OPTIONAL PURE / SURFACES & PARTIAL DIFFERENTIATION

Partial Derivatives, Sections and Contours

Worked solutions 15 questions · 141 marks

EXAM BOARDS OCR A Y545 · OCR MEI Extra Pure Y435

Instructions

OCR MEI Extra Pure (Y435): second partial derivatives are not examined. Questions 3 and 13 to 15 can be left out, or just their part **(b)**.

Questions

Question 1	*8 marks*	2
Question 2	*8 marks*	4
Question 3	*8 marks*	6
Question 4	*9 marks*	7
Question 5	*6 marks*	9
Question 6	*10 marks*	10
Question 7	*10 marks*	12
Question 8	*10 marks*	14
Question 9	*9 marks*	16
Question 10	*10 marks*	18
Question 11	*15 marks*	21
Question 12	*7 marks*	24
Question 13	*10 marks*	26
Question 14	*10 marks*	28
Question 15	*11 marks*	30

1. The surface S is given by

$$z = xy^2 - 4y$$

for $1 \leq x \leq 4$ and $-3 \leq y \leq 3$.

(a) (i) Write down the equation of the section of S in the plane $x = 2$. **[1]**

(ii) Sketch this section, labelling the coordinates (y, z) of its vertex and of the points where it meets the y-axis. **[2]**

(b) Find the equation of the contour of S which passes through the point on S where $x = 2$ and $y = 1$, giving x in terms of y. **[2]**

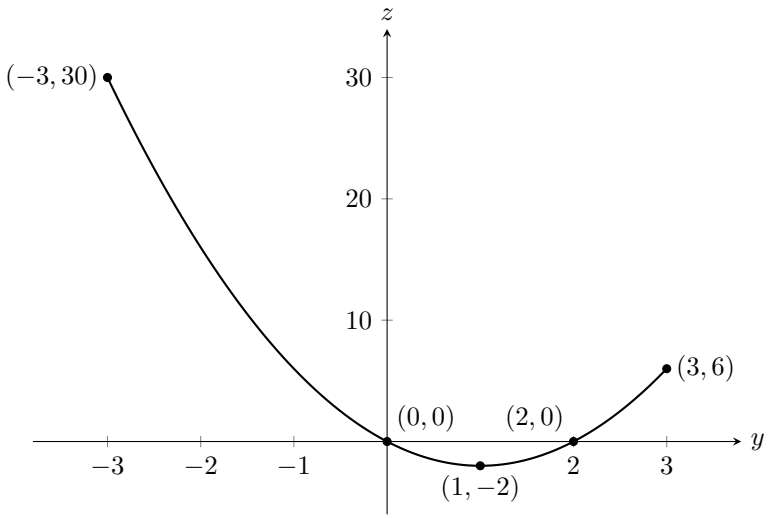
(c) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and evaluate each at the point on S where $x = 2$ and $y = 1$. **[3]**

Solution

(a) (i) Substituting $x = 2$ into $z = xy^2 - 4y$, the section in the plane $x = 2$ is

$$z = 2y^2 - 4y$$

(ii) The sketch should look like:



The section is a U-shaped parabola, drawn for $-3 \leq y \leq 3$. Completing the square:

$$z = 2\left(y^2 - 2y\right) = 2(y-1)^2 - 2$$

so the vertex is at $(1, -2)$. Setting $z = 0$ gives $2y(y-2) = 0$, so $y = 0$ or $y = 2$, and the section meets the y-axis at $(0, 0)$ and $(2, 0)$.

(b) The height of S at the point where $x = 2$ and $y = 1$ is

$$z = 2 \times 1^2 - 4 \times 1 = -2$$

so the contour is the set of points of S at height $z = -2$:

$$xy^2 - 4y = -2$$

$$xy^2 = 4y - 2$$

$$x = \frac{4y - 2}{y^2}$$

(c) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = y^2$$

$$\frac{\partial z}{\partial y} = 2xy - 4$$

At the point where $x = 2$ and $y = 1$:

$$\frac{\partial z}{\partial x} = 1^2 = 1, \qquad \frac{\partial z}{\partial y} = 2 \times 2 \times 1 - 4 = 0$$

This agrees with (a)(ii): the point is the vertex of the section in the plane $x = 2$, where the section's gradient is zero.

2. The surface S is defined for all real x and y by the equation

$$z = x^2y - y^3$$

The intersection of S with a plane gives a section of the surface.

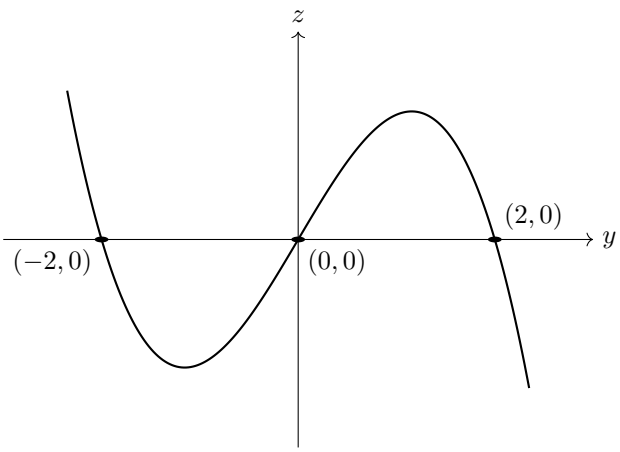
(a) Sketch the section of S by the plane $x = 2$, using y and z axes, and label the points where the section meets the axes. **[3]**

(b) On separate axes, sketch the section of S by the plane $z = 0$, using x and y axes, and label each part of the section with its equation. **[2]**

(c) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and hence find the gradient of the section in part (a) at the point where it meets the positive y-axis. **[3]**

Solution

(a) The sketch should look like:

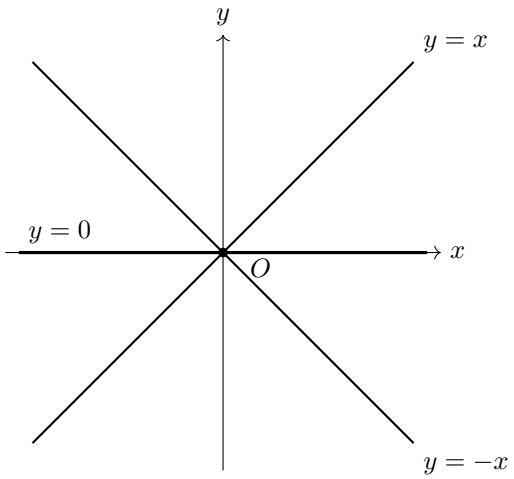


On the plane $x = 2$, the section is

$$z = 4y - y^3 = y\big(4 - y^2\big) = y(2 - y)(2 + y)$$

This is a cubic in y with negative coefficient of y^3. Setting $z = 0$ gives $y = -2$, $y = 0$ or $y = 2$, so the section meets the y-axis at $(-2, 0)$, $(0, 0)$ and $(2, 0)$. Setting $y = 0$ gives $z = 0$, so it meets the z-axis only at the origin.

(b) The sketch should look like:



On the plane $z = 0$, the section is

$$x^2y - y^3 = 0$$
$$y(x^2 - y^2) = 0$$
$$y(x - y)(x + y) = 0$$

So the section is the three straight lines $y = 0$ (the x-axis itself), $y = x$ and $y = -x$, all passing through the origin. Apart from $y = 0$, which lies along the x-axis, the lines meet the axes only at $O(0, 0)$.

(c) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = 2xy$$
$$\frac{\partial z}{\partial y} = x^2 - 3y^2$$

In the plane $x = 2$, the gradient of the section is $\frac{\partial z}{\partial y}$ with $x = 2$. From part (a), the section meets the positive y-axis at $y = 2$, so its gradient there is

$$\frac{\partial z}{\partial y} = 2^2 - 3 \times 2^2 = -8$$

3. A surface has equation $z = f(x, y)$, where

$$f(x, y) = e^{2x} \cos 2y + 2xy$$

Subscripts denote partial derivatives: for example, f_x denotes $\frac{\partial f}{\partial x}$ and f_{xy} denotes $\frac{\partial^2 f}{\partial y \, \partial x}$.

