OPTIONAL PURE / SURFACES & PARTIAL DIFFERENTIATION

Stationary Points of Surfaces

Worked solutions 12 questions · 128 marks

EXAM BOARDS OCR A Y545 · OCR MEI Extra Pure Y435

Instructions

OCR MEI Extra Pure (Y435): the Hessian matrix and second partial derivatives are not examined. Questions 7 and 11 can be left out, or just their part **(a)**; Question 8 can be left out, or just its part **(c)**.

Questions

Question 1	*6 marks*	2
Question 2	*8 marks*	3
Question 3	*9 marks*	5
Question 4	*10 marks*	7
Question 5	*12 marks*	9
Question 6	*13 marks*	11
Question 7	*11 marks*	13
Question 8	*12 marks*	15
Question 9	*13 marks*	17
Question 10	*11 marks*	20
Question 11	*14 marks*	22
Question 12	*9 marks*	24

1. A surface is defined in three dimensions by the equation

$$z = 3x^2y + y^3 - 12y$$

Find the coordinates of all the stationary points on the surface. **[6]**

Solution

At a stationary point on the surface, both partial derivatives are zero:

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0$$

Differentiating partially with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = 6xy$$

$$\frac{\partial z}{\partial y} = 3x^2 + 3y^2 - 12$$

Setting the first to zero gives

$$6xy = 0 \quad \Longrightarrow \quad x = 0 \ \text{ or } \ y = 0$$

Setting the second to zero gives

$$3x^2 + 3y^2 - 12 = 0 \quad \Longrightarrow \quad x^2 + y^2 = 4$$

Case $x = 0$: then $y^2 = 4$, so $y = 2$ or $y = -2$:

$$y = 2: \quad z = 3(0)^2(2) + 2^3 - 12(2) = 8 - 24 = -16$$

$$y = -2: \quad z = 3(0)^2(-2) + (-2)^3 - 12(-2) = -8 + 24 = 16$$

Case $y = 0$: then $x^2 = 4$, so $x = 2$ or $x = -2$. Every term of z contains a factor of y, so

$$z = 3x^2(0) + 0^3 - 12(0) = 0$$

The stationary points are $(0, 2, -16)$, $(0, -2, 16)$, $(2, 0, 0)$ and $(-2, 0, 0)$.

2. A surface S is defined by

$$z = 30 + 6x - 8y - x^2 - 4y^2$$

(a) Show that the point $P(3, -1, 43)$ is the only stationary point on S. **[2]**

(b) (i) On a set of axes, sketch the contour of S corresponding to $z = 27$, stating the exact coordinates of the points where it crosses the coordinate axes. **[3]**

(ii) By using your sketch in part **(b)(i)**, deduce that P must be a maximum point on S. **[3]**

Solution

(a) Stationary points occur where both partial derivatives are zero:

$$\frac{\partial z}{\partial x} = 6 - 2x = 0 \quad \Longrightarrow \quad x = 3$$

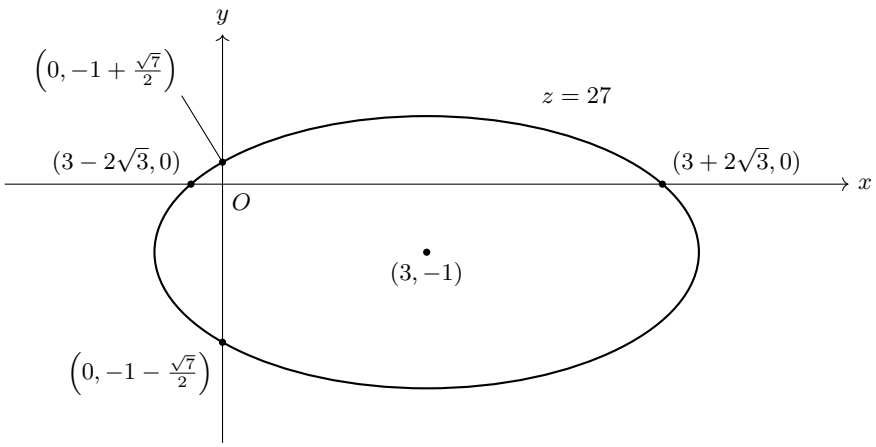
$$\frac{\partial z}{\partial y} = -8 - 8y = 0 \quad \Longrightarrow \quad y = -1$$

Each equation has exactly one solution, so there is exactly one stationary point. At $x = 3$, $y = -1$:

$$z = 30 + 18 + 8 - 9 - 4 = 43$$

so $P(3, -1, 43)$ is the only stationary point on S, as required.

(b) (i) The sketch should look like:



Setting $z = 27$ and completing the square:

$$27 = 30 + 6x - 8y - x^2 - 4y^2$$

$$x^2 - 6x + 4y^2 + 8y = 3$$

$$(x - 3)^2 - 9 + 4(y + 1)^2 - 4 = 3$$

$$(x - 3)^2 + 4(y + 1)^2 = 16$$

so the contour is an ellipse with centre $(3, -1)$, extending 4 units either side of the centre in the x-direction and 2 units either side in the y-direction.

On the x-axis, $y = 0$:

$$(x - 3)^2 + 4 = 16 \quad \Longrightarrow \quad (x - 3)^2 = 12 \quad \Longrightarrow \quad x = 3 \pm 2\sqrt{3}$$

On the y-axis, $x = 0$:

$$9 + 4(y+1)^2 = 16 \quad \Longrightarrow \quad (y+1)^2 = \frac{7}{4} \quad \Longrightarrow \quad y = -1 \pm \frac{\sqrt{7}}{2}$$

The contour crosses the axes at $(3 + 2\sqrt{3}, 0)$, $(3 - 2\sqrt{3}, 0)$, $\left(0, -1 + \frac{\sqrt{7}}{2}\right)$ and $\left(0, -1 - \frac{\sqrt{7}}{2}\right)$.

(ii) The sketch in part **(b)(i)** shows that the contour $z = 27$ is a closed curve, and the point $(3, -1)$, directly below P, lies inside it at its centre.

Everywhere on this contour the surface has height $z = 27$, whereas at P the height is $z = 43 > 27$.

Consider the part of S lying above the region enclosed by the contour. This part of the surface has a highest point. That highest point cannot lie on the contour, since the height there is only 27 and the surface reaches 43 at P. So the highest point lies strictly inside the contour, where the surface has a peak, and a peak is a stationary point.

By part **(a)**, P is the only stationary point on S, so this highest point must be P. Hence P must be a maximum point on S.

