OPTIONAL PURE / VECTORS & MATRICES

Eigenvalues, Eigenvectors and Diagonalisation

Mark scheme

15 questions · 169 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR MEI Extra Pure Y435 · AQA 7367 · CIE 9231

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	Total	**169**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$(3-\lambda)(-2-\lambda)-2k=0$ $\lambda=4 \Rightarrow 6-2k=0$	**M1**	Forms $\det(\mathbf{A}-\lambda\mathbf{I})=0$ with $\lambda=4$ substituted at any stage, or uses $\mathbf{Av}=4\mathbf{v}$, to reach an equation in k. Condone sign slips.
	$k=3$	**A1**	
	$\lambda^2-\lambda-12=0 \Rightarrow \lambda=\ldots$	**M1**	Solves the characteristic equation with their k by any means, calculator allowed, or uses sum of eigenvalues = trace.
	$\lambda=-3$	**A1**	
	$\lambda=4$: $\begin{pmatrix}-1 & \text{“3”}\\ 2 & -6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{0} \Rightarrow x=3y$ $\lambda=-3$: $\begin{pmatrix}6 & \text{“3”}\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{0} \Rightarrow y=-2x$	**M1**	Forms $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ or $\mathbf{Av}=\lambda\mathbf{v}$ with their k and one of their eigenvalues, reaching a relation between x and y.
	$\begin{pmatrix}3\\1\end{pmatrix}$ for $\lambda=4$, $\begin{pmatrix}1\\-2\end{pmatrix}$ for $\lambda=-3$	**A1**	Any non-zero multiples, each matched to its eigenvalue.
	Total for Question 1	**6**	

Question	Scheme	Marks	Notes
2(a)	$\lambda = 1,\ \lambda = -1$	**B1**	Both values, and no others.
		[1]	
2(b)	$(a-\lambda)(b-\lambda) - 0.6^2 = 0$	**M1**	Forms $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ in a, b, λ. Condone sign slips; need not be expanded.
	$\lambda^2 - (a+b)\lambda + ab - 0.36 \equiv \lambda^2 - 1$ $\Rightarrow a + b = 0,\ ab = -0.64 \Rightarrow a = \ldots$	**dM1**	Uses their eigenvalues, by comparing coefficients or substituting $\lambda = \pm 1$, to form two equations in a, b, and solves them simultaneously.
	$a = 0.8,\ b = -0.8$	**A1**	Both, oe. A0 if $a = -0.8$, $b = 0.8$ is not rejected.
		[3]	
2(c)	$\begin{pmatrix} \text{“}-0.2\text{”} & 0.6 \\ 0.6 & \text{“}-1.8\text{”} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow -0.2x + 0.6y = 0$	**M1**	Forms $(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0}$ or $\mathbf{Mv} = \mathbf{v}$ with their numerical a, b, reaching a relation between x and y; or takes the perpendicular to an eigenvector for -1.
	$y = \frac{1}{3}x$	**A1**	Any equivalent equation, such as $x = 3y$. A direction vector alone scores A0.
		[2]	
	Total for Question 2	**6**	

Question	Scheme	Marks	Notes
3(a)	$(2-\lambda)(5-\lambda)-k^2=0$, $\lambda=1$: $4-k^2=0 \Rightarrow k=\ldots$	**M1**	Substitutes $\lambda=1$ into $\det(\mathbf{M}-\lambda\mathbf{I})=0$ and reaches an equation in k. Condone sign slips.
	$k=-2$	**A1**	$k=-2$ only; A0 for $k=\pm 2$ without selecting.
	$\lambda^2-7\lambda+6=0 \Rightarrow \lambda=\ldots$	**M1**	Obtains the other eigenvalue using their k: solves their characteristic quadratic by any means, or uses the trace or determinant.
	$\lambda=6$	**A1**	
	$(\mathbf{M}-\lambda\mathbf{I})\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{0}$ $\lambda=1$: $x-2y=0$; $\lambda=6$: $2x+y=0$	**M1**	Forms $(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ or $\mathbf{M}\mathbf{v}=\lambda\mathbf{v}$ with their k and either eigenvalue, reaching a relation between x and y.
	$\begin{pmatrix}2\\1\end{pmatrix}$, $\begin{pmatrix}1\\-2\end{pmatrix}$	**A1**	Both correct, each any non-zero multiple, matched to its eigenvalue.
		[6]	
3(b)(i)	"$\begin{pmatrix}2\\1\end{pmatrix}$" . "$\begin{pmatrix}1\\-2\end{pmatrix}$" $=\ldots$	**M1**	Forms the scalar product of their two eigenvectors, or the product of the gradients of the two invariant lines.
	$=2-2=0$, so the invariant lines are perpendicular	**A1**	Scalar product 0, or gradient product -1, with the conclusion that the lines are perpendicular.
		[2]	
3(b)(ii)	One-way stretch	**B1**	"Stretch" alone accepted.
	scale factor 6, parallel to $y=-2x$	**B1**	Direction may be given as $\begin{pmatrix}1\\-2\end{pmatrix}$ or as perpendicular to $y=\frac{1}{2}x$.
	line of invariant points $y=\frac{1}{2}x$	**B1**	Any equivalent equation, such as $x=2y$.
		[3]	
Total for Question 3		**11**	