(a) Find f_x and f_y. **[2]**

(b) Find f_{xx}, f_{yy}, f_{xy} and f_{yx}, and hence show that $f_{xx} + f_{yy} = 0$ for all x and y. **[4]**

(c) Find an equation of the section of the surface in the plane $x = 0$, and describe this section as a single transformation of the curve $z = \cos y$. **[2]**

Solution

(a) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$f_x = 2e^{2x} \cos 2y + 2y$$

$$f_y = -2e^{2x} \sin 2y + 2x$$

(b) Differentiating f_x with respect to x, and f_y with respect to y, gives

$$f_{xx} = 4e^{2x} \cos 2y$$

$$f_{yy} = -4e^{2x} \cos 2y$$

Differentiating f_x with respect to y, and f_y with respect to x, gives

$$f_{xy} = -4e^{2x} \sin 2y + 2$$

$$f_{yx} = -4e^{2x} \sin 2y + 2$$

Adding f_{xx} and f_{yy}:

$$f_{xx} + f_{yy} = 4e^{2x} \cos 2y - 4e^{2x} \cos 2y = 0$$

This holds for all x and y, as required.

(c) Substituting $x = 0$ into $z = f(x, y)$:

$$z = e^0 \cos 2y + 2 \times 0 \times y = \cos 2y$$

So the section in the plane $x = 0$ has equation $z = \cos 2y$. Replacing y by $2y$ in $z = \cos y$ gives this curve, so the section is a stretch of $z = \cos y$ parallel to the y-axis with scale factor $\frac{1}{2}$.

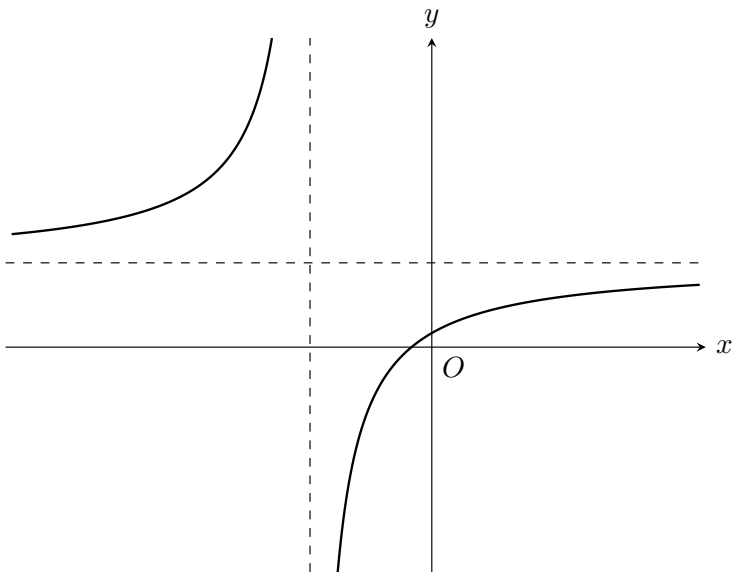
4. A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = 2xy - 6x + 8y$$

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **[2]**

(b) The section of S in the plane $x = a$, where a is a constant, is the line $z = b - 4y$.
Find the value of a and the value of b. **[3]**

(c) Find the equation of the contour of S that passes through the point $(1, 1)$, giving y in terms of x. **[2]**



The diagram above shows this contour, together with its asymptotes.

(d) Find the equations of the asymptotes. **[2]**

Solution

(a) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = 2y - 6, \qquad \frac{\partial z}{\partial y} = 2x + 8$$

(b) In the plane $x = a$, the section is

$$\begin{aligned} z &= 2ay - 6a + 8y \\ &= (2a + 8)y - 6a \end{aligned}$$

This is the line $z = b - 4y$, so equating the coefficients of y and the constant terms gives

$$2a + 8 = -4, \qquad b = -6a$$

The first equation gives $a = -6$, and then $b = -6 \times (-6) = 36$.

(c) At the point $(1, 1)$:

$$z = 2 \times 1 \times 1 - 6 \times 1 + 8 \times 1 = 4$$

So the contour through $(1, 1)$ is $f(x, y) = 4$, which rearranges as

$$2xy - 6x + 8y = 4$$

$$y(2x + 8) = 6x + 4$$

$$y = \frac{6x + 4}{2x + 8} = \frac{3x + 2}{x + 4}$$

(d) The denominator $x + 4$ is zero when $x = -4$, where the numerator is $3 \times (-4) + 2 = -10 \neq 0$, so y is undefined there and the curve has vertical asymptote $x = -4$.

Writing $3x + 2 = 3(x + 4) - 10$ gives

$$y = 3 - \frac{10}{x + 4}$$

As $x \to \pm\infty$, $\frac{10}{x+4} \to 0$, so $y \to 3$. Hence the asymptotes are

$$x = -4, \qquad y = 3$$

5. A surface S has equation $g(x, y, z) = 0$, where

$$g(x, y, z) = (2x + y)(x - z)^3 - 16$$

(a) Show that

$$\frac{\partial g}{\partial x} = (x - z)^2(8x + 3y - 2z) \quad \textbf{[2]}$$

(b) Find the value of the constant k such that, for all values of x, y and z,

$$\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0 \quad \textbf{[4]}$$

Solution

(a) Treat y and z as constants and use the product rule, with $\frac{\partial}{\partial x}(2x + y) = 2$ and $\frac{\partial}{\partial x}(x - z)^3 = 3(x - z)^2$:

$$\begin{aligned}\frac{\partial g}{\partial x} &= 2(x - z)^3 + 3(2x + y)(x - z)^2 \\ &= (x - z)^2[2(x - z) + 3(2x + y)] \\ &= (x - z)^2(2x - 2z + 6x + 3y) \\ &= (x - z)^2(8x + 3y - 2z)\end{aligned}$$

as required.

(b) Treating x and z as constants gives

$$\frac{\partial g}{\partial y} = (x - z)^3$$

Treating x and y as constants, and using the chain rule on $(x - z)^3$, gives

$$\frac{\partial g}{\partial z} = (2x + y) \times 3(x - z)^2 \times (-1) = -3(2x + y)(x - z)^2$$

Substituting these and the result of part (a):

$$\begin{aligned}\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} &= (x - z)^2(8x + 3y - 2z) + k(x - z)^3 - 3(2x + y)(x - z)^2 \\ &= (x - z)^2[8x + 3y - 2z + k(x - z) - 6x - 3y] \\ &= (x - z)^2[2(x - z) + k(x - z)] \\ &= (k + 2)(x - z)^3\end{aligned}$$

This is zero for all values of x, y and z only when $k + 2 = 0$, so

$$k = -2$$

6. A surface S is defined by

$$z = x^2 + 2xy + 3y^2 - 6x + 10$$

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **[2]**

(b) In the section of S given by $y = c$, where c is a constant, the minimum value of z occurs at the point where $x = a$ and $z = 21$.
Find all possible pairs of values of a and c. **[5]**

(c) Sketch the section of S given by $x = 3$ in the y-z plane, stating the coordinates of its minimum point and of the points where it meets the coordinate axes. **[3]**

Solution

(a) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = 2x + 2y - 6$$

$$\frac{\partial z}{\partial y} = 2x + 6y$$

(b) In the section given by $y = c$, the equation of the surface becomes

$$z = x^2 + (2c - 6)x + 3c^2 + 10$$

This is a quadratic in x with positive x^2 coefficient, so its minimum is where $\frac{\partial z}{\partial x} = 0$. Using part (a) with $y = c$:

$$2x + 2c - 6 = 0 \quad \Longrightarrow \quad x = 3 - c$$

So $a = 3 - c$. Substituting $x = 3 - c$ and $y = c$ into the equation of S:

$$\begin{aligned} z &= (3-c)^2 + 2c(3-c) + 3c^2 - 6(3-c) + 10 \\ &= 9 - 6c + c^2 + 6c - 2c^2 + 3c^2 - 18 + 6c + 10 \\ &= 2c^2 + 6c + 1 \end{aligned}$$