3. A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = y^3 - x^2y + 3x^2 - 3y^2 + 20$$

You are given that the point $(0, 0, 20)$ on S is a stationary point.

(a) Determine the coordinates of all the other stationary points on S. **[6]**

(b) By sketching the sections $z = f(x, 0)$ and $z = f(0, y)$ on separate sets of axes, determine the nature of the stationary point $(0, 0, 20)$ on S. **[3]**

Solution

(a) Differentiating partially:

$$\frac{\partial z}{\partial x} = -2xy + 6x = 2x(3 - y)$$

$$\frac{\partial z}{\partial y} = 3y^2 - x^2 - 6y$$

At a stationary point both partial derivatives are zero. From $\frac{\partial z}{\partial x} = 0$, either $x = 0$ or $y = 3$.

If $x = 0$, then $\frac{\partial z}{\partial y} = 0$ gives

$$3y^2 - 6y = 0 \implies 3y(y - 2) = 0 \implies y = 0 \text{ or } y = 2$$

The value $y = 0$ gives the stationary point $(0, 0, 20)$ that the question supplies. When $x = 0$ and $y = 2$:

$$z = 2^3 - 0 + 0 - 3 \times 2^2 + 20 = 16$$

If $y = 3$, then $\frac{\partial z}{\partial y} = 0$ gives

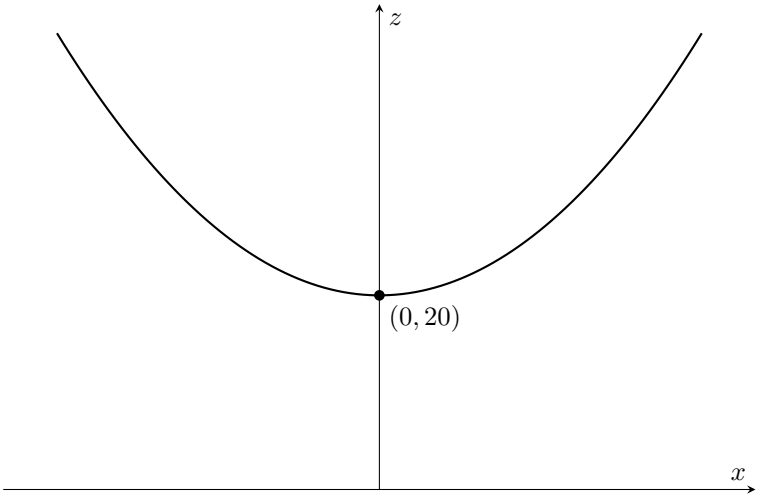
$$27 - x^2 - 18 = 0 \implies x^2 = 9 \implies x = \pm 3$$

When $x = \pm 3$ and $y = 3$:

$$z = 27 - 9 \times 3 + 3 \times 9 - 3 \times 9 + 20 = 20$$

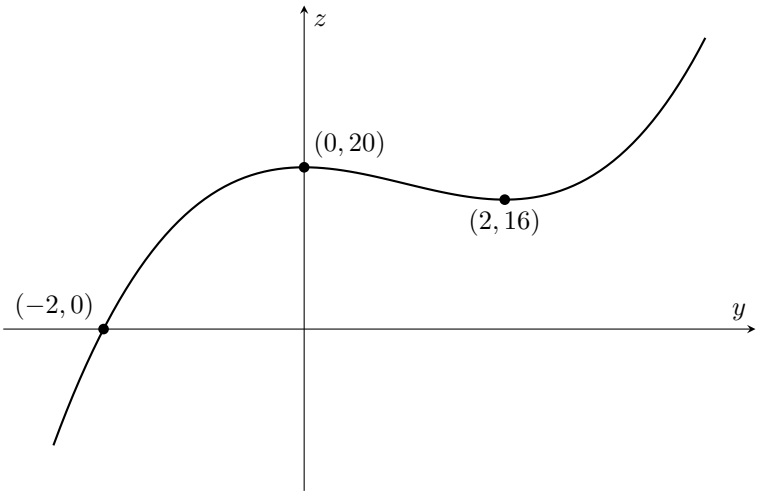
So the other stationary points are $(0, 2, 16)$, $(3, 3, 20)$ and $(-3, 3, 20)$.

(b) The section $z = f(x, 0)$ is $z = 3x^2 + 20$. The sketch should look like:



This is a U-shaped parabola with its vertex at $(0, 20)$, so this section has a minimum at $x = 0$.

The section $z = f(0, y)$ is $z = y^3 - 3y^2 + 20$. The sketch should look like:



Differentiating, $\frac{\mathrm{d}z}{\mathrm{d}y} = 3y^2 - 6y = 3y(y - 2)$, so this section has stationary points at $(0, 20)$ and $(2, 16)$. The cubic has a positive y^3 coefficient, so the first is a maximum and the second a minimum. Also $z = (y + 2)(y^2 - 5y + 10)$, and $y^2 - 5y + 10$ has discriminant $(-5)^2 - 4 \times 1 \times 10 = -15 < 0$, so the section meets the y-axis only at $(-2, 0)$.

Through $(0, 0, 20)$, the section $z = f(x, 0)$ has a minimum, while the section $z = f(0, y)$ has a maximum. Hence $(0, 0, 20)$ is a saddle point.

4. For all real values of x and y, the surface S has equation

$$z = -9x^2 + 12xy - 4y^2 + 15x - 10y + k$$

where k is an integer constant.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **[2]**

(b) Show that S has infinitely many stationary points, and that each of them lies vertically above or below the line $6x-4y = 5$ in the x-y plane. Find, in terms of k, the value of z at these stationary points. **[4]**

(c) Determine the largest value of k for which the whole of S lies below the x-y plane, justifying your answer. **[4]**

Solution

(a) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = -18x + 12y + 15$$

$$\frac{\partial z}{\partial y} = 12x - 8y - 10$$

(b) At a stationary point both partial derivatives are zero:

$$-18x + 12y + 15 = 0 \implies 6x - 4y = 5$$

$$12x - 8y - 10 = 0 \implies 6x - 4y = 5$$

Both conditions reduce to the same equation, $6x - 4y = 5$, so every point (x, y) on this line makes both partial derivatives zero. The line contains infinitely many points, so S has infinitely many stationary points, and each of them lies vertically above or below the line $6x - 4y = 5$ in the x-y plane.