Question	Scheme	Marks	Notes
4(a)	$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**B1**	Eigenvectors in any order as columns, or nonzero multiples of them, with eigenvalues on the diagonal of $\mathbf{D}$ in the same order.
	$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \operatorname{adj} \mathbf{P} = \ldots$ $\left[= \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \right]$	**M1**	Finds $\mathbf{P}^{-1}$ for their $\mathbf{P}$ by cofactors, row reduction or calculator. Condone one slip in a cofactor or sign.
	$\mathbf{A} = \mathbf{PD}$"$\mathbf{P}^{-1}$" $= \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$ "$\mathbf{P}^{-1}$" $= \ldots$	**M1**	Multiplies their $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{P}^{-1}$ in that order to a 3×3 matrix. Condone arithmetic slips.
	$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 2 & -1 & 0 \end{pmatrix}$	**A1**	cao. A correct $\mathbf{A}$ without $\mathbf{P}^{-1}$ and the product $\mathbf{PDP}^{-1}$ seen scores B1 only.
		[4]	
4(b)	$\mathbf{A}^4 = \mathbf{PD}^4$"$\mathbf{P}^{-1}$" $= \begin{pmatrix} 16 & 0 & 1 \\ 0 & 1 & 1 \\ 16 & 1 & 1 \end{pmatrix}$ "$\mathbf{P}^{-1}$" $= \ldots$	**M1**	Uses $\mathbf{A}^4 = \mathbf{PD}^4\mathbf{P}^{-1}$ with their matrices, or multiplies their $\mathbf{A}$ from (a), including repeated squaring. Condone arithmetic slips.
	$\mathbf{A}^4 = \begin{pmatrix} 1 & -15 & 15 \\ 0 & 1 & 0 \\ 0 & -15 & 16 \end{pmatrix}$	**A1**	cao.
		[2]	
	Total for Question 4	**6**	

Question	Scheme	Marks	Notes
5(a)(i)	$\mathbf{M}\mathbf{v}_1 = \begin{pmatrix} 2 \\ a+10 \\ 2b-8 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\Rightarrow \lambda_1 = 2$	**B1**	
		[1]	
5(a)(ii)	$a+10 = 2 \times$ "2", $\quad 2b-8=0$ $\Rightarrow a = \ldots,\ b = \ldots$	**M1**	Equates at least one of the second and third components of $\mathbf{M}\mathbf{v}_1$ with that of their $\lambda_1\mathbf{v}_1$ and solves. Condone a slip in the product.
	$a=-6, \quad b=4$	**A1**	
		[2]	
5(b)	$\mathbf{M}\mathbf{v}_2 = \begin{pmatrix} 4 & -1 & 1 \\ -6 & 5 & -6 \\ -8 & 4 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $= \ldots$	**M1**	Multiplies $\mathbf{M}$, with their a and b, by $\mathbf{v}_2$; or uses the trace or determinant with their other eigenvalues. Condone one slip.
	$\mathbf{M}\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \lambda_2 = -1$	**A1**	
	$\begin{pmatrix} 1 & -1 & 1 \\ -6 & 2 & -6 \\ -8 & 4 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ $\Rightarrow x-y+z=0,\ -6x+2y-6z=0$ $\Rightarrow x:y:z = \ldots$	**M1**	Forms $(\mathbf{M}-3\mathbf{I})\mathbf{v}=\mathbf{0}$ or $\mathbf{M}\mathbf{v}=3\mathbf{v}$ with their a, b and solves two equations for the component ratios, or takes a vector product of two rows. Condone one slip.
	$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	**A1**	Any nonzero multiple.
		[4]	
5(c)	$\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	**B1**	Columns of $\mathbf{U}$ nonzero multiples of the three correct eigenvectors in any order, with the eigenvalues in $\mathbf{D}$ in matching order.
	$\det \mathbf{U} = 1(-1) - 0 + 1(2)\ [=1]$ $\mathbf{U}^{-1} = \frac{1}{\det \mathbf{U}} \operatorname{adj} \mathbf{U} = \ldots$	**M1**	Attempts the inverse of their 3×3 $\mathbf{U}$, e.g. determinant with matrix of cofactors transposed, or row reduction. Condone one slip; calculator inverse accepted.

Question 5 continued

Question	Scheme	Marks	Notes
	$\mathbf{U}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$	**A1**	Correct inverse of their correct **U**, whatever its ordering or scaling.
		[3]	
	Total for Question 5	**10**	

Question	Scheme	Marks	Notes
6(a)	$\det(\mathbf{M}+2\mathbf{I}) = \begin{vmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{vmatrix}$ $= 3(8)+2(-4)+2(-8)$	**M1**	Evaluates $\det(\mathbf{M}+2\mathbf{I})$ by any expansion, or their $\det(\mathbf{M}-\lambda\mathbf{I})$ at $\lambda=-2$, or shows $\mathbf{Mv}=-2\mathbf{v}$ for a nonzero $\mathbf{v}$.
	$=0$, so -2 is an eigenvalue	**A1**	cso, with a conclusion. A0 if the conclusion is missing.
	$-2x+4y=0,\ 2x+2z=0 \Rightarrow \ldots$	**M1**	Forms at least two equations from $(\mathbf{M}+2\mathbf{I})\mathbf{v}=\mathbf{0}$ or $\mathbf{Mv}=-2\mathbf{v}$ and solves for a nonzero vector.
	$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$	**A1**	Any nonzero multiple.
		[4]	
6(b)	$\det(\mathbf{M}-\lambda\mathbf{I}) = (1-\lambda)(\lambda^2-2\lambda)$ $+4\lambda-4(2-\lambda)$	**M1**	Expands $\det(\mathbf{M}-\lambda\mathbf{I})$ by any row or column to a cubic in λ. Condone sign slips.
	$\lambda^3-3\lambda^2-6\lambda+8=0$	**A1**	oe, such as its negative, $=0$ may be implied.
	$(\lambda+2)(\lambda^2-5\lambda+4)=0 \Rightarrow \lambda=\ldots$	**M1**	Obtains a quadratic factor of their cubic using $\lambda+2$ and solves it. Correct roots from a correct cubic imply the mark.
	$\lambda=1,\ 4$	**A1**	
	$\lambda=1$: $-2y+2z=0,\ -2x+y=0 \Rightarrow \ldots$	**M1**	Forms at least two equations from $(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ with their eigenvalue in place of 1 and solves for a nonzero vector.
	$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$	**A1**	Any nonzero multiple.
	$\lambda=4$: $-2x-2y=0,\ 2x-4z=0 \Rightarrow \ldots$	**M1**	Forms at least two equations from $(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ with their eigenvalue in place of 4 and solves for a nonzero vector.
	$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$	**A1**	Any nonzero multiple.
	Divides each by its length: $\sqrt{4+1+4}=\sqrt{1+4+4}=\sqrt{4+4+1}=3$	**M1**	Divides each of their three eigenvectors by its magnitude $\sqrt{x^2+y^2+z^2}$.

