OPTIONAL PURE / SURFACES & PARTIAL DIFFERENTIATION

Tangent Planes and Normal Lines

Worked solutions 10 questions · 103 marks

EXAM BOARDS OCR A Y545 · OCR MEI Extra Pure Y435

Instructions

OCR A (Y545): the specification names the tangent plane only for a surface given as $z = f(x, y)$. Questions 3, 4 and 8 to 10 use ∇g for a surface given as $g(x, y, z) = c$, which the specification does not name, and can be treated as extension.

Questions

Question 1	*5 marks*	2
Question 2	*7 marks*	4
Question 3	*7 marks*	5
Question 4	*9 marks*	7
Question 5	*8 marks*	9
Question 6	*11 marks*	11
Question 7	*12 marks*	13
Question 8	*7 marks*	15
Question 9	*18 marks*	17
Question 10	*19 marks*	19

1. A surface S has equation

$$z = 3y\sqrt{x} - \frac{16x}{y} + xy^2, \qquad x \geq 0,\ y > 0$$

The point P on S has x-coordinate 4 and y-coordinate 2.

Find the z-coordinate of P, and hence determine the equation of the tangent plane to S at P. Give your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. **[5]**

Solution

Substituting $x = 4$ and $y = 2$ into the equation of S:

$$\begin{aligned} z &= 3 \times 2 \times \sqrt{4} - \frac{16 \times 4}{2} + 4 \times 2^2 \\ &= 12 - 32 + 16 \\ &= -4 \end{aligned}$$

So the z-coordinate of P is -4, and P is the point $(4, 2, -4)$.

Write the surface as $g(x, y, z) = 0$, where

$$g(x, y, z) = 3y\sqrt{x} - \frac{16x}{y} + xy^2 - z$$

The partial derivatives are

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{3y}{2\sqrt{x}} - \frac{16}{y} + y^2 \\ \frac{\partial g}{\partial y} &= 3\sqrt{x} + \frac{16x}{y^2} + 2xy \\ \frac{\partial g}{\partial z} &= -1 \end{aligned}$$

At P, with $x = 4$ and $y = 2$:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{6}{4} - 8 + 4 = -\frac{5}{2} \\ \frac{\partial g}{\partial y} &= 6 + \frac{64}{4} + 16 = 38 \end{aligned}$$

A normal vector to S at a point is ∇g, which at P is

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 38 \\ -1 \end{pmatrix}$$

Multiplying by -2 gives the integer normal vector

$$\mathbf{n} = \begin{pmatrix} 5 \\ -76 \\ 2 \end{pmatrix}$$

The tangent plane passes through P, so its equation is $\mathbf{r} \cdot \mathbf{n} = \overrightarrow{OP} \cdot \mathbf{n}$, where

$$\overrightarrow{OP} \cdot \mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -76 \\ 2 \end{pmatrix} = 20 - 152 - 8 = -140$$

With $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the tangent plane to S at P is

$$5x - 76y + 2z = -140$$

This is in the form $ax + by + cz = d$ with $a = 5$, $b = -76$, $c = 2$, $d = -140$.

2. The surface E has equation

$$z = \frac{48}{x^2 - xy + 2y^2}$$

The point P on E has x-coordinate 2 and y-coordinate 3.

(a) Find the z-coordinate of P, and determine the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at P. **[5]**

(b) Find the equation of the tangent plane to E at P, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. **[2]**

Solution

(a) At P, $x = 2$ and $y = 3$, so the denominator is

$$x^2 - xy + 2y^2 = 4 - 6 + 18 = 16$$

Hence the z-coordinate of P is

$$z = \frac{48}{16} = 3$$

Write $z = 48\left(x^2 - xy + 2y^2\right)^{-1}$ and differentiate partially using the chain rule. Holding y constant:

$$\frac{\partial z}{\partial x} = -48\left(x^2 - xy + 2y^2\right)^{-2}(2x - y) = -\frac{48(2x - y)}{\left(x^2 - xy + 2y^2\right)^2}$$