The minimum value of z is 21, so

$$2c^2 + 6c + 1 = 21$$

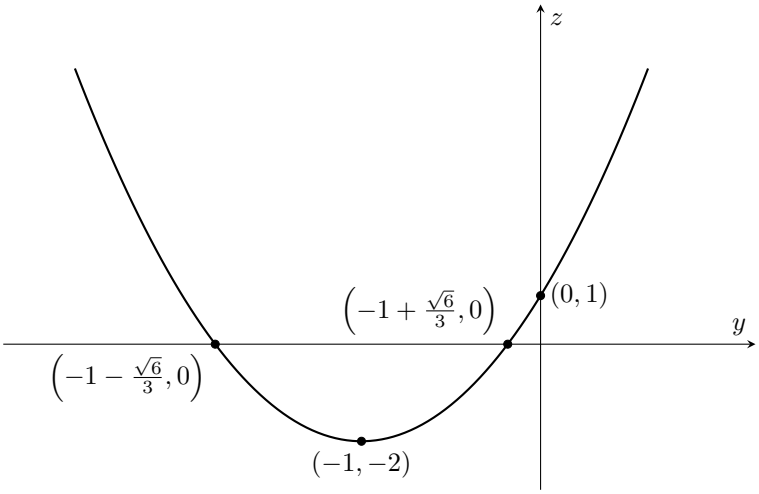
$$c^2 + 3c - 10 = 0$$

$$(c + 5)(c - 2) = 0$$

so $c = 2$ or $c = -5$. Since $a = 3 - c$, the possible pairs are

$$a = 1,\ c = 2 \qquad \text{or} \qquad a = 8,\ c = -5$$

(c) The sketch should look like:



Substituting $x = 3$ into the equation of S:

$$\begin{aligned} z &= 9 + 6y + 3y^2 - 18 + 10 \\ &= 3y^2 + 6y + 1 \end{aligned}$$

This is a U-shaped parabola in the y-z plane. Using part (a) with $x = 3$, $\frac{\partial z}{\partial y} = 6 + 6y$, which is zero when $y = -1$, and then

$$z = 3 - 6 + 1 = -2$$

so the minimum point is $(-1, -2)$.

When $y = 0$, $z = 1$, so the section meets the z-axis at $(0, 1)$.

When $z = 0$:

$$3y^2 + 6y + 1 = 0 \quad \Longrightarrow \quad y = \frac{-6 \pm \sqrt{24}}{6} = -1 \pm \frac{\sqrt{6}}{3}$$

so the section meets the y-axis at $\left(-1 - \frac{\sqrt{6}}{3}, 0\right)$ and $\left(-1 + \frac{\sqrt{6}}{3}, 0\right)$.

7. A surface has equation

$$z = x^3 + 3xy^2 - 3x^2 - 3y^2$$

(a) For a point on the surface at which $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, show that either $y = x$ or $y = x - 2$. **[5]**

(b) Show that the section of the surface by the plane $x = 1$ is a straight line, and describe its position. **[2]**

(c) Sketch, in the x-z plane, the section of the surface by the plane $y = 0$, giving the coordinates of the points where it meets the x-axis and of its turning points. **[3]**

Solution

(a) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 6x, \qquad \frac{\partial z}{\partial y} = 6xy - 6y$$

Setting $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ gives

$$\begin{aligned} 3x^2 + 3y^2 - 6x &= 6xy - 6y \\ 3x^2 - 6xy + 3y^2 - 6x + 6y &= 0 \\ x^2 - 2xy + y^2 - 2x + 2y &= 0 \\ (x-y)^2 - 2(x-y) &= 0 \\ (x-y)(x-y-2) &= 0 \end{aligned}$$

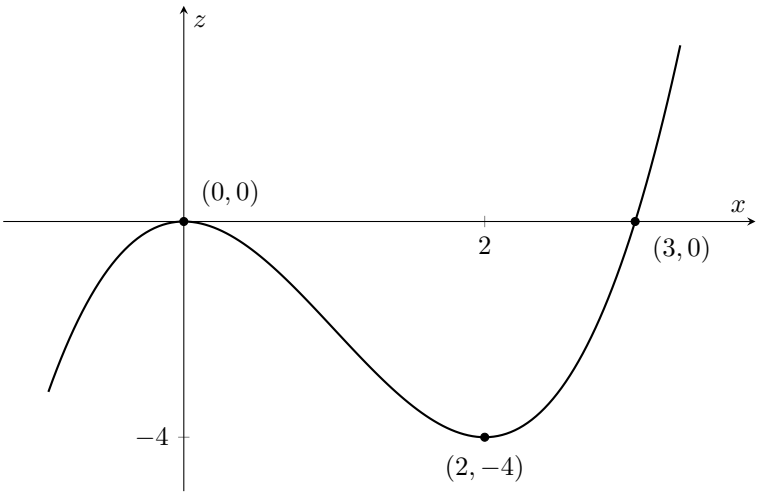
So either $x - y = 0$ or $x - y - 2 = 0$, that is, either $y = x$ or $y = x - 2$, as required.

(b) Substituting $x = 1$ into the equation of the surface:

$$z = 1 + 3y^2 - 3 - 3y^2 = -2$$

So on the plane $x = 1$, the section is $z = -2$ for every value of y: it is a straight line. It lies in the plane $x = 1$, parallel to the y-axis, at a distance 2 below the x-y plane, passing through the point $(1, 0, -2)$.

(c) The sketch should look like:



Substituting $y = 0$ gives the section

$$z = x^3 - 3x^2 = x^2(x - 3)$$

Setting $z = 0$ gives $x = 0$ (a repeated root, so the curve touches the x-axis there) or $x = 3$. The section meets the x-axis at $(0, 0)$ and $(3, 0)$.

For the turning points:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 - 6x = 3x(x - 2) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = 2$$

When $x = 2$, $z = 8 - 12 = -4$. As the sketch shows, $(0, 0)$ is a maximum point and $(2, -4)$ is a minimum point.

8. A surface S has equation

$$z = 2x^3 - 3x^2y + 4y^2$$

(a) Find the equation of the section of S where $x = 2$, giving it in the form $z = g(y)$. Sketch this section, stating the coordinates of its minimum point and of the point where it meets the plane $y = 0$. Find, in three-dimensional vector form, an equation of the line of symmetry of this section. **[5]**

(b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **[2]**

(c) For a constant k, the section of S where $x = k$ is a parabola.
Using your answer to part (b), find, in terms of k, the coordinates of the minimum point of this section. **[3]**

Solution

(a) Substituting $x = 2$ into the equation of S:

$$z = 2 \times 2^3 - 3 \times 2^2y + 4y^2$$

so the section is

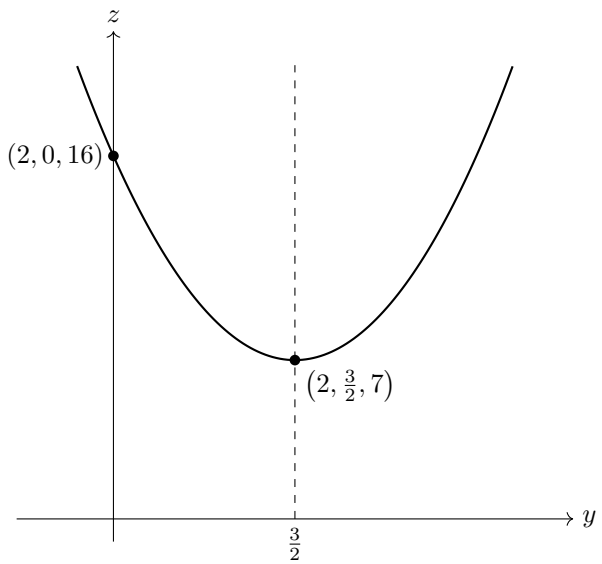
$$z = 4y^2 - 12y + 16$$

Completing the square:

$$\begin{aligned} z &= 4\left(y^2 - 3y\right) + 16 \\ &= 4\left(y - \tfrac{3}{2}\right)^2 - 9 + 16 \\ &= 4\left(y - \tfrac{3}{2}\right)^2 + 7 \end{aligned}$$

So the minimum value of z is 7, when $y = \frac{3}{2}$, and the minimum point is $\left(2, \frac{3}{2}, 7\right)$. When $y = 0$, $z = 16$, so the section meets the plane $y = 0$ at $(2, 0, 16)$.