Question 6 continued

Question	Scheme	Marks	Notes
	$\mathbf{U} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{pmatrix}$	**A1**	Columns in any order, any column negated; decimals awrt 3 s.f. accepted.
	$\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	**B1**	Diagonal entries -2, 1, 4 ordered to match the eigenvectors in their **U**; zeros elsewhere.
		[11]	
	Total for Question 6	**15**	

Question	Scheme	Marks	Notes
7(a)	$\left(\frac{3}{8}-\lambda\right)^2-\frac{5}{16}\times\frac{5}{4}=0$ $\Rightarrow 4\lambda^2-3\lambda-1=0\Rightarrow\lambda=\ldots$	**M1**	Forms $\det(\mathbf{A}-\lambda\mathbf{I})=0$ as a quadratic and solves it by any means, calculator allowed. Condone sign slips in forming it.
	$\lambda=1,\ \lambda=-\frac{1}{4}$	**A1**	
	$\lambda=1$: $-\frac{5}{8}x+\frac{5}{16}y=0\Rightarrow y=\ldots$	**M1**	Uses $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ or $\mathbf{Av}=\lambda\mathbf{v}$ with one of their eigenvalues to obtain a relation between x and y.
	$\begin{pmatrix}1\\2\end{pmatrix}$	**A1**	Any nonzero multiple, paired with $\lambda=1$.
	$\lambda=-\frac{1}{4}$: $\frac{5}{8}x+\frac{5}{16}y=0\Rightarrow\begin{pmatrix}1\\-2\end{pmatrix}$	**A1**	Any nonzero multiple, paired with $\lambda=-\frac{1}{4}$.
		[5]	
7(b)	$\mathbf{P}=\begin{pmatrix}1&1\\2&-2\end{pmatrix},\ \mathbf{D}=\begin{pmatrix}1&0\\0&-\frac{1}{4}\end{pmatrix}$	**B1ft**	ft their eigenvalues and eigenvectors, with the columns of $\mathbf{P}$ in the same order as the eigenvalues in $\mathbf{D}$.
		[1]	
7(c)	$\mathbf{A}^n=\mathbf{PD}^n\mathbf{P}^{-1}$, $\mathbf{D}^n=\begin{pmatrix}1&0\\0&\left(-\frac{1}{4}\right)^n\end{pmatrix}$	**M1**	Uses $\mathbf{A}^n=\mathbf{PD}^n\mathbf{P}^{-1}$ with their $\mathbf{P}$ and $\mathbf{D}$, raising the diagonal entries of $\mathbf{D}$ to the power n.
	$\mathbf{P}^{-1}=-\frac{1}{4}\begin{pmatrix}-2&-1\\-2&1\end{pmatrix}$ $\mathbf{A}^n=$ "$\mathbf{P}$"$\mathbf{D}^n$"$\mathbf{P}^{-1}$"$=\ldots$	**dM1**	Finds their $\mathbf{P}^{-1}$ by any method and multiplies the three matrices in the correct order to a single matrix. Condone arithmetic slips.
	$\mathbf{A}^n=\begin{pmatrix}\frac{1}{2}+\frac{1}{2}\left(-\frac{1}{4}\right)^n&\frac{1}{4}-\frac{1}{4}\left(-\frac{1}{4}\right)^n\\1-\left(-\frac{1}{4}\right)^n&\frac{1}{2}+\frac{1}{2}\left(-\frac{1}{4}\right)^n\end{pmatrix}$	**A1**	oe, such as $(-1)^n4^{-n}$ for $\left(-\frac{1}{4}\right)^n$.
	$\left\|-\frac{1}{4}\right\|<1\Rightarrow\left(-\frac{1}{4}\right)^n\to 0$	**E1**	States $\left(-\frac{1}{4}\right)^n\to 0$ because its modulus is less than 1, or oe. Not for "n is large" alone.
	$\mathbf{L}=\begin{pmatrix}\frac{1}{2}&\frac{1}{4}\\1&\frac{1}{2}\end{pmatrix}$	**A1ft**	ft their $\mathbf{A}^n$ with each $\left(-\frac{1}{4}\right)^n$ term set to 0.
		[5]	
7(d)	"$\mathbf{L}$" $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{2}x+\frac{1}{4}y\\x+\frac{1}{2}y\end{pmatrix}$	**M1**	Finds the image of a general point (x,y) under their $\mathbf{L}$, or uses the columns of their $\mathbf{L}$ as image directions.

