OPTIONAL PURE / SURFACES & PARTIAL DIFFERENTIATION

Stationary Points of Surfaces

Questions by topic 12 questions · 128 marks

EXAM BOARDS OCR A Y545 · OCR MEI Extra Pure Y435

Instructions

OCR MEI Extra Pure (Y435): the Hessian matrix and second partial derivatives are not examined. Questions 7 and 11 can be left out, or just their part **(a)**; Question 8 can be left out, or just its part **(c)**.

Questions

Question 1	*6 marks*	2
Question 2	*8 marks*	3
Question 3	*9 marks*	4
Question 4	*10 marks*	5
Question 5	*12 marks*	6
Question 6	*13 marks*	7
Question 7	*11 marks*	8
Question 8	*12 marks*	9
Question 9	*13 marks*	10
Question 10	*11 marks*	11
Question 11	*14 marks*	12
Question 12	*9 marks*	13

1. A surface is defined in three dimensions by the equation

$$z = 3x^2y + y^3 - 12y$$

Find the coordinates of all the stationary points on the surface. **[6]**

2. A surface S is defined by

$$z = 30 + 6x - 8y - x^2 - 4y^2$$

(a) Show that the point $P(3, -1, 43)$ is the only stationary point on S. **[2]**

(b) (i) On a set of axes, sketch the contour of S corresponding to $z = 27$, stating the exact coordinates of the points where it crosses the coordinate axes. **[3]**

(ii) By using your sketch in part **(b)(i)**, deduce that P must be a maximum point on S. **[3]**

3. A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = y^3 - x^2y + 3x^2 - 3y^2 + 20$$

You are given that the point $(0, 0, 20)$ on S is a stationary point.

(a) Determine the coordinates of all the other stationary points on S. **[6]**

(b) By sketching the sections $z = f(x, 0)$ and $z = f(0, y)$ on separate sets of axes, determine the nature of the stationary point $(0, 0, 20)$ on S. **[3]**

4. For all real values of x and y, the surface S has equation

$$z = -9x^2 + 12xy - 4y^2 + 15x - 10y + k$$

where k is an integer constant.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **[2]**

(b) Show that S has infinitely many stationary points, and that each of them lies vertically above or below the line $6x-4y = 5$ in the x-y plane. Find, in terms of k, the value of z at these stationary points. **[4]**

(c) Determine the largest value of k for which the whole of S lies below the x-y plane, justifying your answer. **[4]**

5. A surface S is defined by

$$z = f(x, y) = x^2y - 2xy + y - 4y^3 + 3$$

(a) Show that there is only one stationary point on S, and find the values of x and y at this point. **[6]**

The value of z at the stationary point is denoted by s.

(b) State the value of s. **[1]**

(c) By factorising $f(x, y) - s$, sketch the contours of S for $z = s$, showing where they meet the coordinate axes. **[3]**

(d) Hence, by considering the sign of $f(x, y) - s$ in the regions separated by these contours, explain whether the stationary point is a maximum point, a minimum point or a saddle point. **[2]**

6. The surface S has equation $z = f(x, y)$, where

$$f(x, y) = 8x^2y - 16xy^2 + 3y^4$$

for all real values of x and y. The surface S has exactly two stationary points: one at the origin, O, and the other at the point $P(a, b, c)$, where $c = f(a, b)$.

(a) Determine the values of a, b and c. **[6]**

(b) Throughout this part, take a and b to be the values found in part (a).

(i) Evaluate f_x at the point of S where $x = a - 0.1$, $y = b$ and at the point of S where $x = a + 0.1$, $y = b$. **[2]**

(ii) Evaluate f_y at the point of S where $x = a$, $y = b - 0.1$ and at the point of S where $x = a$, $y = b + 0.1$. **[2]**

(iii) Use your answers to parts (b)(i) and (b)(ii) to sketch the portions of the sections of S given by

- $z = f(x, b)$, for $|x - a| \leq 0.1$,
- $z = f(a, y)$, for $|y - b| \leq 0.1$.

[2]

(iv) Explain why your sketches in part (b)(iii) are not, on their own, enough to show that P is a local minimum point of S. **[1]**

7. The surface S is given by the equation $z = f(x, y)$, where

$$f(x, y) = 2x^2y + xy^2 + 9y$$

and x and y take all real values.
Subscripts denote partial derivatives: for example, f_x denotes $\frac{\partial f}{\partial x}$ and f_{xy} denotes $\frac{\partial^2 f}{\partial y\, \partial x}$.