The sketch, in the plane $x = 2$, should look like:



The section is a U-shaped parabola, symmetrical about the line in the plane $x = 2$ through the

minimum point, parallel to the z-axis. This line passes through $\left(2, \frac{3}{2}, 0\right)$ and has direction $\mathbf{k}$, so an equation of it is

$$\mathbf{r} = \begin{pmatrix} 2 \\ \frac{3}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) Differentiating with respect to x, treating y as a constant, and then with respect to y, treating x as a constant:

$$\frac{\partial z}{\partial x} = 6x^2 - 6xy$$

$$\frac{\partial z}{\partial y} = -3x^2 + 8y$$

(c) On the section $x = k$, z is a function of y alone, with $z = 4y^2 - 3k^2y + 2k^3$. Its coefficient of y^2 is $4 > 0$, so the parabola is U-shaped and its minimum point is where its gradient $\frac{\partial z}{\partial y}$ is zero. Using part (b), with $x = k$:

$$-3k^2 + 8y = 0 \quad \Longrightarrow \quad y = \frac{3k^2}{8}$$

Substituting $x = k$ and $y = \frac{3k^2}{8}$ into the equation of S:

$$\begin{aligned} z &= 2k^3 - 3k^2\left(\frac{3k^2}{8}\right) + 4\left(\frac{3k^2}{8}\right)^2 \\ &= 2k^3 - \frac{9k^4}{8} + \frac{9k^4}{16} \\ &= 2k^3 - \frac{9k^4}{16} \end{aligned}$$

So the minimum point is

$$\left(k,\ \frac{3k^2}{8},\ 2k^3 - \frac{9k^4}{16}\right)$$

9. A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = y^2 e^{-y(x^2-6x+10)}$$

(a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. **[4]**

(b) Specify a sequence of transformations which maps the graph of $z = e^{-x^2}$ onto the section of S in the plane $y = 2$. **[3]**

(c) Hence, or otherwise, sketch the section of S in the plane $y = 2$, stating the exact coordinates of its highest point. **[2]**

Solution

(a) Differentiating with respect to x, treating y as a constant, and using the chain rule:

$$\frac{\partial f}{\partial x} = y^2 \times (-y(2x-6))e^{-y(x^2-6x+10)}$$

$$= -2y^3(x-3)e^{-y(x^2-6x+10)}$$

Differentiating with respect to y, treating x as a constant, and using the product rule:

$$\frac{\partial f}{\partial y} = 2ye^{-y(x^2-6x+10)} + y^2 \times \big(-(x^2-6x+10)\big)e^{-y(x^2-6x+10)}$$

$$= y\big(2 - y(x^2-6x+10)\big)e^{-y(x^2-6x+10)}$$

(b) The section of S in the plane $y = 2$ is

$$z = 4e^{-2(x^2-6x+10)}$$

Completing the square, $x^2 - 6x + 10 = (x-3)^2 + 1$, so

$$z = 4e^{-2\left((x-3)^2+1\right)} = 4e^{-2}e^{-2(x-3)^2}$$

Start from $z = e^{-x^2}$. A stretch parallel to the x-axis with scale factor $\frac{1}{\sqrt{2}}$ replaces x by $\sqrt{2}\,x$, giving

$$z = e^{-2x^2}$$

A translation of 3 units in the positive x-direction then replaces x by $x - 3$, giving

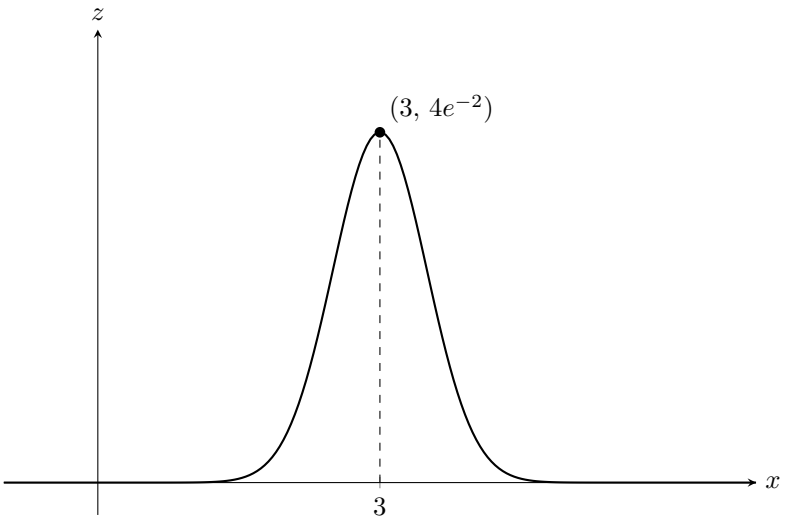
$$z = e^{-2(x-3)^2}$$

Finally, a stretch parallel to the z-axis with scale factor $4e^{-2}$ gives

$$z = 4e^{-2}e^{-2(x-3)^2}$$

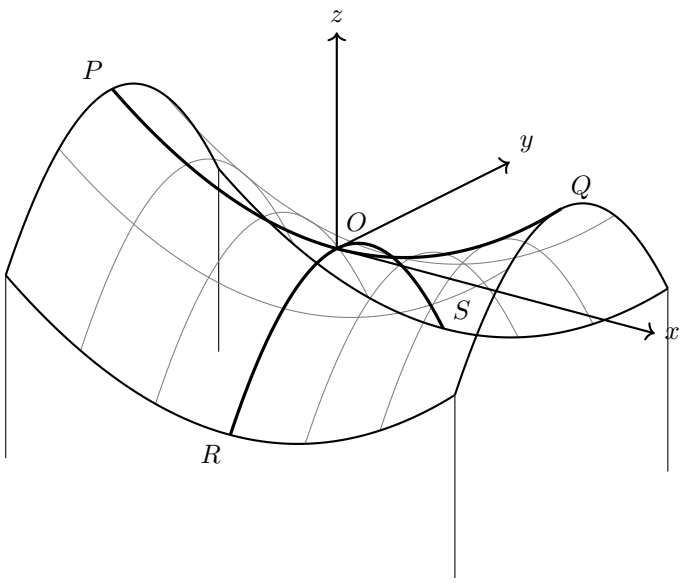
which is the section of S in the plane $y = 2$. The translation must come after the stretch parallel to the x-axis.

(c) The sketch, in the plane $y = 2$, should look like:



The maximum point of $z = e^{-x^2}$ is $(0, 1)$. The stretch parallel to the x-axis leaves it at $(0, 1)$, the translation moves it to $(3, 1)$, and the stretch parallel to the z-axis moves it to $(3, 4e^{-2})$. The curve is symmetric about $x = 3$, lies above the x-axis for all x, and $z \to 0$ as $x \to \pm\infty$, so $z = 0$ is an asymptote.

In three dimensions, with $y = 2$, the highest point of the section is $(3, 2, 4e^{-2})$.



10. An architect is designing the roof of a pavilion. The roof is modelled by a surface with equation $z = f(x, y)$ which has a saddle point at the origin O.
The z-axis is vertically upwards, and the x- and y-axes lie in a horizontal plane, with the x-axis along the length of the pavilion and the y-axis across it, as shown in the diagram above.
The arc POQ is part of a parabola which lies in the x-z plane, rising from O to the points P and Q. The arc ROS is part of a parabola which lies in the y-z plane, falling from O to the points R and S, where S has a positive y-coordinate. The roof is symmetric in both the x-z plane and the y-z plane.
The distance PQ is to be 12 m, with both P and Q at a height of 1.2 m above O. The distance RS is to be 8 m, and the gradient of the arc ROS at S is to be -0.8.