Question 7 continued

Question	Scheme	Marks	Notes
	$y = 2x$	**A1**	oe, such as $2x - y = 0$.
		[2]	
	Total for Question 7	**13**	

Question	Scheme	Marks	Notes
8(a)	$\det(\mathbf{A}-\lambda\mathbf{I})$ $=(-4-\lambda)[(-7-\lambda)(16-\lambda)+132]$ $+13[-6(16-\lambda)+72]$ $+8[132-12(7+\lambda)]$	**M1**	Expands $\det(\mathbf{A}-\lambda\mathbf{I})$ along any row or column, each of three terms an element times its 2×2 minor, or by Sarrus' rule. Condone sign and arithmetic slips.
	$-\lambda^3+5\lambda^2-2\lambda-8=0$ $\Rightarrow \lambda^3-5\lambda^2+2\lambda+8=0$	**A1**	Printed equation from a correct expansion without error; collected steps accepted.
	$\lambda=-1$: $-1-5-2+8=0$ $(\lambda+1)(\lambda^2-6\lambda+8)=0\Rightarrow\lambda=\ldots$	**M1**	Shows a root by substitution and obtains a quadratic factor, by division or comparing coefficients, with an attempt to solve it. Condone coefficient slips.
	$\lambda=-1,\ 2,\ 4$	**A1**	
		[4]	
8(b)	$\lambda=-1$: $-3x-13y+8z=0$, $-6x-6y+6z=0$ $\Rightarrow x:y:z=\ldots$	**M1**	Forms $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ for one of their eigenvalues and solves two equations to a ratio $x:y:z$, or takes a vector product of two rows. Condone arithmetic slips.
	$\lambda=-1$: $\begin{pmatrix}1\\1\\2\end{pmatrix}$	**A1**	Each A1: any nonzero multiple; eigenvectors in any order.
	$\lambda=2$: $\begin{pmatrix}1\\2\\4\end{pmatrix}$	**A1**	
	$\lambda=4$: $\begin{pmatrix}1\\0\\1\end{pmatrix}$	**A1**	
	$\mathbf{P}=\begin{pmatrix}1&1&1\\1&2&0\\2&4&1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&4\end{pmatrix}$	**B1ft**	ft their eigenvalues and eigenvectors: eigenvectors as the columns of $\mathbf{P}$, eigenvalues in the same order on the diagonal of $\mathbf{D}$.
		[5]	
8(c)	$\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ $\mathbf{D}^n=\begin{pmatrix}(-1)^n&0&0\\0&2^n&0\\0&0&4^n\end{pmatrix}$	**M1**	Uses $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ with their diagonal $\mathbf{D}$ raised to the power n entrywise. No proof of the result needed.
	$\mathbf{P}^{-1}=\begin{pmatrix}2&3&-2\\-1&-1&1\\0&-2&1\end{pmatrix}$	**B1**	Correct inverse of their valid $\mathbf{P}$.

Question 8 continued

Question	Scheme	Marks	Notes
	$(-1)^n \times 2 + 2^n \times (-1) + 4^n \times 0$ $= 2(-1)^n - 2^n$	**A1 AG**	Correct substituted sum of three products from valid $\mathbf{P}$, $\mathbf{D}^n$, $\mathbf{P}^{-1}$, leading without error to $2(-1)^n - 2^n$; full matrix product accepted.
		[3]	
	Total for Question 8	**12**	

Question	Scheme	Marks	Notes
9(a)	$\det(\mathbf{M}-\lambda\mathbf{I})$ $=(3-\lambda)[(3-\lambda)(-2-\lambda)+4]$ $\quad -4[(-2-\lambda)+1]-4[4-(3-\lambda)]$	**M1**	Expands $\det(\mathbf{M}-\lambda\mathbf{I})$ or $\det(\lambda\mathbf{I}-\mathbf{M})$ along any row or column, with three terms each a 2×2 cofactor. Condone sign slips.
	$=(3-\lambda)(\lambda^2-\lambda-2)-4(-1-\lambda)-4(1+\lambda)$	**A1**	Correct expansion, need not be simplified.
	$-\lambda^3+4\lambda^2-\lambda-6=0$ $\Rightarrow \lambda^3-4\lambda^2+\lambda+6=0$	**A1 AG**	Correct expansion simplified without error to the printed equation, set equal to zero; combined steps accepted.
		[3]	
9(b)	$f(-1)=-1-4-1+6=0$ $\Rightarrow(\lambda+1)(\lambda^2-5\lambda+6)=0\Rightarrow\lambda=\ldots$	**M1**	Shows a root of the cubic and factorises to a linear factor times a quadratic, by inspection, division or comparing coefficients, then solves the quadratic.
	$\lambda=-1,\ 2,\ 3$	**A1**	
		[2]	
9(c)	$(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$, e.g. $\lambda=-1$: $x+y-z=0,\ x+4y-z=0\Rightarrow x:y:z=\ldots$	**M1**	Forms $(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ or $\mathbf{Mv}=\lambda\mathbf{v}$ for one of their eigenvalues and solves for a ratio $x:y:z$, or takes the vector product of two rows. Condone arithmetic slips.
	$\lambda=-1$: $\begin{pmatrix}1\\0\\1\end{pmatrix}$	**A1**	Any one correct eigenvector, any nonzero multiple. The three A1s may be earned in any order.
	$\lambda=2$: $\begin{pmatrix}0\\1\\1\end{pmatrix}$	**A1**	
	$\lambda=3$: $\begin{pmatrix}1\\1\\1\end{pmatrix}$	**A1**	
		[4]	
9(d)	$\mathbf{P}=\begin{pmatrix}1&0&1\\0&1&1\\1&1&1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&3\end{pmatrix}$	**B1ft**	ft their eigenvalues and eigenvectors: eigenvectors as columns of $\mathbf{P}$, eigenvalues on the diagonal of $\mathbf{D}$ in the same order.