Holding x constant:

$$\frac{\partial z}{\partial y} = -48\left(x^2 - xy + 2y^2\right)^{-2}(-x + 4y) = -\frac{48(4y - x)}{\left(x^2 - xy + 2y^2\right)^2}$$

At P, $2x - y = 1$, $4y - x = 10$ and $\left(x^2 - xy + 2y^2\right)^2 = 16^2 = 256$, so

$$\frac{\partial z}{\partial x} = -\frac{48 \times 1}{256} = -\frac{3}{16}$$

$$\frac{\partial z}{\partial y} = -\frac{48 \times 10}{256} = -\frac{15}{8}$$

(b) At a point (x_0, y_0, z_0) on the surface, with the partial derivatives evaluated there, the tangent plane is

$$z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

For the surface $z = f(x, y)$ and the point $\left(a, b, f(a, b)\right)$, this is also written as

$$z = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$$

Using part (a), with $P(2, 3, 3)$:

$$z - 3 = -\frac{3}{16}(x - 2) - \frac{15}{8}(y - 3)$$

$$16z - 48 = -3x + 6 - 30y + 90$$

$$3x + 30y + 16z = 144$$

This is in the form $ax + by + cz = d$ with $a = 3$, $b = 30$, $c = 16$, $d = 144$.

3. The function g is defined by

$$g(x, y, z) = (x^2 + yz)e^{xy+z}$$

The surface S has equation $g(x, y, z) = 2$, and the point $P(2, 1, -2)$ lies on S.

(a) Find $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$. **[4]**

(b) Find a vector equation of the normal line to S at P, and hence find the coordinates of the two points on this normal line whose distance from P is 14. **[3]**

Solution

(a) Using the product rule, with $\frac{\partial}{\partial x}(e^{xy+z}) = ye^{xy+z}$:

$$\begin{aligned}\frac{\partial g}{\partial x} &= 2x\, e^{xy+z} + (x^2 + yz)\, y\, e^{xy+z} \\ &= e^{xy+z}\left(2x + x^2y + y^2z\right)\end{aligned}$$

Similarly, with $\frac{\partial}{\partial y}(e^{xy+z}) = xe^{xy+z}$:

$$\begin{aligned}\frac{\partial g}{\partial y} &= z\, e^{xy+z} + (x^2 + yz)\, x\, e^{xy+z} \\ &= e^{xy+z}\left(z + x^3 + xyz\right)\end{aligned}$$

With $\frac{\partial}{\partial z}(e^{xy+z}) = e^{xy+z}$:

$$\begin{aligned}\frac{\partial g}{\partial z} &= y\, e^{xy+z} + (x^2 + yz)\, e^{xy+z} \\ &= e^{xy+z}\left(x^2 + yz + y\right)\end{aligned}$$

(b) At $P(2, 1, -2)$, $xy + z = 2 - 2 = 0$, so $e^{xy+z} = 1$. Substituting into the results of part (a):

$$\frac{\partial g}{\partial x} = 2 \times 2 + 2^2 \times 1 + 1^2 \times (-2) = 6$$

$$\frac{\partial g}{\partial y} = -2 + 2^3 + 2 \times 1 \times (-2) = 2$$

$$\frac{\partial g}{\partial z} = 2^2 + 1 \times (-2) + 1 = 3$$

A normal vector to S at a point is ∇g, which at P is

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

So a vector equation of the normal line to S at P is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

The direction vector has length

$$\left|\begin{pmatrix}6\\2\\3\end{pmatrix}\right| = \sqrt{36+4+9} = 7$$

so the point with parameter λ is at distance $7|\lambda|$ from P. For a distance of 14:

$$7|\lambda| = 14 \quad \Longrightarrow \quad \lambda = \pm 2$$

With $\lambda = 2$:

$$\mathbf{r} = \begin{pmatrix}2\\1\\-2\end{pmatrix} + 2\begin{pmatrix}6\\2\\3\end{pmatrix} = \begin{pmatrix}14\\5\\4\end{pmatrix}$$

With $\lambda = -2$:

$$\mathbf{r} = \begin{pmatrix}2\\1\\-2\end{pmatrix} - 2\begin{pmatrix}6\\2\\3\end{pmatrix} = \begin{pmatrix}-10\\-3\\-8\end{pmatrix}$$

The two points are $(14, 5, 4)$ and $(-10, -3, -8)$.