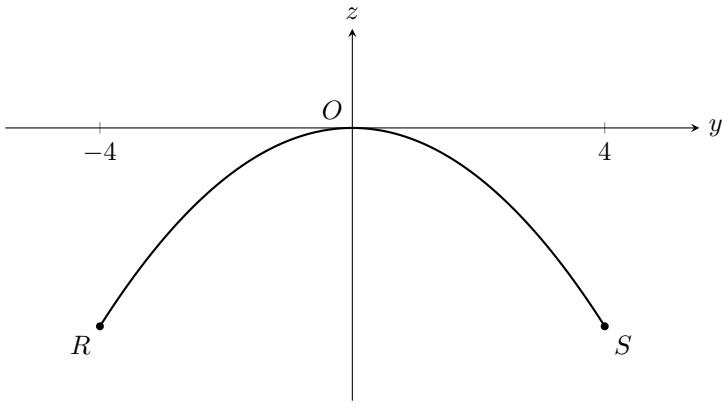
(a) On separate diagrams, sketch the sections $x = 0$ and $y = 0$. **[2]**

(b) Determine a function f that describes the roof. You do not need to state the domain of f. **[5]**

(c) Find the equations of the contour of the surface at the height of O, and find the acute angle between the two lines that form this contour. **[3]**

Solution

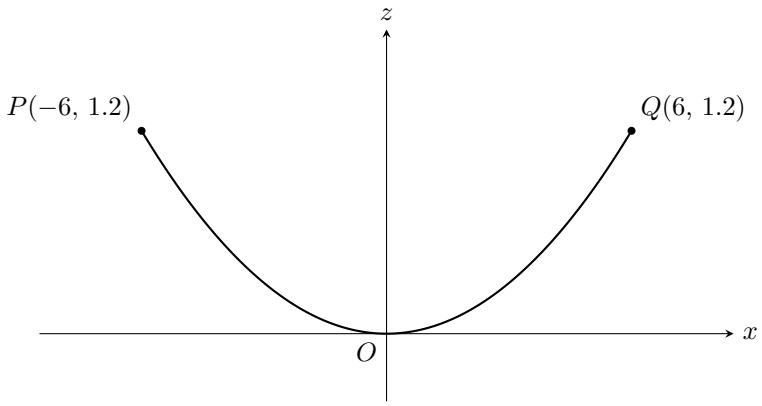
(a) The section $x = 0$ should look like:



This section is the arc ROS: a ∩-shaped parabola with its maximum at O, symmetric about the

z-axis because the roof is symmetric in the x-z plane. Since $RS = 8\,\text{m}$, the points R and S are at $y = -4$ and $y = 4$.

The section $y = 0$ should look like:



This section is the arc POQ: a $\cup$-shaped parabola with its minimum at O, symmetric about the z-axis because the roof is symmetric in the y-z plane. Since $PQ = 12\,\text{m}$, the points P and Q are at $x = -6$ and $x = 6$, both at height $z = 1.2$.

(b) The roof is symmetric in the y-z plane, so the section $y = 0$ is a parabola symmetric about the z-axis. It passes through O and rises from there, so it is $z = ax^2$ with $a > 0$. In the same way, the section $x = 0$ falls from O and is $z = -by^2$ with $b > 0$. A function with both of these sections is

$$f(x, y) = ax^2 - by^2$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 2ax, \qquad \frac{\partial f}{\partial y} = -2by$$

Both are zero at O, and the surface rises from O in the x-direction but falls in the y-direction, so O is a saddle point.

Since the roof is symmetric and $PQ = 12$, Q is the point $(6, 0, 1.2)$. Substituting into $z = f(x, y)$:

$$36a = 1.2 \quad \Longrightarrow \quad a = \frac{1}{30}$$

Since $RS = 8$, S has y-coordinate 4. The arc ROS is the section $x = 0$, so its gradient at S is $\frac{\partial f}{\partial y}$ at $(0, 4)$:

$$-2b \times 4 = -0.8 \quad \Longrightarrow \quad b = \frac{1}{10}$$

Hence

$$f(x, y) = \frac{x^2}{30} - \frac{y^2}{10}$$

(c) O is at height $z = 0$, so the contour is given by

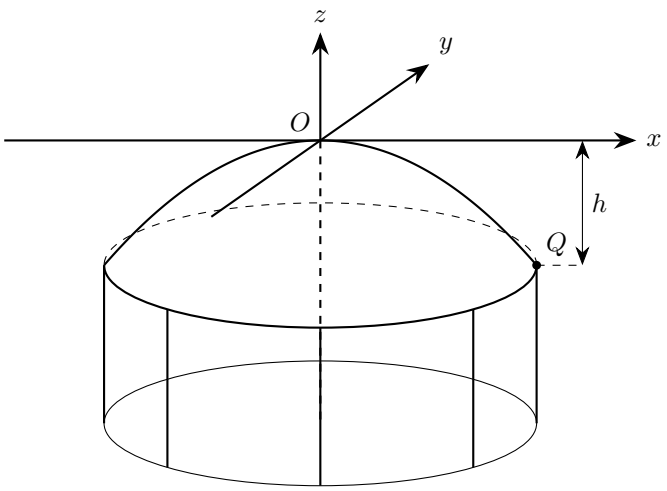
$$\frac{x^2}{30} - \frac{y^2}{10} = 0 \quad \Longrightarrow \quad x^2 = 3y^2$$

so in the plane $z = 0$ the contour is the pair of lines

$$y = \frac{x}{\sqrt{3}} \qquad \text{and} \qquad y = -\frac{x}{\sqrt{3}}$$

The lines have gradients $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$, so each makes an angle of $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ with the x-axis, on opposite sides of it. The acute angle between the lines is therefore

$$2 \times \frac{\pi}{6} = \frac{\pi}{3} = 60^\circ$$



11. The diagram above shows a bandstand with a curved roof. The roof is modelled by the surface generated by rotating part of a parabola about the vertical z-axis.
The model is part of the surface with equation $z = f(x, y)$, and the highest point of the roof is the origin O, which lies on the surface.
The lower edge of the roof is a horizontal circle. The point Q is on this edge, directly below the positive x-axis, at a depth h below O.

(a) The vertical plane containing both Q and the z-axis meets the surface of the model in a section.

(i) Sketch this section, marking the positions of O and Q. **[1]**

(ii) State a suitable equation for this section, in terms of x, z and a constant. **[1]**

(b) The horizontal plane containing Q meets the surface of the model in a contour, and the horizontal plane $z = -\frac{4h}{9}$ meets it in a second contour.

(i) Sketch these two contours on a single set of axes in the x-y plane. **[1]**

(ii) Using your answer to part **(a)(ii)**, state an equation for each contour, and hence show that the radius of the second contour is $\frac{2}{3}$ of the radius of the first. **[2]**

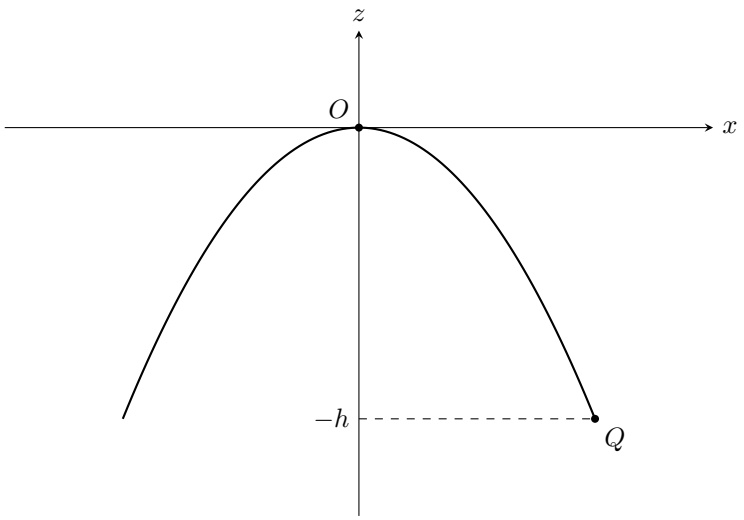
(c) A proposed equation for the surface is $z = ax^2 + by^2 + cxy$. What can you say about the constants a, b and c? Justify your answers. **[4]**