Question 9 continued

Question	Scheme	Marks	Notes
	$\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ $\mathbf{D}^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$, $\mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$	**M1**	Uses $\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ with $\mathbf{D}^n$ the diagonal of powers of their eigenvalues and an attempt at the inverse of their $\mathbf{P}$. Condone one slip in $\mathbf{P}^{-1}$.
	$\mathbf{P}\mathbf{D}^n = \begin{pmatrix} (-1)^n & 0 & 3^n \\ 0 & 2^n & 3^n \\ (-1)^n & 2^n & 3^n \end{pmatrix}$ $\Rightarrow \mathbf{M}^n =$ "$\mathbf{P}\mathbf{D}^n$" "$\mathbf{P}^{-1}$" $= \ldots$	**dM1**	Multiplies their $\mathbf{P}$, $\mathbf{D}^n$ and $\mathbf{P}^{-1}$ in that order to a single 3×3 matrix. Condone arithmetic slips.
	$\mathbf{M}^n = \begin{pmatrix} 3^n & 3^n - (-1)^n & (-1)^n - 3^n \\ 3^n - 2^n & 3^n & 2^n - 3^n \\ 3^n - 2^n & 3^n - (-1)^n & (-1)^n + 2^n - 3^n \end{pmatrix}$	**A1**	oe, e.g. $(-1)^{n+1}$ for $-(-1)^n$.
		[4]	
Total for Question 9		**13**	

Question	Scheme	Marks	Notes
10(a)	$(-1-\lambda)\big[\lambda(1+\lambda)-6\big]+(3\lambda-6k)$ $+2\big[-3+k(1+\lambda)\big]=0$	**M1**	Forms $\det(\mathbf{M}-\lambda\mathbf{I})$ or $\det(\lambda\mathbf{I}-\mathbf{M})$ and expands by any row or column, with all three terms attempted. Condone sign slips.
	$\lambda^3+2\lambda^2-(2k+8)\lambda+4k=0$	**A1**	oe, such as $-\lambda^3-2\lambda^2+(2k+8)\lambda-4k=0$.
	$2^3+2\times2^2-2(2k+8)+4k$ $=16-16-4k+4k=0$ for all k	**A1 AG**	Shows $\lambda=2$ satisfies the correct equation for every k, by substitution, $\det(\mathbf{M}-2\mathbf{I})=0$ or the factor $\lambda-2$, with a conclusion for every k.
		[3]	
10(b)	$(\lambda-2)(\lambda^2+4\lambda-2k)=0$	**M1 A1**	**M1** Divides their cubic by $\lambda-2$, by any method, reaching a quadratic factor. Condone coefficient slips. **A1** Correct quadratic factor.
	$4^2+8k=0\Rightarrow k=\ldots$ or $2^2+4\times2-2k=0\Rightarrow k=\ldots$	**M1**	Uses either condition on their quadratic, discriminant zero or $\lambda=2$ a root, and solves for k. Condone sign slips.
	$k=-2$: eigenvalues $2,-2,-2$	**A1**	
	$k=6$: eigenvalues $2,2,-6$	**A1**	
		[5]	
	Total for Question 10	**8**	

Question	Scheme	Marks	Notes
11(a)	$\mathbf{B}(\mathbf{M}^{-1}\mathbf{e}) = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{M}^{-1}\mathbf{e}$	**M1**	Replaces $\mathbf{B}$ by $\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ in $\mathbf{B}\mathbf{M}^{-1}\mathbf{e}$, or premultiplies $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ by $\mathbf{M}^{-1}$ and inserts $\mathbf{M}\mathbf{M}^{-1}$. Matrix order must be correct.
	$= \mathbf{M}^{-1}\mathbf{A}\mathbf{e} = \mathbf{M}^{-1}(\lambda\mathbf{e})$	**A1**	Uses both $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$ and $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$; combined steps accepted.
	$= \lambda(\mathbf{M}^{-1}\mathbf{e})$ $\mathbf{e} \neq \mathbf{0}$, $\mathbf{M}$ non-singular $\Rightarrow \mathbf{M}^{-1}\mathbf{e} \neq \mathbf{0}$	**A1 AG**	$\lambda(\mathbf{M}^{-1}\mathbf{e})$ reached without error, and $\mathbf{M}^{-1}\mathbf{e} \neq \mathbf{0}$ stated with a reason. Nothing for working backwards from the result.
		[3]	
11(b)	$\lambda = 3,\ -2,\ 1$	**B1**	
	$\lambda = 3$: $x - 5y = 0,\ 2x + 5y - 2z = 0 \Rightarrow \mathbf{v} = \ldots$	**M1**	Forms $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ or $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ with one of their eigenvalues and attempts a non-zero solution. Condone arithmetic slips.
	$\lambda = 3$: $\begin{pmatrix}10\\2\\15\end{pmatrix}$; $\lambda = -2$: $\begin{pmatrix}0\\3\\-5\end{pmatrix}$	**A1**	Any two correct, each paired with its eigenvalue; any non-zero multiples.
	$\lambda = 1$: $\begin{pmatrix}0\\0\\1\end{pmatrix}$	**A1**	All three correct, each paired with its eigenvalue; any non-zero multiples.
		[4]	
11(c)	$\det\mathbf{M} = 1$, cofactors $\begin{pmatrix}1&0&0\\-1&1&-2\\0&0&1\end{pmatrix}$	**M1**	Finds $\det\mathbf{M}$ and attempts the cofactor matrix, transposing and dividing; condone sign slips. Any valid method, e.g. row reduction. Implied by correct $\mathbf{M}^{-1}$.
	$\mathbf{M}^{-1} = \begin{pmatrix}1&-1&0\\0&1&0\\0&-2&1\end{pmatrix}$	**A1**	
	Eigenvalues of $\mathbf{B}$: $3,\ -2,\ 1$	**B1**	Not if found from $\mathbf{B}$ explicitly.
	"$\mathbf{M}^{-1}$" "$\begin{pmatrix}10\\2\\15\end{pmatrix}$" $= \ldots$, etc.	**M1**	Multiplies their 3×3 $\mathbf{M}^{-1}$ by at least one of their non-zero eigenvectors of $\mathbf{A}$. Not for eigenvectors found from $\mathbf{B}$ explicitly.