4. A surface S is defined by $g(x, y, z) = 2$, where

$$g(x, y, z) = x^2y - 2yz^2 + 3xz$$

The point $A(1, 1, a)$, where a is an integer, and the point $B(1, 2, 0)$ both lie on S.

(a) Find the value of a. **[2]**

(b) The normal line to S at A and the tangent plane to S at B intersect at the point P. Determine the position vector of P. **[7]**

Solution

(a) Since $A(1, 1, a)$ lies on S, $g(1, 1, a) = 2$:

$$1^2 \times 1 - 2 \times 1 \times a^2 + 3 \times 1 \times a = 2$$
$$2a^2 - 3a + 1 = 0$$
$$(2a - 1)(a - 1) = 0$$

So $a = \frac{1}{2}$ or $a = 1$. Since a is an integer, $a = 1$.

(b) The partial derivatives of g are

$$\frac{\partial g}{\partial x} = 2xy + 3z, \qquad \frac{\partial g}{\partial y} = x^2 - 2z^2, \qquad \frac{\partial g}{\partial z} = 3x - 4yz$$

A normal vector to S at a point is ∇g, which is

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix}$$

At $A(1, 1, 1)$ the partial derivatives are $2 + 3 = 5$, $1 - 2 = -1$ and $3 - 4 = -1$, so the normal line to S at A is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$$

At $B(1, 2, 0)$ the partial derivatives are $4 + 0 = 4$, $1 - 0 = 1$ and $3 - 0 = 3$, so the tangent plane at B has normal vector with components 4, 1, 3 and passes through B:

$$\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = 4 + 2 + 0 = 6$$

In Cartesian form the tangent plane is $4x + y + 3z = 6$. A general point on the normal line at A is $(1 + 5\lambda, 1 - \lambda, 1 - \lambda)$. Substituting into the plane:

$$4(1 + 5\lambda) + (1 - \lambda) + 3(1 - \lambda) = 6$$
$$8 + 16\lambda = 6$$
$$\lambda = -\frac{1}{8}$$

Therefore

$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{8}\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{9}{8} \\ \frac{9}{8} \end{pmatrix}$$

5. A surface has equation

$$z = x^2y - 3x^2 + y^3 - 6y^2 + 24y + 5$$

Show that there are exactly four points on the surface at which the normal line is parallel to the vector $24\mathbf{j} - \mathbf{k}$, and find the coordinates of these points. Hence write down a vector equation of the normal line at the one of these points with the greatest z-coordinate. **[8]**

Solution

Write the surface as $g(x, y, z) = 0$, where

$$g(x, y, z) = x^2y - 3x^2 + y^3 - 6y^2 + 24y + 5 - z$$

A normal vector to the surface at a point is ∇g, which is

$$\nabla g = \begin{pmatrix} \dfrac{\partial g}{\partial x} \\ \dfrac{\partial g}{\partial y} \\ \dfrac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy - 6x \\ x^2 + 3y^2 - 12y + 24 \\ -1 \end{pmatrix}$$

The normal line is parallel to $24\mathbf{j} - \mathbf{k}$ when $\nabla g = \mu(24\mathbf{j} - \mathbf{k})$ for some scalar μ. Equating the z-components gives $-1 = -\mu$, so $\mu = 1$, and therefore

$$2xy - 6x = 0 \qquad \text{and} \qquad x^2 + 3y^2 - 12y + 24 = 24$$

The first equation factorises:

$$2x(y - 3) = 0 \quad \Longrightarrow \quad x = 0 \ \text{ or } \ y = 3$$

The second equation simplifies to

$$x^2 + 3y^2 - 12y = 0$$

Case $x = 0$:

$$3y^2 - 12y = 0 \quad \Longrightarrow \quad 3y(y - 4) = 0 \quad \Longrightarrow \quad y = 0 \ \text{ or } \ y = 4$$