To find z at these points, note that the quadratic terms form a perfect square:

$$-9x^2 + 12xy - 4y^2 = -(3x - 2y)^2$$

Also $15x - 10y = 5(3x - 2y)$, so

$$z = -(3x - 2y)^2 + 5(3x - 2y) + k$$

On the line $6x - 4y = 5$ we have $3x - 2y = \frac{5}{2}$, so at every stationary point

$$z = -\left(\frac{5}{2}\right)^2 + 5 \times \frac{5}{2} + k = -\frac{25}{4} + \frac{25}{2} + k = k + \frac{25}{4}$$

(c) Using the form of z from part (b) and completing the square in $3x - 2y$:

$$z = -\left[(3x - 2y)^2 - 5(3x - 2y)\right] + k$$

$$= -\left[\left(3x - 2y - \frac{5}{2}\right)^2 - \frac{25}{4}\right] + k$$

$$= k + \frac{25}{4} - \left(3x - 2y - \frac{5}{2}\right)^2$$

The squared term is never negative, so $z \leq k + \frac{25}{4}$ for all x and y, with equality exactly when $3x - 2y = \frac{5}{2}$, that is, at the stationary points on the line $6x - 4y = 5$. Hence these stationary points are all maxima, and the greatest value of z on S is $k + \frac{25}{4}$.

The whole of S lies below the x-y plane when $z < 0$ for all x and y, which happens exactly when this greatest value is negative:

$$k + \frac{25}{4} < 0 \quad \Longrightarrow \quad k < -\frac{25}{4} = -6.25$$

Since k is an integer, the largest value is $k = -7$.

5. A surface S is defined by

$$z = f(x, y) = x^2y - 2xy + y - 4y^3 + 3$$

(a) Show that there is only one stationary point on S, and find the values of x and y at this point. **[6]**

The value of z at the stationary point is denoted by s.

(b) State the value of s. **[1]**

(c) By factorising $f(x, y) - s$, sketch the contours of S for $z = s$, showing where they meet the coordinate axes. **[3]**

(d) Hence, by considering the sign of $f(x, y) - s$ in the regions separated by these contours, explain whether the stationary point is a maximum point, a minimum point or a saddle point. **[2]**

Solution

(a) The first partial derivatives are

$$\frac{\partial f}{\partial x} = 2xy - 2y = 2y(x - 1)$$

$$\frac{\partial f}{\partial y} = x^2 - 2x + 1 - 12y^2 = (x - 1)^2 - 12y^2$$

At a stationary point both partial derivatives are zero. From $\frac{\partial f}{\partial x} = 0$, either $y = 0$ or $x = 1$.

If $y = 0$, then $\frac{\partial f}{\partial y} = 0$ gives

$$(x - 1)^2 = 0 \quad \Longrightarrow \quad x = 1$$

If $x = 1$, then $\frac{\partial f}{\partial y} = 0$ gives

$$-12y^2 = 0 \quad \Longrightarrow \quad y = 0$$

Both cases give the same point, so there is only one stationary point on S, where $x = 1$ and $y = 0$.

(b) Substituting $x = 1$, $y = 0$:

$$s = f(1, 0) = 0 - 0 + 0 - 0 + 3 = 3$$

(c) Subtracting $s = 3$ and factorising:

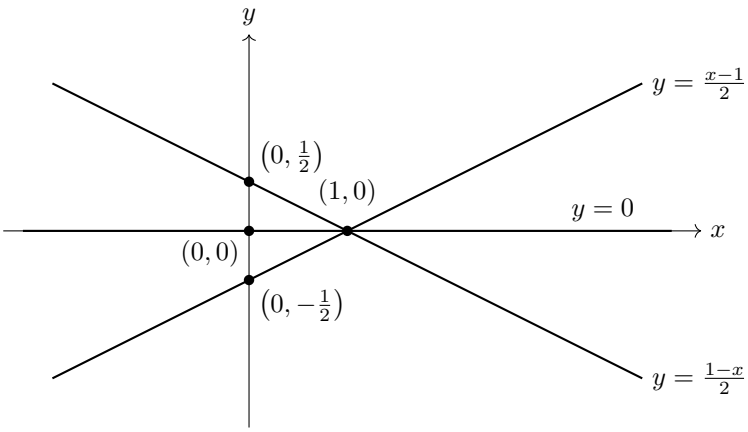
$$\begin{aligned} f(x, y) - 3 &= x^2y - 2xy + y - 4y^3 \\ &= y\big(x^2 - 2x + 1 - 4y^2\big) \\ &= y\big((x - 1)^2 - (2y)^2\big) \\ &= y(x - 1 - 2y)(x - 1 + 2y) \end{aligned}$$

The contour $z = 3$ is where $f(x, y) - 3 = 0$, so it consists of the three straight lines

$$y = 0, \qquad y = \frac{x - 1}{2}, \qquad y = \frac{1 - x}{2}$$

The line $y = 0$ is the x-axis itself. Setting $y = 0$ in the other two lines gives $x = 1$, so all three lines pass through $(1, 0)$, the stationary point. Setting $x = 0$ gives $y = -\frac{1}{2}$ on $y = \frac{x-1}{2}$ and $y = \frac{1}{2}$ on $y = \frac{1-x}{2}$.

The sketch should look like:



The contours meet the x-axis at $(1, 0)$ (the contour $y = 0$ lies along the whole x-axis), and meet the y-axis at $\left(0, \frac{1}{2}\right)$, $\left(0, -\frac{1}{2}\right)$ and $(0, 0)$.

(d) Using part (c):

$$f(x, y) - 3 = y(x - 1 - 2y)(x - 1 + 2y)$$

The three lines through $(1, 0)$ divide the plane into six regions around the stationary point. Crossing one of the lines (away from $(1, 0)$) changes the sign of exactly one factor, so the sign of $f(x, y) - 3$ alternates from one region to the next. For example, on the line $x = 1$:

$$f(1, y) - 3 = y(-2y)(2y) = -4y^3$$

so $f(x, y) - 3 < 0$ in the region directly above $(1, 0)$ ($y > 0$) and $f(x, y) - 3 > 0$ in the region directly below it ($y < 0$). Going round $(1, 0)$ the signs are therefore $+, -, +, -, +, -$ in turn.

Each of the six regions has a corner at $(1, 0)$, so however close to the stationary point we look, there are points where $z > s$ and points where $z < s$. Hence the stationary point is neither a maximum nor a minimum: it is a saddle point.

6. The surface S has equation $z = f(x, y)$, where

$$f(x, y) = 8x^2y - 16xy^2 + 3y^4$$

for all real values of x and y. The surface S has exactly two stationary points: one at the origin, O, and the other at the point $P(a, b, c)$, where $c = f(a, b)$.

(a) Determine the values of a, b and c. **[6]**

(b) Throughout this part, take a and b to be the values found in part (a).