(d) In the real bandstand, the lower edge of the roof is a circle of diameter 8 m. A lamp hangs from a point L on the roof at a horizontal distance of 2 m from the z-axis and 0.5 m below O. Using the equation in part **(c)**, with distances in metres, find the values of a, b, c and h. **[3]**

(e) Using your answers to part **(d)**, find the value of $\frac{\partial z}{\partial x}$ at Q. Hence find the angle that the roof makes with the horizontal at Q, measured in the vertical plane containing Q and the z-axis. **[3]**

Solution

(a) (i) The sketch should look like:

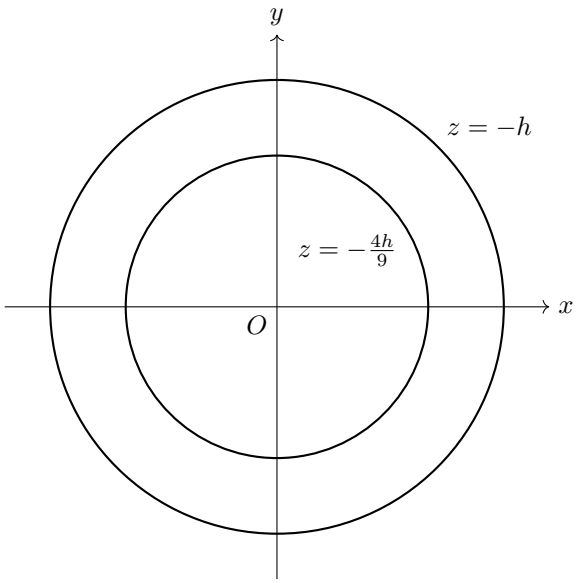


The section lies in the plane $y = 0$. It is a parabola, concave down and symmetric about the z-axis, with its highest point at O, and it passes through Q, which is at depth h below O on the positive x side.

(ii) The section is a parabola with its vertex at O, opening downwards, so a suitable equation is

$$z = -kx^2, \quad \text{where } k > 0 \text{ is a constant}$$

(b) (i) The sketch should look like:



The roof is a surface of revolution about the z-axis, so both contours are circles centred on O. The plane $z = -\frac{4h}{9}$ is above the plane of Q, where the roof is narrower, so its contour is the smaller circle.

(ii) A point (x, y) is at distance $\sqrt{x^2 + y^2}$ from the z-axis. Rotating the section $z = -kx^2$ about the z-axis replaces x^2 by $x^2 + y^2$, so the surface is

$$z = -k\left(x^2 + y^2\right)$$

The contour through Q is in the plane $z = -h$:

$$-h = -k(x^2 + y^2) \quad \Longrightarrow \quad x^2 + y^2 = \frac{h}{k}$$

This is a circle of radius $\sqrt{\frac{h}{k}}$. The second contour is in the plane $z = -\frac{4h}{9}$:

$$-\frac{4h}{9} = -k(x^2 + y^2) \quad \Longrightarrow \quad x^2 + y^2 = \frac{4h}{9k}$$

This is a circle of radius

$$\sqrt{\frac{4h}{9k}} = \frac{2}{3}\sqrt{\frac{h}{k}}$$

so the radius of the second contour is $\frac{2}{3}$ of the radius of the first, as required.

(c) The surface is a surface of revolution about the z-axis, so every section by a vertical plane through the z-axis is the same curve, with z depending only on the distance from the z-axis.

The section in the plane $y = 0$ is $z = ax^2$, and the section in the plane $x = 0$ is $z = by^2$. These must be the same curve, so $a = b$.

In the plane $y = 0$, a point at distance d from the z-axis has $z = ad^2$. In the plane $y = x$, take the point with $x = y = t$, which is at distance $d = \sqrt{2}\,|t|$ from the z-axis. Its height is

$$z = at^2 + at^2 + ct^2 = (2a + c)t^2$$

and this must equal $ad^2 = 2at^2$ for every t, so $2a + c = 2a$, giving $c = 0$.

The section in the plane $y = 0$ is then $z = ax^2$, which must match $z = -kx^2$ from part **(a)(ii)**, with its highest point at O and the roof curving downwards. So $a = -k < 0$.

Hence $a = b$, with a and b negative, and $c = 0$.

(d) With $a = b$ and $c = 0$, the surface is $z = a(x^2 + y^2)$. At L, $x^2 + y^2 = 2^2 = 4$ and $z = -\frac{1}{2}$:

$$-\frac{1}{2} = 4a \quad \Longrightarrow \quad a = -\frac{1}{8}$$

The lower edge has radius 4 m, so at Q, $x^2 + y^2 = 16$ and

$$z = -\frac{1}{8} \times 16 = -2$$

so Q is 2 m below O. Hence $a = -\frac{1}{8}$, $b = -\frac{1}{8}$, $c = 0$ and $h = 2$.

(e) The surface is $z = -\frac{1}{8}(x^2 + y^2)$, so

$$\frac{\partial z}{\partial x} = -\frac{x}{4}$$

At Q, $x = 4$ and $y = 0$, so

$$\frac{\partial z}{\partial x} = -\frac{4}{4} = -1$$

The vertical plane containing Q and the z-axis is $y = 0$, and in it $\frac{\partial z}{\partial x}$ is the gradient of the section $z = -\frac{1}{8}x^2$. The angle θ that the roof makes with the horizontal at Q therefore satisfies

$$\tan\theta = |-1| = 1 \quad \Longrightarrow \quad \theta = 45^\circ$$

12. A surface, S, is given by the equation $z = f(x, y)$ for all real values of x and y, where f is continuous and $f(-x, y) = f(x, y) = f(x, -y)$ for all x and y.

You are given that S has the following properties.

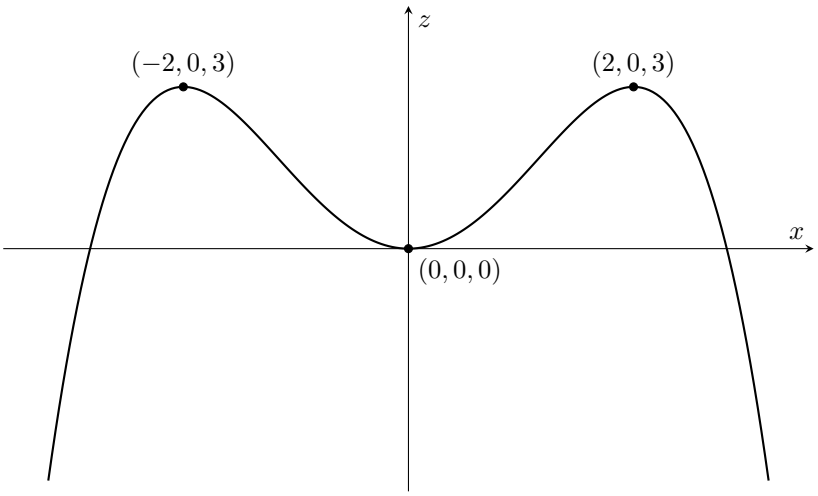
- For $a > 3$, the contour $z = a$ is empty.
- The contour $z = 3$ consists of the two points $(2, 0, 3)$ and $(-2, 0, 3)$.
- For $0 < a < 3$, the contour $z = a$ consists of two separate closed curves: one lies in the region $x > 0$ and encloses the point $(2, 0, a)$, and the other is its reflection in the plane $x = 0$.
- The contour $z = 0$ consists of two closed curves which meet only at the origin: one lies in the region $x \geq 0$ and encloses the point $(2, 0, 0)$, and the other is its reflection in the plane $x = 0$.
- For $a < 0$, the contour $z = a$ is a single closed curve which encloses both of the points $(2, 0, a)$ and $(-2, 0, a)$.