Question 11 continued

Question	Scheme	Marks	Notes
	3: $\begin{pmatrix}8\\2\\11\end{pmatrix}$; -2: $\begin{pmatrix}-3\\3\\-11\end{pmatrix}$; 1: $\begin{pmatrix}0\\0\\1\end{pmatrix}$	**A1**	All three correct, each paired with its eigenvalue; any non-zero multiples.
		[5]	
	Total for Question 11	**12**	

Question	Scheme	Marks	Notes
12(a)	$\mathbf{A}^2\mathbf{e} = \mathbf{A}(\lambda\mathbf{e})$ $\mathbf{A}^{-1}\mathbf{A}\mathbf{e} = \lambda\mathbf{A}^{-1}\mathbf{e}$	**M1**	Premultiplies $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ by $\mathbf{A}$ or $\mathbf{A}^{-1}$ to start either proof. Not for stating the results alone.
	$\mathbf{A}^2\mathbf{e} = \lambda\mathbf{A}\mathbf{e} = \lambda^2\mathbf{e}$	**A1**	One complete correct proof, for $\mathbf{A}^2$ or $\mathbf{A}^{-1}$, identifying eigenvalue and eigenvector; for the inverse, justify $\lambda \neq 0$ before dividing.
	$\mathbf{A}$ non-singular, $\mathbf{e} \neq \mathbf{0} \Rightarrow \lambda \neq 0$ $\mathbf{e} = \lambda\mathbf{A}^{-1}\mathbf{e} \Rightarrow \mathbf{A}^{-1}\mathbf{e} = \lambda^{-1}\mathbf{e}$	**A1 AG**	Both proofs complete without error, identifying eigenvalues and eigenvector. Needs $\lambda \neq 0$ justified from $\mathbf{A}$ non-singular and $\mathbf{e} \neq \mathbf{0}$ before dividing.
		[3]	
12(b)	$(\mathbf{A}^2 + 2\mathbf{A}^{-1})\mathbf{e} = \lambda^2\mathbf{e} + 2\lambda^{-1}\mathbf{e}$ $= (\lambda^2 + 2\lambda^{-1})\mathbf{e}$	**B1**	Applies the matrix to $\mathbf{e}$, uses both results of (a) and factorises, with a conclusion.
		[1]	
12(c)	$\begin{pmatrix} 3 & 1 & -1 \\ 3 & 3 & -3 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ $3x + y - z = 0,\ x + y - z = 0 \Rightarrow \ldots$	**M1**	Forms equations from $(\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{0}$ or $\mathbf{A}\mathbf{v} = -\mathbf{v}$ and eliminates to relate the components, or crosses two rows of $\mathbf{A} + \mathbf{I}$. Condone one arithmetic slip.
	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	**A1**	Any non-zero multiple.
		[2]	
12(d)	$\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{A}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$	**M1**	Multiplies $\mathbf{A}$ by each given vector and compares with a multiple of it. Condone one arithmetic slip.
	Eigenvalues 1 and 2	**A1**	Each matched to its vector.
		[2]	
12(e)	$\mu = \lambda^2 + 2\lambda^{-1}$, $\lambda = -1, 1, 2 \Rightarrow \mu = \ldots$	**M1**	Substitutes at least two of their eigenvalues of $\mathbf{A}$ from (c) and (d) into $\lambda^2 + 2\lambda^{-1}$.
	$\mu = -1,\ 3,\ 5$	**A1**	
	"-1"^{2}, "3"^{2}, "5"^{2} $[= 1,\ 9,\ 25]$	**M1**	Squares their eigenvalues of $\mathbf{A}^2 + 2\mathbf{A}^{-1}$, or evaluates $(\lambda^2 + 2\lambda^{-1})^2$, for the diagonal of $\mathbf{D}$.