(i) Evaluate f_x at the point of S where $x = a - 0.1$, $y = b$ and at the point of S where $x = a + 0.1$, $y = b$. **[2]**

(ii) Evaluate f_y at the point of S where $x = a$, $y = b - 0.1$ and at the point of S where $x = a$, $y = b + 0.1$. **[2]**

(iii) Use your answers to parts (b)(i) and (b)(ii) to sketch the portions of the sections of S given by

- $z = f(x, b)$, for $|x - a| \leq 0.1$,
- $z = f(a, y)$, for $|y - b| \leq 0.1$.

[2]

(iv) Explain why your sketches in part (b)(iii) are not, on their own, enough to show that P is a local minimum point of S. **[1]**

Solution

(a) Stationary points of S occur where $f_x = 0$ and $f_y = 0$. Differentiating partially:

$$f_x = 16xy - 16y^2 = 16y(x - y)$$

$$f_y = 8x^2 - 32xy + 12y^3$$

So $f_x = 0$ when $y = 0$ or $x = y$.

If $y = 0$, then $f_y = 8x^2 = 0$, so $x = 0$. This is the origin O.

If $x = y$, then

$$f_y = 8y^2 - 32y^2 + 12y^3 = 12y^3 - 24y^2 = 12y^2(y - 2) = 0$$

so $y = 0$ or $y = 2$. The value $y = 0$ gives the origin again, so P has $y = 2$ and $x = 2$. Then

$$c = f(2, 2) = 8 \times 4 \times 2 - 16 \times 2 \times 4 + 3 \times 16 = 64 - 128 + 48 = -16$$

so $a = 2$, $b = 2$ and $c = -16$.

(b) (i) Using $f_x = 16y(x - y)$ with $y = 2$:

$$f_x(1.9, 2) = 16 \times 2 \times (1.9 - 2) = -3.2$$

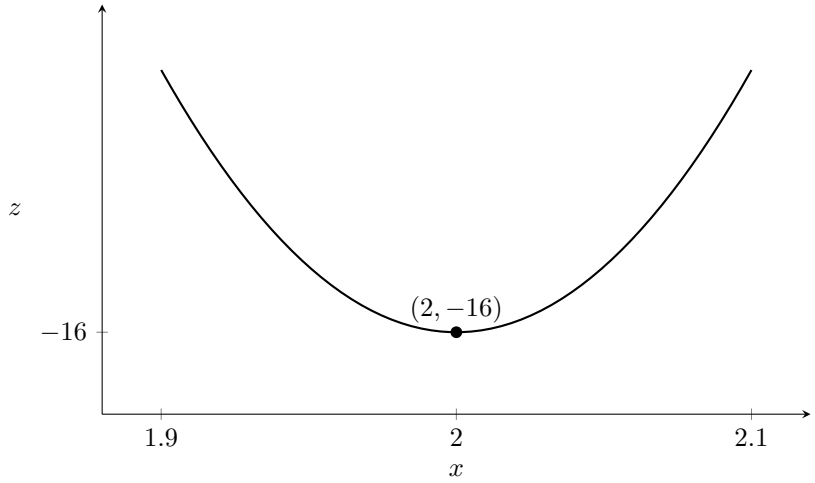
$$f_x(2.1, 2) = 16 \times 2 \times (2.1 - 2) = 3.2$$

(ii) Using $f_y = 8x^2 - 32xy + 12y^3$ with $x = 2$, so that $f_y = 32 - 64y + 12y^3$:

$$f_y(2, 1.9) = 32 - 121.6 + 82.308 = -7.292$$

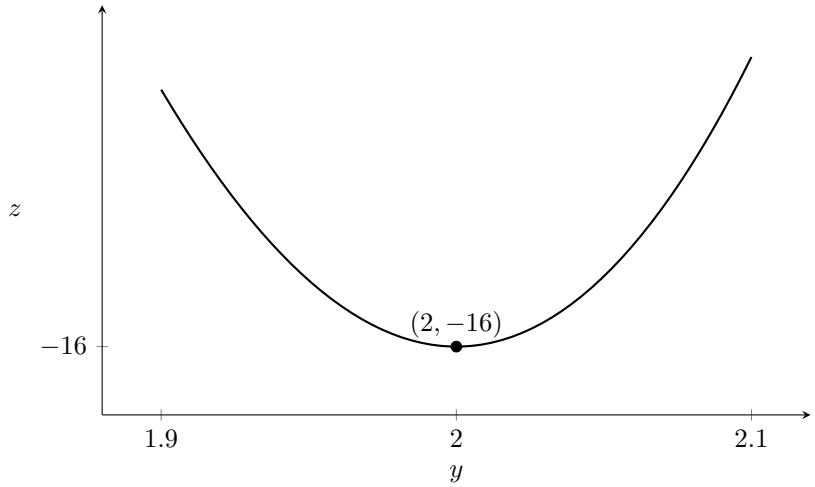
$$f_y(2, 2.1) = 32 - 134.4 + 111.132 = 8.732$$

(iii) The section $z = f(x, 2)$, for $1.9 \leq x \leq 2.1$, should look like:



From part (b)(i), the gradient of this section is -3.2 at $x = 1.9$ and 3.2 at $x = 2.1$, and it is 0 at $x = 2$ since P is a stationary point. So the section decreases to P and then increases: it is U-shaped with a minimum at $(2, -16)$.

The section $z = f(2, y)$, for $1.9 \leq y \leq 2.1$, should look like:



From part (b)(ii), the gradient of this section is -7.292 at $y = 1.9$ and 8.732 at $y = 2.1$, and it is 0 at $y = 2$. So this section is also U-shaped, with a minimum at $(2, -16)$.

(iv) The two sections only show how f behaves as we move away from P parallel to the x-axis or parallel to the y-axis. The surface could still decrease from P in some other direction (as at a saddle point), so a minimum in both of these sections does not show that P is a local minimum of S.

7. The surface S is given by the equation $z = f(x, y)$, where

$$f(x, y) = 2x^2y + xy^2 + 9y$$

and x and y take all real values.
Subscripts denote partial derivatives: for example, f_x denotes $\frac{\partial f}{\partial x}$ and f_{xy} denotes $\frac{\partial^2 f}{\partial y\,\partial x}$.

(a) Show that S has exactly two stationary points. Find, in exact form, the coordinates of the stationary point Q that has a positive y-coordinate. **[6]**

(b) Find the value of the determinant of $\mathbf{H}$, the Hessian matrix of S, at Q. **[4]**

(c) Deduce the nature of the stationary point Q. **[1]**

Solution

(a) Differentiating partially:

$$f_x = 4xy + y^2 = y(4x + y)$$
$$f_y = 2x^2 + 2xy + 9$$

At a stationary point both partial derivatives are zero. From $f_x = 0$, either $y = 0$ or $y = -4x$.

