenvectors in $\mathbf{P}$.
		[6]	
	Total for Question 9	**14**	

Question	Scheme	Marks	Notes
10(a)(i)	$\mathbf{A}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}18\\p-14\\1-2p\end{pmatrix}$ $18 = 2\lambda \Rightarrow \lambda = \ldots$	**M1**	Multiplies $\mathbf{A}$ by the eigenvector and equates the first component to 2λ. Condone arithmetic slips.
	$\lambda = 9$	**A1**	
		[2]	
10(a)(ii)	$p - 14 = -2 \times$ "9" or $1 - 2p =$ "9" $\Rightarrow p = \ldots$	**M1**	Equates the second or third component of $\mathbf{A}$ times the eigenvector to their eigenvalue times that component of the eigenvector.
	$p - 14 = -18 \Rightarrow p = -4$	**A1 AG**	Correct equation from either component, leading without error to $p = -4$; one component is enough.
		[2]	
10(a)(iii)	$\det(\mathbf{A} - \lambda\mathbf{I})$ $= (5-\lambda)\left[(3-\lambda)(1-\lambda) - 16\right] - 16(1-\lambda)$	**M1**	Expands $\det(\mathbf{A} - \lambda\mathbf{I})$ with $p = -4$ to a cubic, or uses the trace and determinant of $\mathbf{A}$ with eigenvalue 9. Condone sign slips.
	$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$	**A1**	oe, any form equated to zero, or a correct quadratic factor $\lambda^2 - 9$.
	$(\lambda - 9)(\lambda^2 - 9) = 0 \Rightarrow \lambda = \ldots$	**dM1**	Uses the factor $\lambda - 9$, by division or inspection, or solves their cubic by any means, reaching values of λ.
	$\lambda = 3,\ \lambda = -3$	**A1**	
	$\lambda = 3$: $2\alpha - 4\beta = 0,\ -4\beta - 2\gamma = 0$ $\Rightarrow \alpha = 2\beta,\ \gamma = -2\beta$	**M1**	Forms $(\mathbf{A} - \lambda\mathbf{I})$ times a general vector equal to $\mathbf{0}$ for one of their eigenvalues and solves two equations, or takes the vector product of two rows.
	$\begin{pmatrix}2\\1\\-2\end{pmatrix}$	**A1**	Any nonzero multiple.
	$\lambda = -3$: $8\alpha - 4\beta = 0,\ -4\beta + 4\gamma = 0$ $\Rightarrow \begin{pmatrix}1\\2\\2\end{pmatrix}$	**A1**	Any nonzero multiple.
		[7]	

Question 10 continued

Question	Scheme	Marks	Notes
10(b)	$\mathbf{P} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$	**B1ft**	ft their eigenvectors, each divided by its magnitude, as the columns, in any order and with either sign.
	$\mathbf{D} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$	**B1ft**	ft their eigenvalues on the diagonal, in the order of the columns of their $\mathbf{P}$.
		[2]	
10(c)(i)	$\int \frac{1}{u}\,\mathrm{d}u = \int k\,\mathrm{d}t \Rightarrow \ln\lvert u\rvert = kt + c$	**M1**	Separates the variables and integrates, at least one side correct, or uses an integrating factor or auxiliary equation.
	$u = Ke^{kt}$	**A1**	oe, such as $u = e^{kt+c}$; needs an arbitrary constant.
		[2]	
10(c)(ii)	$\mathbf{P}^{\mathrm{T}}\mathbf{A} = \mathbf{P}^{\mathrm{T}}\mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}} = \mathbf{D}\mathbf{P}^{\mathrm{T}}$ $\Rightarrow \dot{u} = 9u,\ \dot{v} = 3v,\ \dot{w} = -3w$ $\Rightarrow u = \ldots$	**M1**	Uses $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{I}$ to reach three separate equations in their eigenvalues, and solves at least one by (c)(i).
	$u = K_1e^{9t},\ v = K_2e^{3t},\ w = K_3e^{-3t}$	**A1**	Any constants, or their values.
	$t = 0$: $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{\mathrm{T}}\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \ldots\ [= \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}]$	**M1**	Applies their $\mathbf{P}^{\mathrm{T}}$ to the initial position, or sets $t = 0$ in their x, y, z, to find the constants.
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{P}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \ldots$	**dM1**	Multiplies their u, v, w by their $\mathbf{P}$, reaching three expressions. Depends on the first M1.
	$x = 2e^{9t} + 2e^{3t} + e^{-3t}$ $y = -2e^{9t} + e^{3t} + 2e^{-3t}$ $z = e^{9t} - 2e^{3t} + 2e^{-3t}$	**A1**	As a column vector or three separate expressions.
		[5]	
	Total for Question 10	**20**	