Substituting into the equation of the surface:

$$x = 0,\ y = 0: \quad z = 5$$

$$x = 0,\ y = 4: \quad z = 64 - 96 + 96 + 5 = 69$$

Case $y = 3$:

$$x^2 + 27 - 36 = 0 \quad \Longrightarrow \quad x^2 = 9 \quad \Longrightarrow \quad x = \pm 3$$

Substituting into the equation of the surface, with $x^2 = 9$ in both cases:

$$z = 9 \times 3 - 3 \times 9 + 27 - 54 + 72 + 5 = 50$$

These two cases cover every solution of the pair of equations, and they give four distinct points, so there are exactly four points at which the normal line is parallel to $24\mathbf{j} - \mathbf{k}$. They are

$$(0, 0, 5), \qquad (0, 4, 69), \qquad (3, 3, 50), \qquad (-3, 3, 50)$$

The point with the greatest z-coordinate is $(0, 4, 69)$. The normal line there passes through $(0, 4, 69)$ and has direction $24\mathbf{j} - \mathbf{k}$, so a vector equation is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 69 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 24 \\ -1 \end{pmatrix}$$

6. A surface has equation

$$z = 2x^3 - 6xy + 3y^2 + 5$$

The point Q with coordinates $(2, 1, 12)$ lies on the surface.

(a) Show that the equation of the tangent plane to the surface at Q is $18x - 6y - z = 18$. **[4]**

(b) There is exactly one other point R on the surface at which the tangent plane is parallel to the tangent plane at Q.
Find the coordinates of R and the equation of the tangent plane at R. **[7]**

Solution

(a) Differentiating partially:

$$\frac{\partial z}{\partial x} = 6x^2 - 6y$$

$$\frac{\partial z}{\partial y} = -6x + 6y$$

At Q, where $x = 2$ and $y = 1$:

$$\frac{\partial z}{\partial x} = 6 \times 2^2 - 6 \times 1 = 18$$

$$\frac{\partial z}{\partial y} = -6 \times 2 + 6 \times 1 = -6$$

The tangent plane at a point on the surface has normal vector

$$\mathbf{n} = \begin{pmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \\ -1 \end{pmatrix}$$

so at Q the normal is

$$\mathbf{n} = \begin{pmatrix} 18 \\ -6 \\ -1 \end{pmatrix}$$

The tangent plane is $18x - 6y - z = d$ for some constant d. It passes through $Q(2, 1, 12)$, so

$$d = 18 \times 2 - 6 \times 1 - 12$$

$$= 18$$

so the tangent plane at Q is

$$18x - 6y - z = 18$$

as required.

(b) Parallel planes have parallel normal vectors. The third component of the normal at every point is -1, so the normal at R must be exactly $18\mathbf{i} - 6\mathbf{j} - \mathbf{k}$. Hence at R:

$$6x^2 - 6y = 18$$

$$-6x + 6y = -6$$

The second equation gives $y = x - 1$. Substituting into the first:

$$6x^2 - 6(x - 1) = 18$$

$$6x^2 - 6x - 12 = 0$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

The root $x = 2$ gives $y = 1$, which is the point Q. So at R, $x = -1$ and $y = -2$, and

$$\begin{aligned} z &= 2 \times (-1)^3 - 6 \times (-1) \times (-2) + 3 \times (-2)^2 + 5 \\ &= -2 - 12 + 12 + 5 \\ &= 3 \end{aligned}$$

So R has coordinates $(-1, -2, 3)$.