If $y = 0$, then $f_y = 0$ gives

$$2x^2 + 9 = 0$$

which has no real solutions, since $2x^2 + 9 \geq 9$.

If $y = -4x$, then $f_y = 0$ gives

$$2x^2 + 2x(-4x) + 9 = 0$$
$$-6x^2 + 9 = 0$$
$$x^2 = \frac{3}{2} \quad \Longrightarrow \quad x = \pm\frac{\sqrt{6}}{2}$$

Each value of x gives one value $y = -4x = \mp 2\sqrt{6}$, so there are exactly two stationary points, at $\left(\frac{\sqrt{6}}{2}, -2\sqrt{6}\right)$ and $\left(-\frac{\sqrt{6}}{2}, 2\sqrt{6}\right)$ in the x-y plane.

For Q, $y > 0$, so $x = -\frac{\sqrt{6}}{2}$ and $y = 2\sqrt{6}$. Then $2x^2 = 3$ and $xy = -\frac{\sqrt{6}}{2} \times 2\sqrt{6} = -6$, so

$$z = y\big(2x^2 + xy + 9\big) = 2\sqrt{6}\,(3 - 6 + 9) = 12\sqrt{6}$$

The coordinates of Q are $\left(-\frac{\sqrt{6}}{2},\, 2\sqrt{6},\, 12\sqrt{6}\right)$.

(b) The Hessian matrix is

$$\mathbf{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

The second partial derivatives are

$$f_{xx} = 4y$$
$$f_{xy} = f_{yx} = 4x + 2y$$
$$f_{yy} = 2x$$

At Q, with $x = -\frac{\sqrt{6}}{2}$ and $y = 2\sqrt{6}$:

$$f_{xx} = 8\sqrt{6}$$
$$f_{xy} = -2\sqrt{6} + 4\sqrt{6} = 2\sqrt{6}$$
$$f_{yy} = -\sqrt{6}$$

So

$$\begin{aligned}\det(\mathbf{H}) &= \begin{vmatrix} 8\sqrt{6} & 2\sqrt{6} \\ 2\sqrt{6} & -\sqrt{6} \end{vmatrix} \\ &= 8\sqrt{6} \times \left(-\sqrt{6}\right) - \left(2\sqrt{6}\right)^2 \\ &= -48 - 24 \\ &= -72\end{aligned}$$

(c) Since $\det(\mathbf{H}) = -72 < 0$ at Q, the stationary point Q is a saddle point.

8. The surface S has equation

$$z = y \ln x - x \sin y$$

for $x > 0$ and $0 < y < \pi$.

(a) Determine, as a function of x and y, the determinant of $\mathbf{H}$, the Hessian matrix of S. **[6]**

(b) Given that S has exactly one stationary point, P, use your answer to part (a) to deduce the nature of P. **[2]**

(c) The coordinates of P are (α, β, γ). Show that

$$\ln\left(\frac{\beta}{\sin\beta}\right) = \beta \cot\beta$$

and that $\gamma = \beta(\beta \cot\beta - 1)$. **[4]**

Solution

(a) The first partial derivatives of $z = y \ln x - x \sin y$ are

$$\frac{\partial z}{\partial x} = \frac{y}{x} - \sin y, \qquad \frac{\partial z}{\partial y} = \ln x - x \cos y$$

Differentiating again:

$$\frac{\partial^2 z}{\partial x^2} = -\frac{y}{x^2}$$

$$\frac{\partial^2 z}{\partial y^2} = x \sin y$$

$$\frac{\partial^2 z}{\partial x \, \partial y} = \frac{1}{x} - \cos y$$

The Hessian matrix is

$$\mathbf{H} = \begin{pmatrix} \dfrac{\partial^2 z}{\partial x^2} & \dfrac{\partial^2 z}{\partial x \, \partial y} \\ \dfrac{\partial^2 z}{\partial y \, \partial x} & \dfrac{\partial^2 z}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{y}{x^2} & \dfrac{1}{x} - \cos y \\ \dfrac{1}{x} - \cos y & x \sin y \end{pmatrix}$$

so its determinant is

$$\det(\mathbf{H}) = \left(-\frac{y}{x^2}\right)(x \sin y) - \left(\frac{1}{x} - \cos y\right)^2$$

$$= -\frac{y \sin y}{x} - \left(\frac{1}{x} - \cos y\right)^2$$

(b) For $x > 0$ and $0 < y < \pi$, we have $y > 0$ and $\sin y > 0$, so $\frac{y \sin y}{x} > 0$. Also $\left(\frac{1}{x} - \cos y\right)^2 \geq 0$. Hence

$$\det(\mathbf{H}) = -\frac{y \sin y}{x} - \left(\frac{1}{x} - \cos y\right)^2 < 0$$

at every point of S, and in particular at P. A stationary point at which $\det(\mathbf{H}) < 0$ is a saddle point, so P is a saddle point.

(c) At P, $x = \alpha$, $y = \beta$ and both first partial derivatives are zero. From $\frac{\partial z}{\partial x} = 0$:

$$\frac{\beta}{\alpha} - \sin\beta = 0 \quad \Longrightarrow \quad \alpha = \frac{\beta}{\sin\beta}$$

which is valid since $\sin\beta \neq 0$ for $0 < \beta < \pi$. From $\frac{\partial z}{\partial y} = 0$:

$$\ln\alpha - \alpha\cos\beta = 0 \quad \Longrightarrow \quad \ln\alpha = \alpha\cos\beta$$

Substituting $\alpha = \frac{\beta}{\sin\beta}$:

$$\ln\left(\frac{\beta}{\sin\beta}\right) = \frac{\beta\cos\beta}{\sin\beta} = \beta\cot\beta$$

as required.

For γ, substitute $x = \alpha$, $y = \beta$ into the equation of S, using $\ln\alpha = \beta\cot\beta$ and $\alpha\sin\beta = \beta$:

$$\begin{aligned}\gamma &= \beta\ln\alpha - \alpha\sin\beta \\ &= \beta(\beta\cot\beta) - \beta \\ &= \beta(\beta\cot\beta - 1)\end{aligned}$$

as required.