Question 12 continued

Question	Scheme	Marks	Notes
	$\mathbf{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix}$	**A1**	cso. Columns of $\mathbf{P}$ any non-zero multiples of the three eigenvectors, with $1, 9, 25$ in $\mathbf{D}$ in matching order.
		[4]	
	Total for Question 12	**12**	

Question	Scheme	Marks	Notes
13(a)(i)	$\det(\mathbf{M}-\lambda\mathbf{I}) = \frac{1}{27}\big[(2-3\lambda)\big((2-3\lambda)^2+2\big)$ $+\big(2(2-3\lambda)-1\big)+2\big(4+(2-3\lambda)\big)\big]$	**M1**	Expands the determinant of $\mathbf{M}-\lambda\mathbf{I}$ or $\lambda\mathbf{I}-\mathbf{M}$ along any row or column to a cubic in λ. Condone sign slips and a lost factor $\frac{1}{27}$.
	$\lambda^3-2\lambda^2+2\lambda-1=0$	**A1**	Must have leading coefficient 1 as printed; $-\lambda^3+2\lambda^2-2\lambda+1=0$ alone scores A0.
	$1-2+2-1=0 \Rightarrow 1$ is an eigenvalue	**A1 AG**	Substitutes $\lambda=1$ into the correct equation, shows it gives 0 and concludes 1 is an eigenvalue.
		[3]	
13(a)(ii)	$(\lambda-1)(\lambda^2-\lambda+1)=0 \Rightarrow \lambda=\ldots$	**M1**	Obtains a quadratic factor $\lambda^2+b\lambda+c$ of their cubic and solves it by formula, completing the square or calculator. Condone slips in the factor.
	$\lambda=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	**A1**	Both; exact or awrt $0.5\pm0.866i$.
	$b^2-4ac=-3<0$, so 1 is the only real eigenvalue	**E1**	States the discriminant of the correct quadratic is negative, or its roots are non-real, and concludes 1 is the only real eigenvalue.
		[3]	
13(b)	Diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{M}$: $1,\ \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	**E1**	States the diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{M}$, or that $\mathbf{M}$ and $\mathbf{D}$ share eigenvalues. No proof of this needed.
	Two are not real, so $\mathbf{D}$ cannot be real: the student is wrong	**E1**	States two eigenvalues are non-real, so $\mathbf{D}$ would have non-real entries, contradicting the claim.
		[2]	
13(c)(i)	$(\mathbf{M}-\mathbf{I})\mathbf{e}=\mathbf{0}$: $-x-y+2z=0,\ 2x-y-z=0$ $\Rightarrow x:y:z=\ldots$	**M1**	Forms $(\mathbf{M}-\mathbf{I})\mathbf{e}=\mathbf{0}$ or $\mathbf{M}\mathbf{e}=\mathbf{e}$ and solves two equations for the ratio $x:y:z$, or takes a vector product of two rows of $\mathbf{M}-\mathbf{I}$. Condone slips.

Question 13 continued

Question	Scheme	Marks	Notes
	$\mathbf{e} = \begin{pmatrix}1\\1\\1\end{pmatrix}$	**A1**	Any non-zero multiple.
		[2]	
13(c)(ii)	$\mathbf{Me} = \mathbf{e}$: every point on the line through O along $\mathbf{e}$ is fixed, so this line is the axis	**E1**	States that eigenvalue 1 means each point on the line along $\mathbf{e}$ is fixed, not merely the line invariant, so that line is the axis.
	$\mathbf{r} = t\begin{pmatrix}1\\1\\1\end{pmatrix}$	**B1ft**	ft their $\mathbf{e}$ from (c)(i). Needs $\mathbf{r} =$; any parameter.
		[2]	
13(c)(iii)	An eigenvector maps to a parallel vector; an acute rotation turns every vector off the axis to a non-parallel direction	**E1**	States an eigenvector maps to a parallel vector, and that a rotation through an acute angle changes the direction of every vector not along the axis. An algebraic argument alone scores E0.
		[1]	
	Total for Question 13	**13**	

Question	Scheme	Marks	Notes
14(a)	$\det(\mathbf{M}-\lambda\mathbf{I})$ $= (1-\lambda)^2(-2-\lambda) - \big(3-(1-\lambda)\big)$	**M1**	Forms $\det(\mathbf{M}-\lambda\mathbf{I})$ or $\det(\lambda\mathbf{I}-\mathbf{M})$ and expands along any row or column, all terms attempted. Condone sign slips.
	$\lambda^3 - 2\lambda + 4 = 0$	**A1**	Any equivalent equation, such as $-\lambda^3 + 2\lambda - 4 = 0$.
	$\det(\mathbf{M}) = -4$	**B1**	cao, by any method.
		[3]	
14(b)	$\lambda = -2$	**B1**	
	$\lambda^3 - 2\lambda + 4 = (\lambda+2)(\lambda^2 + \ldots)$	**M1**	Divides their cubic by λ minus their real eigenvalue, by any method, reaching a three-term quadratic factor.
	$= (\lambda+2)(\lambda^2 - 2\lambda + 2)$	**A1**	
	$(-2)^2 - 4\times 1\times 2 = \ldots$	**dM1**	Finds the discriminant of their quadratic factor, completes its square, or solves it to complex roots.
	$-4 < 0$, so the other two eigenvalues are not real	**A1**	Discriminant -4 from the correct quadratic factor, or $(\lambda-1)^2+1$, or roots $1\pm i$, with no incorrect working to here and the conclusion stated.
	$(\mathbf{M}+2\mathbf{I})\mathbf{v} = \mathbf{0}$ $\Rightarrow 3a - c = 0,\ a + b = 0 \Rightarrow \mathbf{v} = \ldots$	**M1**	Forms at least two equations from $\mathbf{Mv} = \lambda\mathbf{v}$ with their real eigenvalue and solves them, or takes a vector product of two rows of $\mathbf{M}-\lambda\mathbf{I}$.
	$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$	**A1**	Any nonzero multiple.
	$\begin{pmatrix} 0.4 \\ -0.4 \\ 1.2 \end{pmatrix} = 0.4\,\mathbf{v}$ and $\mathbf{Mv} = -2\mathbf{v}$ $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{0.4}{-2}\,\mathbf{v} = \ldots$	**M1**	Writes the right-hand side as a multiple of their eigenvector and divides by their eigenvalue, or solves the system by any other complete method.
	$x = -0.2,\ y = 0.2,\ z = -0.6$	**A1**	oe, such as $-\frac{1}{5}$, $\frac{1}{5}$, $-\frac{3}{5}$, or as a vector.