The tangent plane at R has the same normal, so it is $18x - 6y - z = d'$ for some constant d'. It passes through R, so

$$d' = 18 \times (-1) - 6 \times (-2) - 3 = -9$$

so the tangent plane at R is

$$18x - 6y - z = -9$$

7. A surface has equation

$$z = 2y(x-y)^2 - 2y^2 + 14y$$

(a) The point $P(3,1,20)$ lies on the surface. Find a vector equation of the normal line to the surface at P, and find the coordinates of the point where this normal line meets the plane $x = y$. **[5]**

(b) Show that there is exactly one point on the surface at which the tangent plane has an equation of the form $6y - z = d$, where d is a constant. Find the coordinates of this point and the value of d. **[7]**

Solution

The partial derivatives are

$$\frac{\partial z}{\partial x} = 4y(x-y)$$

$$\frac{\partial z}{\partial y} = 2(x-y)^2 + 2y \times 2(x-y) \times (-1) - 4y + 14 = 2(x-y)^2 - 4y(x-y) - 4y + 14$$

At a point (x_0, y_0, z_0) on the surface, with the partial derivatives evaluated there, the tangent plane is

$$z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

For the surface $z = f(x,y)$ and the point $\big(a, b, f(a,b)\big)$, this is also written as

$$z = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$

So a normal vector to the surface at that point is

$$\begin{pmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \\ -1 \end{pmatrix}$$

(a) At $P(3,1,20)$, $x - y = 2$ and $y = 1$, so

$$\frac{\partial z}{\partial x} = 4 \times 1 \times 2 = 8$$

$$\frac{\partial z}{\partial y} = 2 \times 4 - 4 \times 1 \times 2 - 4 + 14 = 10$$

A normal vector at P is therefore $8\mathbf{i} + 10\mathbf{j} - \mathbf{k}$, and the normal line is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 10 \\ -1 \end{pmatrix}$$

A general point on this line is $(3 + 8\lambda,\ 1 + 10\lambda,\ 20 - \lambda)$. It lies in the plane $x = y$ when

$$3 + 8\lambda = 1 + 10\lambda \quad \Longrightarrow \quad \lambda = 1$$

so the normal line meets the plane $x = y$ at $(11, 11, 19)$.

(b) The plane $6y - z = d$ has normal vector

$$\begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix}$$

The tangent plane at a point has this form exactly when its normal vector $\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}$ is a multiple of $6\mathbf{j} - \mathbf{k}$. Both $\mathbf{k}$-components are -1, so the multiple is 1, and the conditions are

$$\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial y} = 6$$

The first condition gives

$$4y(x - y) = 0 \quad \Longrightarrow \quad y = 0 \text{ or } x = y$$

If $y = 0$, the second condition becomes

$$2x^2 + 14 = 6 \quad \Longrightarrow \quad x^2 = -4$$

which has no real solutions.

If $x = y$, the second condition becomes

$$-4y + 14 = 6 \quad \Longrightarrow \quad y = 2$$

so $x = 2$, and then

$$z = 2 \times 2 \times 0^2 - 2 \times 2^2 + 14 \times 2 = 20$$

The conditions are satisfied at only this one point, so there is exactly one point on the surface at which the tangent plane has the form $6y - z = d$, namely $(2, 2, 20)$.

The tangent plane at $(2, 2, 20)$ is

$$z - 20 = 0 \times (x - 2) + 6(y - 2)$$

$$z = 6y + 8$$

$$6y - z = -8$$

so $d = -8$.

8. A surface S has equation

$$x^2 + y^2 + 6z = 54$$

C is the set of points on S at which the tangent plane to S makes an angle of $\frac{1}{3}\pi$ radians with the x-y plane.