9. A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = ye^{-(x^2-6x+10)y^2}$$

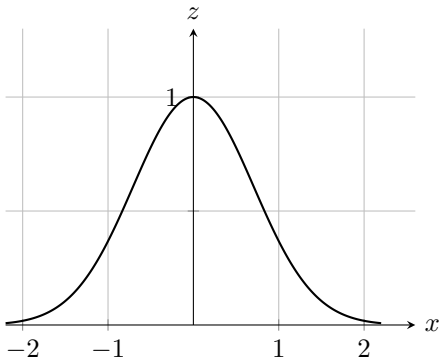
(a) (i) Find $\frac{\partial f}{\partial x}$. **[1]**

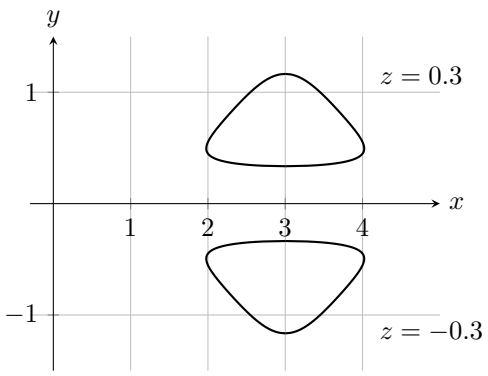
(ii) Show that

$$\frac{\partial f}{\partial y} = \left(1 - 2(x^2 - 6x + 10)y^2\right) e^{-(x^2-6x+10)y^2}$$ **[1]**

(iii) Find the exact coordinates of each of the stationary points on S. **[4]**

The first diagram below shows the graph of $z = e^{-x^2}$, and the second diagram below shows the contours of S defined by $z = 0.3$ and $z = -0.3$.





(b) Specify a sequence of transformations which transforms the graph of $z = e^{-x^2}$ onto the graph of the section defined by $z = f\left(x, \frac{1}{\sqrt{2}}\right)$. **[3]**

(c) Hence, or otherwise, sketch the section defined by $z = f\left(x, \frac{1}{\sqrt{2}}\right)$. **[1]**

(d) Using the second diagram above and your answer to part (c), classify each of the stationary points on S, justifying your answer. **[3]**

Solution

(a) (i) Treating y as a constant and using the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial x} &= y \times \left(-(2x-6)y^2\right)e^{-(x^2-6x+10)y^2} \\ &= -2(x-3)y^3 e^{-(x^2-6x+10)y^2}\end{aligned}$$

(ii) Treating x as a constant, use the product rule with $u = y$ and $v = e^{-(x^2-6x+10)y^2}$, so that

$$\frac{\partial u}{\partial y} = 1, \qquad \frac{\partial v}{\partial y} = -2(x^2-6x+10)y\, e^{-(x^2-6x+10)y^2}$$

Then

$$\begin{aligned}\frac{\partial f}{\partial y} &= v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} \\ &= e^{-(x^2-6x+10)y^2} - 2(x^2-6x+10)y^2 e^{-(x^2-6x+10)y^2} \\ &= \left(1-2(x^2-6x+10)y^2\right)e^{-(x^2-6x+10)y^2}\end{aligned}$$

as required.

(iii) At a stationary point, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. The exponential factor is always positive, so $\frac{\partial f}{\partial x} = 0$ gives

$$(x-3)y^3 = 0 \quad \Longrightarrow \quad x = 3 \text{ or } y = 0$$

If $y = 0$, then $\frac{\partial f}{\partial y} = (1-0)e^0 = 1 \neq 0$, so there is no stationary point with $y = 0$. Hence $x = 3$, so that $x^2 - 6x + 10 = 9 - 18 + 10 = 1$, and $\frac{\partial f}{\partial y} = 0$ gives

$$(1-2y^2)e^{-y^2} = 0 \quad \Longrightarrow \quad y^2 = \frac{1}{2} \quad \Longrightarrow \quad y = \pm\frac{1}{\sqrt{2}}$$

The corresponding values of z are

$$z = f\left(3, \pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} = \pm\frac{1}{\sqrt{2e}}$$

The stationary points are $\left(3, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ and $\left(3, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$.

(b) With $y = \frac{1}{\sqrt{2}}$, so that $y^2 = \frac{1}{2}$, completing the square gives

$$\begin{aligned}f\left(x, \frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}}e^{-\frac{1}{2}(x^2-6x+10)} \\ &= \frac{1}{\sqrt{2}}e^{-\frac{1}{2}\left((x-3)^2+1\right)} \\ &= \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}e^{-\frac{1}{2}(x-3)^2} \\ &= \frac{1}{\sqrt{2e}}e^{-\left(\frac{x-3}{\sqrt{2}}\right)^2}\end{aligned}$$

Starting from $z = e^{-x^2}$, a stretch parallel to the x-axis with scale factor $\sqrt{2}$ replaces x by $\frac{x}{\sqrt{2}}$:

$$z = e^{-\left(\frac{x}{\sqrt{2}}\right)^2}$$

Next, a translation of 3 units in the positive x-direction replaces x by $x-3$:

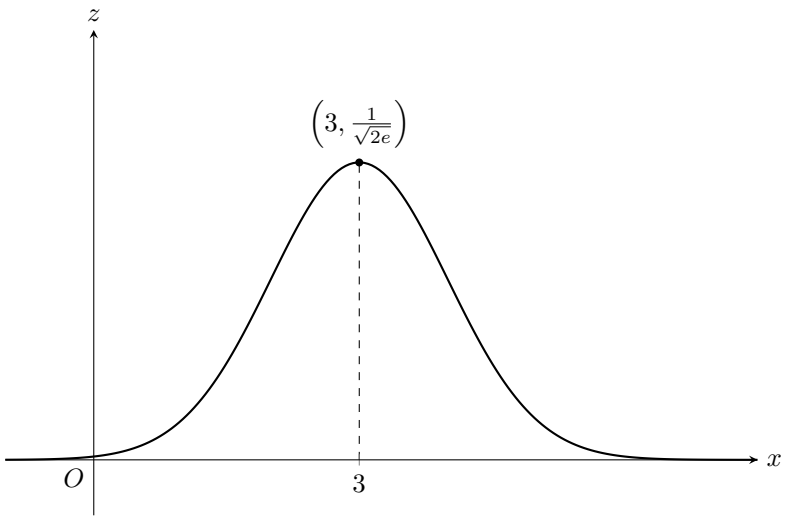
$$z = e^{-\left(\frac{x-3}{\sqrt{2}}\right)^2}$$

Finally, a stretch parallel to the z-axis with scale factor $\frac{1}{\sqrt{2e}}$ multiplies z by $\frac{1}{\sqrt{2e}}$:

$$z = \frac{1}{\sqrt{2e}} e^{-\left(\frac{x-3}{\sqrt{2}}\right)^2} = f\left(x, \frac{1}{\sqrt{2}}\right)$$

So the sequence is: a stretch parallel to the x-axis, scale factor $\sqrt{2}$; then a translation of 3 units in the positive x-direction; then a stretch parallel to the z-axis, scale factor $\frac{1}{\sqrt{2e}}$.