Question 14 continued

Question	Scheme	Marks	Notes
	$\det(\mathbf{M}) = -4 \neq 0$, so $\mathbf{M}^{-1}$ exists and the solution is unique	**E1**	States $\det(\mathbf{M}) \neq 0$, $\mathbf{M}$ non-singular or invertible, or 0 not an eigenvalue of $\mathbf{M}$, so the solution is unique. Not for "only one solution" without a reason.
		[10]	
	Total for Question 14	**13**	

Question	Scheme	Marks	Notes
15(a)(i)	$\lambda = 3$	**B1**	
		[1]	
15(a)(ii)	$-2 + 4 - p = 3 \times 2$	**M1**	Equates the second component of $\mathbf{A}$ times the eigenvector to 2 times their eigenvalue. Condone a slip in forming the component.
	$p = -4$	**A1 AG**	Correct equation in p leading without error to $p = -4$.
		[2]	
15(a)(iii)	$\det(\mathbf{A} - \lambda\mathbf{I}) = (-3-\lambda)[(2-\lambda)(3-\lambda) - 20]$ $- 2[-2(3-\lambda) + 40]$ $- 2[10 - 10(2-\lambda)]$	**M1**	Expands $\det(\mathbf{A} - \lambda\mathbf{I})$ with $p = -4$ along any row or column, or by any valid method. Condone sign and arithmetic slips.
	$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$	**A1**	oe, such as the negated cubic.
	$(\lambda - 3)(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda = \ldots$	**M1**	Solves their cubic by any means, such as dividing by $\lambda - 3$ and solving the quadratic; a calculator is allowed.
	$\lambda = 1,\ \lambda = -2$	**A1**	
	$\lambda = 1$: $-4\alpha + 2\beta - 2\gamma = 0$ $-2\alpha + \beta - 4\gamma = 0 \Rightarrow \ldots$	**M1**	Forms $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ for one of their other eigenvalues and solves at least two equations, or takes a cross product of two rows.
	$\lambda = 1$: $(1,\ 2,\ 0)^{\mathrm{T}}$	**A1**	Or any non-zero multiple.
	$\lambda = -2$: $(0,\ 1,\ 1)^{\mathrm{T}}$	**A1**	Or any non-zero multiple.
		[7]	
15(b)	$\mathbf{P}$ with columns $(1,2,-1)^{\mathrm{T}}, (1,2,0)^{\mathrm{T}}, (0,1,1)^{\mathrm{T}}$ $\mathbf{D} = \mathrm{diag}(3,\ 1,\ -2)$	**B1ft**	ft their eigenvectors and eigenvalues: columns of $\mathbf{P}$ in the same order as the eigenvalues on the diagonal of $\mathbf{D}$. Any order.
		[1]	
15(c)(i)	$\int \frac{1}{u}\,\mathrm{d}u = \int k\,\mathrm{d}t \Rightarrow \ln\lvert u\rvert = kt + c$	**M1**	Separates the variables and integrates both sides, reaching $\ln u$ or $\ln\lvert u\rvert$.

Question 15 continued

Question	Scheme	Marks	Notes
	$u = Ke^{kt}$	**A1**	$u = Ke^{kt}$, K any real constant, including zero; oe. $u = e^{kt+c}$ alone is not the general solution.
		[2]	
15(c)(ii)	Uses $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$: $\dot{u} = 3u,\ \dot{v} = v,\ \dot{w} = -2w$	**M1**	Uses $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ to reach three separate equations of the form $\dot{u} = \lambda u$ with their eigenvalues.
	$u = C_1e^{3t},\ v = C_2e^{t},\ w = C_3e^{-2t}$	**A1**	Three distinct arbitrary constants; rates in an order consistent with their **D**.
	$x = u + v,\ y = 2u + 2v + w,\ z = -u + w$	**dM1**	Multiplies their **P** by their exponential u, v, w to obtain x, y, z. Not for using $\mathbf{P}^{-1}$.
	$x = C_1e^{3t} + C_2e^{t}$ $y = 2C_1e^{3t} + 2C_2e^{t} + C_3e^{-2t}$ $z = -C_1e^{3t} + C_3e^{-2t}$	**A1**	oe, including vector form; consistent with their order of eigenvectors.
		[4]	
15(c)(iii)	$C_1 + C_2 = 3,\ 2C_1 + 2C_2 + C_3 = 7$ $-C_1 + C_3 = -1 \Rightarrow C_1 = \ldots$	**M1**	Sets $t = 0$ in their general solution with three constants, equates to $(3, 7, -1)$ and solves; a calculator is allowed.
	$x = 2e^{3t} + e^{t}$ $y = 4e^{3t} + 2e^{t} + e^{-2t}$ $z = -2e^{3t} + e^{-2t}$	**A1**	
		[2]	
	Total for Question 15	**19**	