(a) Show that C lies in a plane, Π, and find the equation of Π. **[5]**

(b) Show that the tangent planes to S at all points of C pass through a common point, and find the coordinates of this point. **[2]**

Solution

(a) Write the surface as $g(x, y, z) = 54$, where $g(x, y, z) = x^2 + y^2 + 6z$. The partial derivatives are

$$\frac{\partial g}{\partial x} = 2x, \qquad \frac{\partial g}{\partial y} = 2y, \qquad \frac{\partial g}{\partial z} = 6$$

A normal vector to S at the point (x, y, z) is ∇g, which is

$$\mathbf{n} = \nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 6 \end{pmatrix}$$

The x-y plane has normal $\mathbf{k}$. The acute angle between two planes equals the acute angle between their normals. Using the scalar product:

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{|\mathbf{n}||\mathbf{k}|}$$

With $\theta = \frac{1}{3}\pi$, $\mathbf{n} \cdot \mathbf{k} = 6$ and $|\mathbf{n}| = \sqrt{4x^2 + 4y^2 + 36} = 2\sqrt{x^2 + y^2 + 9}$:

$$\frac{6}{2\sqrt{x^2 + y^2 + 9}} = \frac{1}{2}$$

$$\sqrt{x^2 + y^2 + 9} = 6$$

$$x^2 + y^2 + 9 = 36$$

$$x^2 + y^2 = 27$$

These points lie on S, so substituting into $x^2 + y^2 + 6z = 54$:

$$27 + 6z = 54 \quad \Longrightarrow \quad z = \frac{9}{2}$$

Every point of C has the same z-coordinate, so C lies in the plane

$$\Pi : \ z = \frac{9}{2}$$

(b) Let $(p, q, \frac{9}{2})$ be any point of C, so that $p^2 + q^2 = 27$. From part (a), the tangent plane there has normal $2p\mathbf{i} + 2q\mathbf{j} + 6\mathbf{k}$, so its equation is

$$\begin{aligned} 2px + 2qy + 6z &= 2p \times p + 2q \times q + 6 \times \frac{9}{2} \\ &= 2(p^2 + q^2) + 27 \\ &= 2 \times 27 + 27 \\ &= 81 \end{aligned}$$

that is, $px + qy + 3z = \frac{81}{2}$. Setting $x = 0$ and $y = 0$ removes p and q:

$$3z = \frac{81}{2} \quad \Longrightarrow \quad z = \frac{27}{2}$$

This holds whatever the values of p and q, so every tangent plane to S at a point of C passes through the common point $\left(0, 0, \frac{27}{2}\right)$.

9. A surface S has equation

$$x^2 + 2xz - y^2 + 4yz - z^2 + 10 = 0$$

(a) Find a normal vector to S at the general point (x, y, z) on S. **[3]**

(b) Find a Cartesian equation of the tangent plane to S at the point $Q(1, 6, 1)$. **[3]**

(c) Show that there is no point on S at which the normal line to S is parallel to the x-axis. **[4]**

(d) The plane with equation $x + 2y - 2z = k$, where k is a constant, is a tangent plane to S. Find the two possible values of k and, for each value, the coordinates of the point at which the plane touches S. **[8]**

Solution

(a) Write the surface as $g(x, y, z) = 0$, where

$$g(x, y, z) = x^2 + 2xz - y^2 + 4yz - z^2 + 10$$

The partial derivatives are

$$\frac{\partial g}{\partial x} = 2x + 2z$$

$$\frac{\partial g}{\partial y} = -2y + 4z$$

$$\frac{\partial g}{\partial z} = 2x + 4y - 2z$$

A normal vector to S at the point (x, y, z) is ∇g, which is

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x + 2z \\ 4z - 2y \\ 2x + 4y - 2z \end{pmatrix} = 2\begin{pmatrix} x + z \\ 2z - y \\ x + 2y - z \end{pmatrix}$$

So a normal vector to S at (x, y, z) is $(x + z)\mathbf{i} + (2z - y)\mathbf{j} + (x + 2y - z)\mathbf{k}$, or any non-zero multiple of it.

(b) At $Q(1, 6, 1)$, using part (a):

$$\nabla g = \begin{pmatrix} 2 + 2 \\ 4 - 12 \\ 2 + 24 - 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 24 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

So the tangent plane has equation $x - 2y + 6z = d$ for some constant d. It passes through Q, so

$$d = 1 - 2 \times 6 + 6 \times 1 = -5$$

The tangent plane at Q is $x - 2y + 6z = -5$.