(c) The sketch should look like:



By part (b), the section is the bell-shaped curve $z = e^{-x^2}$ widened, moved so that it is symmetrical about $x = 3$, and scaled down so that its maximum point is $\left(3, \frac{1}{\sqrt{2e}}\right)$, where $\frac{1}{\sqrt{2e}} \approx 0.429$. As $x \to \pm\infty$, $e^{-\frac{1}{2}(x-3)^2} \to 0$, so $z \to 0$, and $z > 0$ for all x. Hence the x-axis is a horizontal asymptote in both directions.

(d) For $\left(3, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$: in the second diagram, the contour $z = 0.3$ is a closed loop surrounding the point $\left(3, \frac{1}{\sqrt{2}}\right)$, at which $z = \frac{1}{\sqrt{2e}} \approx 0.429 > 0.3$. So, moving out from this point in any direction, the surface comes down to the level 0.3 on the loop. A saddle point has no closed contour loop around it, so the stationary point is a local maximum or a local minimum. By part (c), along the section $z = f\left(x, \frac{1}{\sqrt{2}}\right)$ the surface has a maximum at $x = 3$, so the surface falls away from the point. Hence $\left(3, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ is a local maximum.

For $\left(3, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$: replacing y by $-y$ gives

$$f(x, -y) = -ye^{-(x^2-6x+10)y^2} = -f(x, y)$$

So the contour $z = -0.3$ is the reflection of the contour $z = 0.3$ in the x-axis, as the second diagram shows: a closed loop surrounding $\left(3, -\frac{1}{\sqrt{2}}\right)$, at which $z = -\frac{1}{\sqrt{2e}} \approx -0.429 < -0.3$. Also, the section $z = f\left(x, -\frac{1}{\sqrt{2}}\right)$ is $z = -f\left(x, \frac{1}{\sqrt{2}}\right)$, the curve from part (c) reflected in the x-axis, which has a minimum at $x = 3$. So, moving out from this point in any direction, the surface rises to the level -0.3 on the loop. Hence $\left(3, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ is a local maximum and $\left(3, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ is a local minimum.

10. A trading company deals in two goods. The formula used to estimate z, the total weekly cost to the company of trading the two goods, in tens of thousands of pounds, is

$$z = 0.4xy + \frac{7.2}{x} - 0.3y^2$$

where x and y are the masses, in thousands of tonnes, of the two goods.

You are given that $x > 0$ and $y > 0$.

(a) In the first week of trading, it was found that the values of x and y corresponded to the stationary value of z. Find these values of x and y, and determine the total cost to the company for this week. **[6]**

(b) For the second week, the company intends to make a small change to exactly one of x and y, keeping the other fixed, in order to reduce the total weekly cost. Determine which of x and y the company should change, and whether it matters if that variable is increased or decreased. **[5]**

Solution

(a) Differentiating partially with respect to x and to y:

$$\frac{\partial z}{\partial x} = 0.4y - \frac{7.2}{x^2}$$

$$\frac{\partial z}{\partial y} = 0.4x - 0.6y$$

At a stationary point both partial derivatives are zero. From $\frac{\partial z}{\partial y} = 0$:

$$0.4x = 0.6y \quad \Longrightarrow \quad y = \frac{2x}{3}$$

Substituting into $\frac{\partial z}{\partial x} = 0$:

$$0.4 \times \frac{2x}{3} = \frac{7.2}{x^2}$$

$$\frac{4}{15}x^3 = 7.2$$

$$x^3 = 27$$

So $x = 3$, which satisfies $x > 0$, and then $y = \frac{2\times 3}{3} = 2$.

The stationary value of z is

$$z = 0.4 \times 3 \times 2 + \frac{7.2}{3} - 0.3 \times 2^2 = 2.4 + 2.4 - 1.2 = 3.6$$

So $x = 3$ and $y = 2$, and the total cost for the first week is 3.6 tens of thousands of pounds, that is £36 000.

(b) First keep $y = 2$ fixed and vary x. Then

$$z = 0.8x + \frac{7.2}{x} - 1.2$$

so

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 0.8 - \frac{7.2}{x^2}, \qquad \frac{\mathrm{d}^2z}{\mathrm{d}x^2} = \frac{14.4}{x^3}$$

At $x = 3$, $\frac{\mathrm{d}z}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = \frac{14.4}{27} > 0$, so along this section z has a minimum at $x = 3$. Any small change in x alone would increase the cost.

Now keep $x = 3$ fixed and vary y. Then, completing the square:

$$\begin{aligned} z &= 1.2y + 2.4 - 0.3y^2 \\ &= 2.4 - 0.3\big(y^2 - 4y\big) \\ &= 2.4 - 0.3\big[(y-2)^2 - 4\big] \\ &= 3.6 - 0.3(y-2)^2 \end{aligned}$$

Along this section z has a maximum at $y = 2$. If y is changed to $2 + h$ or $2 - h$, where h is small, then in both cases

$$z = 3.6 - 0.3h^2$$

so the cost falls by $0.3h^2$ either way.

The company should change y, and it does not matter whether y is increased or decreased: a change of the same size in either direction reduces the cost by the same amount.

11. The surface C has equation $z = f(x, y)$, where

$$f(x, y) = x^4 + y^2 + 2axy$$

for all real x and y, and a is a non-zero real constant.
Subscripts denote partial derivatives: for example, f_x denotes $\frac{\partial f}{\partial x}$ and f_{xy} denotes $\frac{\partial^2 f}{\partial y\,\partial x}$.

(a) Show that C has exactly three stationary points, one of which is at the origin, and find the coordinates of the other two in terms of a. **[6]**

(b) Determine the nature of each of these stationary points of C. **[5]**

(c) The surface D has equation $z = g(x, y)$, where $g(x, y) = x^4 + y^2$ for all real x and y. Find the stationary point of D, explain why the test you used in part (b) cannot determine its nature, and use another method to determine its nature. **[3]**

Solution

(a) Differentiating partially with respect to x and to y:

$$f_x = 4x^3 + 2ay$$
$$f_y = 2y + 2ax$$

At a stationary point both partial derivatives are zero. From $f_y = 0$:

$$2y + 2ax = 0 \quad \Longrightarrow \quad y = -ax$$

Substituting into $f_x = 0$:

$$4x^3 + 2a(-ax) = 0$$
$$4x^3 - 2a^2x = 0$$
$$2x\left(2x^2 - a^2\right) = 0$$

so $x = 0$ or $x^2 = \frac{a^2}{2}$, that is $x = \pm\frac{a}{\sqrt{2}}$. Since $a \neq 0$, these are three distinct values of x, and each gives exactly one value of $y = -ax$, so C has exactly three stationary points.