(a) Sketch a possible section of S corresponding to $y = 0$, labelling the points where the section reaches $z = 3$ and the point where it meets the z-axis. **[4]**

(b) Sketch a possible section of S corresponding to $x = 0$, explaining how the contours show where its highest point lies. **[3]**

Solution

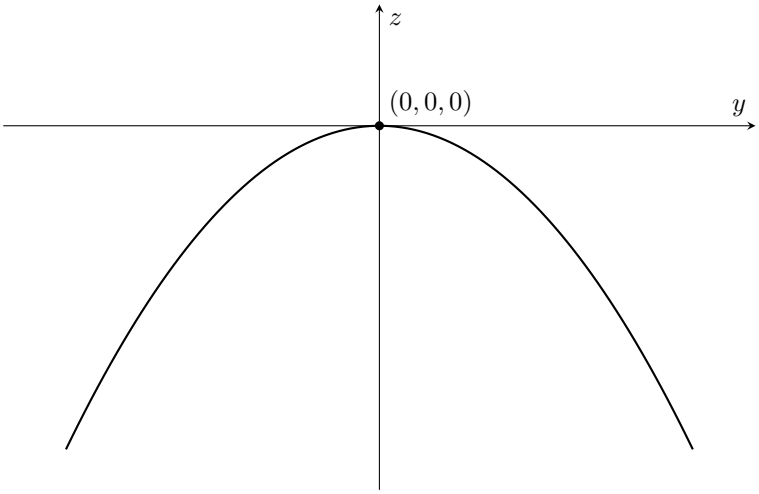
(a) The sketch should look like:



Since $f(-x, 0) = f(x, 0)$, the section is symmetric about the z-axis. No contour with $a > 3$ exists and the contour $z = 3$ is the two points $(\pm 2, 0, 3)$, so these are the highest points of the section.

The contour $z = 0$ passes through the origin, so the section meets the z-axis at $(0, 0, 0)$. For $0 < a < 3$, the closed curve around $(2, 0, a)$ lies in $x > 0$, so it crosses the line $y = 0$ twice, once on each side of $x = 2$. So the section reaches each height between 0 and 3 twice for $x > 0$: it rises to the peak at $(2, 0, 3)$ and falls back to $z = 0$ at the origin, and also at a point with $x > 2$ where the $z = 0$ curve crosses $y = 0$ again. The contours $z = a$ with $a < 0$ enclose both peaks, so beyond that point the section falls below $z = 0$, giving an M-shape.

(b) The sketch should look like:



Since $f(0,-y) = f(0,y)$, the section is symmetric about the z-axis.

The points of the contour $z = 3$ have $x = \pm 2$, and for $0 < a < 3$ each contour $z = a$ lies in $x > 0$ or $x < 0$. So no contour with $a > 0$ meets the plane $x = 0$, and $f(0,y) \leq 0$ for all y. The contour $z = 0$ meets $x = 0$ only at the origin, so $f(0,0) = 0$ and $f(0,y) < 0$ for $y \neq 0$. Each contour $z = a$ with $a < 0$ is a closed curve enclosing $(2,0,a)$ and $(-2,0,a)$, so it crosses $x = 0$, and the section takes every negative value.

Hence the section is an inverted U-shape with highest point $(0,0,0)$, falling as $|y|$ increases.

13. For $x, y \in \mathbb{R}$ with $x \neq 0$, the function f is given by

$$f(x,y) = \left(x^3 + y^3\right) e^{y/x}$$

Subscripts denote partial derivatives: for example, f_x denotes $\frac{\partial f}{\partial x}$ and f_{xy} denotes $\frac{\partial^2 f}{\partial y\, \partial x}$.

(a) Show that $xf_x + yf_y = nf$, where n is a positive integer to be determined. **[5]**

(b) By differentiating the result of part (a) with respect to x and with respect to y, or otherwise, show that

$$x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = kf$$

where k is a constant to be found. **[5]**

Solution

(a) Differentiating partially with respect to x, treating y as a constant, and using the product and chain rules with $\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2}$:

$$f_x = 3x^2 e^{y/x} - \frac{y}{x^2}\left(x^3 + y^3\right) e^{y/x}$$

Differentiating partially with respect to y, treating x as a constant, with $\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}$:

$$f_y = 3y^2 e^{y/x} + \frac{1}{x}\left(x^3 + y^3\right) e^{y/x}$$

Multiplying the first by x and the second by y, then adding:

$$\begin{aligned} xf_x + yf_y &= 3x^3 e^{y/x} - \frac{y}{x}\left(x^3 + y^3\right) e^{y/x} + 3y^3 e^{y/x} + \frac{y}{x}\left(x^3 + y^3\right) e^{y/x} \\ &= 3\left(x^3 + y^3\right) e^{y/x} \\ &= 3f \end{aligned}$$

So $xf_x + yf_y = nf$ with $n = 3$.

(b) From part (a):

$$xf_x + yf_y = 3f$$

Differentiating both sides partially with respect to x, using the product rule on xf_x and treating y as a constant:

$$f_x + xf_{xx} + yf_{yx} = 3f_x$$

The mixed derivatives are equal, $f_{yx} = f_{xy}$, since f and its partial derivatives are continuous for $x \neq 0$, so

$$xf_{xx} + yf_{xy} = 2f_x$$

Differentiating both sides partially with respect to y, using the product rule on yf_y and treating x as a constant:

$$xf_{xy} + f_y + yf_{yy} = 3f_y \quad \Longrightarrow \quad xf_{xy} + yf_{yy} = 2f_y$$

Multiplying the first of these by x and the second by y, then adding:

$$\begin{aligned} x^2 f_{xx} + xy f_{xy} + xy f_{xy} + y^2 f_{yy} &= 2x f_x + 2y f_y \\ x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} &= 2(x f_x + y f_y) \\ &= 2 \times 3f \\ &= 6f \end{aligned}$$

So $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = kf$ with $k = 6$.

14. The surface S has equation

$$z^3 + 3xz = y^2 + 2$$

(a) Show that

$$2y\frac{\partial z}{\partial x} + 3z\frac{\partial z}{\partial y} = 0$$ **[4]**

(b) Show that

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2y\left(z^2 - x\right)}{3\left(z^2 + x\right)^3}$$ **[3]**

(c) Find an equation for the contour of S on which $z = 2$. Sketch this contour in the x-y plane, stating the coordinates of the points where it meets the coordinate axes. **[3]**

Solution

(a) Differentiating $z^3 + 3xz = y^2 + 2$ implicitly with respect to x, treating y as a constant, and using the product rule on $3xz$:

$$3z^2\frac{\partial z}{\partial x} + 3z + 3x\frac{\partial z}{\partial x} = 0$$

$$3\left(z^2 + x\right)\frac{\partial z}{\partial x} = -3z$$

$$\frac{\partial z}{\partial x} = -\frac{z}{z^2 + x}$$

Differentiating implicitly with respect to y, treating x as a constant:

$$3z^2\frac{\partial z}{\partial y} + 3x\frac{\partial z}{\partial y} = 2y$$

$$3\left(z^2 + x\right)\frac{\partial z}{\partial y} = 2y$$

$$\frac{\partial z}{\partial y} = \frac{2y}{3(z^2 + x)}$$

Substituting both into the left-hand side:

$$2y\frac{\partial z}{\partial x} + 3z\frac{\partial z}{\partial y} = 2y\left(-\frac{z}{z^2 + x}\right) + 3z \times \frac{2y}{3(z^2 + x)}$$

$$= -\frac{2yz}{z^2 + x} + \frac{2yz}{z^2 + x}$$

$$= 0$$

as required.