(c) The normal line is parallel to the x-axis when the normal vector is a non-zero multiple of $\mathbf{i}$. Using part (a), the y- and z-components must both be zero:

$$2z - y = 0 \quad \Longrightarrow \quad y = 2z$$

$$x + 2y - z = 0 \quad \Longrightarrow \quad x = z - 4z = -3z$$

The point must also lie on S. Substituting $x = -3z$ and $y = 2z$ into the equation of S:

$$9z^2 + 2(-3z)z - (2z)^2 + 4(2z)z - z^2 + 10 = 0$$

$$9z^2 - 6z^2 - 4z^2 + 8z^2 - z^2 + 10 = 0$$

$$6z^2 + 10 = 0$$

Since $6z^2 + 10 \geq 10 > 0$ for all real z, this has no real solutions. Hence there is no point on S at which the normal line to S is parallel to the x-axis.

(d) The plane $x + 2y - 2z = k$ has normal $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. At the point of contact the normal to S from part (a) must be parallel to this, so for some $\lambda \neq 0$:

$$\begin{pmatrix} x + z \\ 2z - y \\ x + 2y - z \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

This gives

$$x + z = \lambda$$

$$2z - y = 2\lambda$$

$$x + 2y - z = -2\lambda$$

From the first two equations, $x = \lambda - z$ and $y = 2z - 2\lambda$. Substituting into the third:

$$(\lambda - z) + 2(2z - 2\lambda) - z = -2\lambda$$

$$2z - 3\lambda = -2\lambda$$

$$z = \tfrac{1}{2}\lambda$$

so $x = \frac{1}{2}\lambda$ and $y = -\lambda$. The point lies on S. Substituting:

$$\tfrac{1}{4}\lambda^2 + 2\left(\tfrac{1}{2}\lambda\right)\left(\tfrac{1}{2}\lambda\right) - \lambda^2 + 4(-\lambda)\left(\tfrac{1}{2}\lambda\right) - \tfrac{1}{4}\lambda^2 + 10 = 0$$

$$\tfrac{1}{4}\lambda^2 + \tfrac{1}{2}\lambda^2 - \lambda^2 - 2\lambda^2 - \tfrac{1}{4}\lambda^2 + 10 = 0$$

$$-\tfrac{5}{2}\lambda^2 + 10 = 0$$

$$\lambda^2 = 4$$

so $\lambda = 2$ or $\lambda = -2$.

When $\lambda = 2$, the point is $(1, -2, 1)$. It lies in the plane, so

$$k = 1 + 2 \times (-2) - 2 \times 1 = -5$$

When $\lambda = -2$, the point is $(-1, 2, -1)$. It lies in the plane, so

$$k = -1 + 2 \times 2 - 2 \times (-1) = 5$$

So $k = -5$, with the plane touching S at $(1, -2, 1)$, or $k = 5$, with the plane touching S at $(-1, 2, -1)$.

10. A surface S has equation $g(x, y, z) = 0$, where

$$g(x, y, z) = x^2 + 3y^2 + 3z^2 - 2xy + 4yz - 2xz - 54$$

The point P has coordinates $(1, 2, 3)$.

(a) Verify that P lies on S, and find an equation of the tangent plane to S at P, giving your answer in the form $ax + by + cz = d$. **[5]**

(b) Find the coordinates of the two points on S at which the tangent plane is parallel to the plane $z = 0$, and write down the equations of the tangent planes at these points. **[6]**

(c) Show that the plane $4x + 4y - 3z = 54$ is a tangent plane to S, and find the coordinates of its point of contact with S. **[8]**

Solution

(a) Substituting $x = 1$, $y = 2$, $z = 3$:

$$g(1, 2, 3) = 1 + 12 + 27 - 4 + 24 - 6 - 54 = 0$$

so P lies on S.