When $x = 0$, $y = 0$ and $z = 0$, so one stationary point is the origin.

When $x = \frac{a}{\sqrt{2}}$, $y = -\frac{a^2}{\sqrt{2}}$, and

$$z = \left(\frac{a}{\sqrt{2}}\right)^4 + \left(-\frac{a^2}{\sqrt{2}}\right)^2 + 2a\left(\frac{a}{\sqrt{2}}\right)\left(-\frac{a^2}{\sqrt{2}}\right)$$
$$= \frac{a^4}{4} + \frac{a^4}{2} - a^4$$
$$= -\frac{a^4}{4}$$

When $x = -\frac{a}{\sqrt{2}}$, $y = \frac{a^2}{\sqrt{2}}$, and each term of z takes the same value as before, so again $z = -\frac{a^4}{4}$.

The other two stationary points are

$$\left(\frac{a}{\sqrt{2}}, -\frac{a^2}{\sqrt{2}}, -\frac{a^4}{4}\right) \quad \text{and} \quad \left(-\frac{a}{\sqrt{2}}, \frac{a^2}{\sqrt{2}}, -\frac{a^4}{4}\right)$$

(b) The second partial derivatives are

$$f_{xx} = 12x^2, \qquad f_{yy} = 2, \qquad f_{xy} = 2a$$

The Hessian matrix is

$$\mathbf{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

so its determinant is

$$\det(\mathbf{H}) = f_{xx}f_{yy} - (f_{xy})^2$$

At a stationary point: if $\det(\mathbf{H}) > 0$ and $f_{xx} > 0$ it is a minimum; if $\det(\mathbf{H}) > 0$ and $f_{xx} < 0$ it is a maximum; if $\det(\mathbf{H}) < 0$ it is a saddle point; if $\det(\mathbf{H}) = 0$ the test gives no conclusion. Here

$$\det(\mathbf{H}) = 12x^2 \times 2 - (2a)^2 = 24x^2 - 4a^2$$

At the origin, $\det(\mathbf{H}) = -4a^2 < 0$ since $a \neq 0$, so the origin is a saddle point.

At the other two points, $x^2 = \frac{a^2}{2}$, so

$$\det(\mathbf{H}) = 24 \times \frac{a^2}{2} - 4a^2 = 8a^2 > 0, \qquad f_{xx} = 12 \times \frac{a^2}{2} = 6a^2 > 0$$

so $\left(\frac{a}{\sqrt{2}}, -\frac{a^2}{\sqrt{2}}, -\frac{a^4}{4}\right)$ and $\left(-\frac{a}{\sqrt{2}}, \frac{a^2}{\sqrt{2}}, -\frac{a^4}{4}\right)$ are both minimum points.

(c) For D:

$$g_x = 4x^3, \qquad g_y = 2y$$

Both are zero only when $x = 0$ and $y = 0$, giving $z = 0$, so the stationary point of D is $(0, 0, 0)$.

At the origin, $g_{xx} = 12x^2 = 0$, $g_{yy} = 2$ and $g_{xy} = 0$, so

$$\det(\mathbf{H}) = 0 \times 2 - 0^2 = 0$$

and the test used in part (b) gives no conclusion.

Instead, $x^4 \geq 0$ and $y^2 \geq 0$ for all real x and y, so $z = x^4 + y^2 \geq 0$, with $z = 0$ only when $x = y = 0$. Every other point of D is higher than the origin, so $(0, 0, 0)$ is a minimum point of D.

12. The surface S has equation

$$x^2 + 2y^2 + z^3 = 3xyz + 8$$

(a) Show that

$$3\left(z^2 - xy\right)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 2\left(z^3 - 8\right) \quad \textbf{[6]}$$

(b) Hence show that S has exactly one stationary point, and find its coordinates. **[3]**

Solution

(a) Differentiate the equation of S partially with respect to x. Treat y as a constant and z as a function of x and y:

$$2x + 3z^2\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x}$$

$$3\big(z^2 - xy\big)\frac{\partial z}{\partial x} = 3yz - 2x$$

Now differentiate partially with respect to y, treating x as a constant:

$$4y + 3z^2\frac{\partial z}{\partial y} = 3xz + 3xy\frac{\partial z}{\partial y}$$

$$3\big(z^2 - xy\big)\frac{\partial z}{\partial y} = 3xz - 4y$$

Multiplying the first result by x and the second by y, then adding:

$$\begin{aligned} 3\big(z^2 - xy\big)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) &= x(3yz - 2x) + y(3xz - 4y) \\ &= 6xyz - 2x^2 - 4y^2 \\ &= -2\big(x^2 + 2y^2 - 3xyz\big) \end{aligned}$$

The equation of S rearranges to $x^2 + 2y^2 - 3xyz = 8 - z^3$, so

$$3\big(z^2 - xy\big)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = -2\big(8 - z^3\big) = 2\big(z^3 - 8\big)$$

as required.

(b) At a stationary point of S, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$. The left-hand side of the result of part (a) is then zero. Since z is real, this gives

$$2\big(z^3 - 8\big) = 0 \quad \Longrightarrow \quad z^3 = 8 \quad \Longrightarrow \quad z = 2$$

In part (a), we found $3\big(z^2 - xy\big)\frac{\partial z}{\partial x} = 3yz - 2x$ and $3\big(z^2 - xy\big)\frac{\partial z}{\partial y} = 3xz - 4y$. Setting both partial derivatives to zero and taking $z = 2$:

$$3yz - 2x = 0 \quad \Longrightarrow \quad 6y = 2x \quad \Longrightarrow \quad x = 3y$$

$$3xz - 4y = 0 \quad \Longrightarrow \quad 6x = 4y \quad \Longrightarrow \quad y = \frac{3}{2}x$$

Substituting $x = 3y$ into the second equation gives $y = \frac{9}{2}y$. So $y = 0$, and then $x = 0$.

At $(0, 0, 2)$, the equation of S gives $0 + 0 + 8 = 0 + 8$, so this point lies on S. Also $z^2 - xy = 4 \neq 0$ there, so the partial derivatives are defined at this point. Each step above gave a single value, so S has exactly one stationary point, at $(0, 0, 2)$.