(b) The mixed derivative is

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{3}\left(z^2 + x\right)^{-1}\right)$$

Treating y as a constant, the chain rule gives

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{2y}{3}\left(z^2+x\right)^{-2}\left(2z\frac{\partial z}{\partial x}+1\right)$$

Using $\frac{\partial z}{\partial x} = -\frac{z}{z^2+x}$ from part (a):

$$2z\frac{\partial z}{\partial x}+1 = -\frac{2z^2}{z^2+x}+\frac{z^2+x}{z^2+x} = \frac{x-z^2}{z^2+x}$$

Therefore

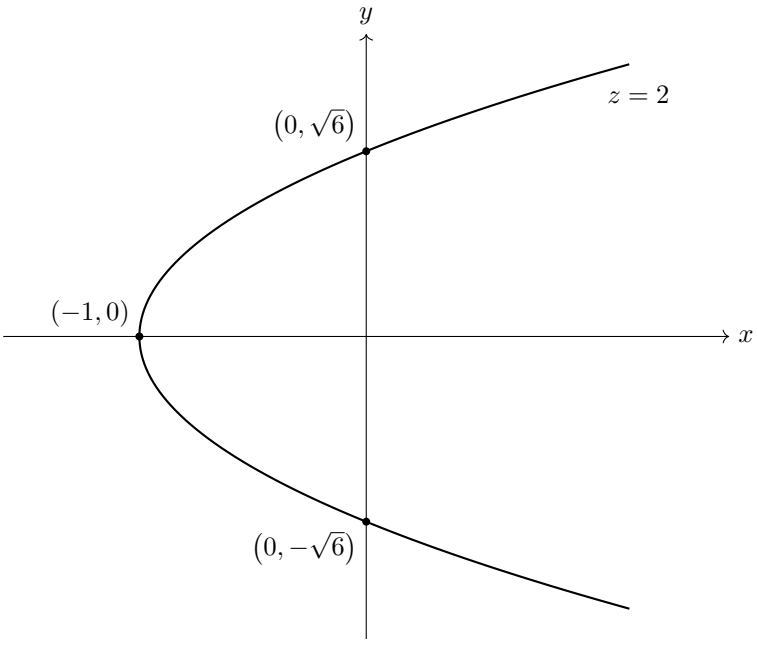
$$\begin{aligned}\frac{\partial^2 z}{\partial x\,\partial y} &= -\frac{2y}{3\left(z^2+x\right)^2}\times\frac{x-z^2}{z^2+x}\\ &= \frac{2y\left(z^2-x\right)}{3\left(z^2+x\right)^3}\end{aligned}$$

as required.

(c) Setting $z=2$ in the equation of S:

$$\begin{aligned}2^3+3x\times 2 &= y^2+2\\ 8+6x &= y^2+2\\ y^2 &= 6x+6\end{aligned}$$

The sketch should look like:



The contour is a parabola, symmetric about the x-axis since only y^2 appears, opening in the positive x-direction with vertex at $(-1,0)$.

On the x-axis, $y=0$, so $6x+6=0$, giving $x=-1$. On the y-axis, $x=0$, so $y^2=6$, giving $y=\pm\sqrt{6}$.

The contour meets the coordinate axes at $(-1,0)$, $\left(0,\sqrt{6}\right)$ and $\left(0,-\sqrt{6}\right)$.

15. The surface S has equation

$$x^2e^z + 2x - 4y^2 = 8(z+1)$$

The point $P(4, 2, 0)$ lies on S.

(a) Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at P. **[4]**

(b) Find the value of $\frac{\partial^2 z}{\partial x^2}$ at P. **[4]**

(c) Show that the contour of S on which $z = 0$ has equation $(x+1)^2 - 4y^2 = 9$. Sketch this contour in the x-y plane, stating the coordinates of the points where it meets the x-axis. **[3]**

Solution

(a) On S, z is a function of x and y. Differentiating the equation of S partially with respect to x, treating y as a constant and using the product rule on x^2e^z, gives

$$2xe^z + x^2e^z\frac{\partial z}{\partial x} + 2 = 8\frac{\partial z}{\partial x}$$

At P, $x = 4$, $y = 2$ and $z = 0$, so $e^z = 1$:

$$8 + 16\frac{\partial z}{\partial x} + 2 = 8\frac{\partial z}{\partial x}$$

$$8\frac{\partial z}{\partial x} = -10$$

$$\frac{\partial z}{\partial x} = -\frac{5}{4}$$

Differentiating partially with respect to y, treating x as a constant, gives

$$x^2e^z\frac{\partial z}{\partial y} - 8y = 8\frac{\partial z}{\partial y}$$

At P:

$$16\frac{\partial z}{\partial y} - 16 = 8\frac{\partial z}{\partial y}$$

$$8\frac{\partial z}{\partial y} = 16$$

$$\frac{\partial z}{\partial y} = 2$$

(b) Differentiate the x-equation from part (a) partially with respect to x again. The term $2xe^z$ gives $2e^z + 2xe^z\frac{\partial z}{\partial x}$, and the term $x^2e^z\frac{\partial z}{\partial x}$ is a product of three factors, giving $2xe^z\frac{\partial z}{\partial x} + x^2e^z\left(\frac{\partial z}{\partial x}\right)^2 + x^2e^z\frac{\partial^2 z}{\partial x^2}$. Hence

$$2e^z + 4xe^z\frac{\partial z}{\partial x} + x^2e^z\left(\frac{\partial z}{\partial x}\right)^2 + x^2e^z\frac{\partial^2 z}{\partial x^2} = 8\frac{\partial^2 z}{\partial x^2}$$

At P, with $\frac{\partial z}{\partial x} = -\frac{5}{4}$ from part (a):

$$2 + 4 \times 4 \times \left(-\frac{5}{4}\right) + 16 \times \frac{25}{16} + 16\frac{\partial^2 z}{\partial x^2} = 8\frac{\partial^2 z}{\partial x^2}$$

$$2 - 20 + 25 = -8\frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{7}{8}$$

(c) Setting $z = 0$, so that $e^z = 1$, the contour is

$$x^2 + 2x - 4y^2 = 8$$

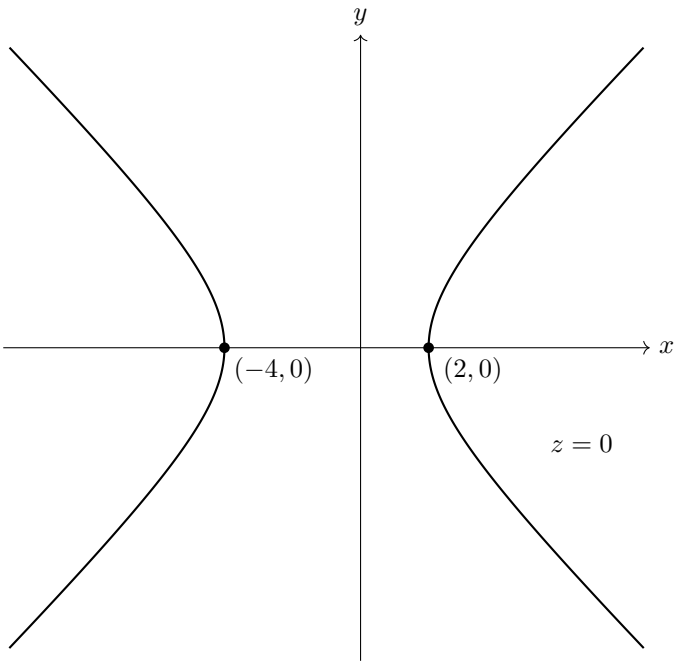
Completing the square in x:

$$(x+1)^2 - 1 - 4y^2 = 8$$

$$(x+1)^2 - 4y^2 = 9$$

as required.

The sketch should look like:



Rearranging, $4y^2 = (x+1)^2 - 9$, so points of the contour have $(x+1)^2 \geq 9$: there are none with $-4 < x < 2$. The contour is symmetric about the x-axis and about the line $x = -1$, and for large $|x|$ its branches spread out along the lines $y = \pm\frac{1}{2}(x+1)$.

On the x-axis, $y = 0$, so $(x+1)^2 = 9$, giving $x = 2$ or $x = -4$. The contour meets the x-axis at $(2, 0)$ and $(-4, 0)$.