The partial derivatives of g are

$$\frac{\partial g}{\partial x} = 2x - 2y - 2z$$

$$\frac{\partial g}{\partial y} = 6y - 2x + 4z$$

$$\frac{\partial g}{\partial z} = 6z + 4y - 2x$$

At $P(1, 2, 3)$ these are $2 - 4 - 6 = -8$, $12 - 2 + 12 = 22$ and $18 + 8 - 2 = 24$. A normal vector to S at a point is ∇g, which at P is

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} -8 \\ 22 \\ 24 \end{pmatrix} = 2\begin{pmatrix} -4 \\ 11 \\ 12 \end{pmatrix}$$

The tangent plane is therefore $-4x + 11y + 12z = d$. Substituting the coordinates of P:

$$d = -4 + 22 + 36 = 54$$

so the tangent plane at P is

$$-4x + 11y + 12z = 54$$

This is in the form $ax + by + cz = d$ with $a = -4$, $b = 11$, $c = 12$, $d = 54$.

(b) The plane $z = 0$ has normal $\mathbf{k}$, so the tangent plane is parallel to it where

$$\frac{\partial g}{\partial x} = 0, \qquad \frac{\partial g}{\partial y} = 0, \qquad \frac{\partial g}{\partial z} \neq 0$$

Using part (a), the first two conditions give

$$x - y - z = 0$$
$$-x + 3y + 2z = 0$$

Adding these gives $2y + z = 0$, so $z = -2y$, and then $x = y + z = -y$.

Substituting $x = -y$, $z = -2y$ into $g(x, y, z) = 0$:

$$y^2 + 3y^2 + 12y^2 + 2y^2 - 8y^2 - 4y^2 - 54 = 0$$
$$6y^2 = 54$$
$$y = \pm 3$$

When $y = 3$, the point is $(-3, 3, -6)$, and there

$$\frac{\partial g}{\partial z} = -36 + 12 + 6 = -18 \neq 0$$

When $y = -3$, the point is $(3, -3, 6)$, and there

$$\frac{\partial g}{\partial z} = 36 - 12 - 6 = 18 \neq 0$$

At each point the normal is a multiple of $\mathbf{k}$, so the tangent plane is $z =$ constant, the constant being the point's z-coordinate. The points are $(-3, 3, -6)$, with tangent plane $z = -6$, and $(3, -3, 6)$, with tangent plane $z = 6$.

(c) The plane $4x + 4y - 3z = 54$ has normal $4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$. Look for points of S where the normal to S is parallel to this. Using part (a), the partial derivatives are $2(x - y - z)$, $2(-x + 3y + 2z)$ and $2(-x + 2y + 3z)$, so for some constant k the conditions are

$$x - y - z = 4k$$
$$-x + 3y + 2z = 4k$$
$$-x + 2y + 3z = -3k$$

Adding the first equation to each of the others:

$$2y + z = 8k$$
$$y + 2z = k$$

So $3y = 2(8k) - k = 15k$, giving $y = 5k$, $z = 8k - 10k = -2k$ and $x = 4k + y + z = 7k$.

The point $(7k, 5k, -2k)$ lies on S when

$$49k^2 + 75k^2 + 12k^2 - 70k^2 - 40k^2 + 28k^2 - 54 = 0$$
$$54k^2 = 54$$
$$k = \pm 1$$

Take $k = 1$, giving the point $(7, 5, -2)$ on S. There the partial derivatives are

$$\begin{pmatrix} 2(7 - 5 + 2) \\ 2(-7 + 15 - 4) \\ 2(-7 + 10 - 6) \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix}$$

so the tangent plane at $(7, 5, -2)$ is $4x + 4y - 3z = d$, where

$$d = 4 \times 7 + 4 \times 5 - 3 \times (-2) = 28 + 20 + 6 = 54$$

Hence the tangent plane to S at $(7, 5, -2)$ is $4x + 4y - 3z = 54$, so this plane is a tangent plane to S, as required. (The value $k = -1$ gives the point $(-7, -5, 2)$, where the tangent plane is $4x + 4y - 3z = -54$, a different plane.)

The point of contact is $(7, 5, -2)$.