(a) Show that S has exactly two stationary points. Find, in exact form, the coordinates of the stationary point Q that has a positive y-coordinate. **[6]**

(b) Find the value of the determinant of $\mathbf{H}$, the Hessian matrix of S, at Q. **[4]**

(c) Deduce the nature of the stationary point Q. **[1]**

8. The surface S has equation

$$z = y \ln x - x \sin y$$

for $x > 0$ and $0 < y < \pi$.

(a) Determine, as a function of x and y, the determinant of $\mathbf{H}$, the Hessian matrix of S. **[6]**

(b) Given that S has exactly one stationary point, P, use your answer to part (a) to deduce the nature of P. **[2]**

(c) The coordinates of P are (α, β, γ). Show that

$$\ln\left(\frac{\beta}{\sin\beta}\right) = \beta \cot \beta$$

and that $\gamma = \beta(\beta \cot \beta - 1)$. **[4]**

9. A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = ye^{-(x^2-6x+10)y^2}$$

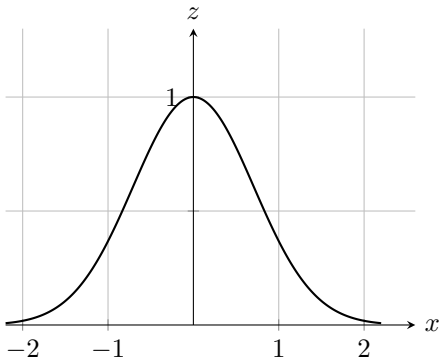
(a) (i) Find $\frac{\partial f}{\partial x}$. **[1]**

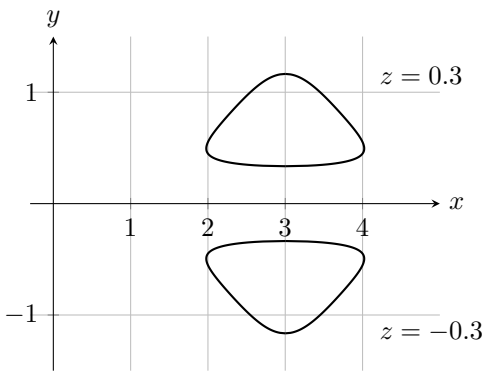
(ii) Show that

$$\frac{\partial f}{\partial y} = \left(1 - 2(x^2 - 6x + 10)y^2\right) e^{-(x^2-6x+10)y^2}$$ **[1]**

(iii) Find the exact coordinates of each of the stationary points on S. **[4]**

The first diagram below shows the graph of $z = e^{-x^2}$, and the second diagram below shows the contours of S defined by $z = 0.3$ and $z = -0.3$.





(b) Specify a sequence of transformations which transforms the graph of $z = e^{-x^2}$ onto the graph of the section defined by $z = f\left(x, \frac{1}{\sqrt{2}}\right)$. **[3]**

(c) Hence, or otherwise, sketch the section defined by $z = f\left(x, \frac{1}{\sqrt{2}}\right)$. **[1]**

(d) Using the second diagram above and your answer to part (c), classify each of the stationary points on S, justifying your answer. **[3]**

10. A trading company deals in two goods. The formula used to estimate z, the total weekly cost to the company of trading the two goods, in tens of thousands of pounds, is

$$z = 0.4xy + \frac{7.2}{x} - 0.3y^2$$

where x and y are the masses, in thousands of tonnes, of the two goods.

You are given that $x > 0$ and $y > 0$.

(a) In the first week of trading, it was found that the values of x and y corresponded to the stationary value of z. Find these values of x and y, and determine the total cost to the company for this week. **[6]**

(b) For the second week, the company intends to make a small change to exactly one of x and y, keeping the other fixed, in order to reduce the total weekly cost. Determine which of x and y the company should change, and whether it matters if that variable is increased or decreased. **[5]**

11. The surface C has equation $z = f(x, y)$, where

$$f(x, y) = x^4 + y^2 + 2axy$$

for all real x and y, and a is a non-zero real constant.
Subscripts denote partial derivatives: for example, f_x denotes $\frac{\partial f}{\partial x}$ and f_{xy} denotes $\frac{\partial^2 f}{\partial y\, \partial x}$.

(a) Show that C has exactly three stationary points, one of which is at the origin, and find the coordinates of the other two in terms of a. **[6]**

(b) Determine the nature of each of these stationary points of C. **[5]**

(c) The surface D has equation $z = g(x, y)$, where $g(x, y) = x^4 + y^2$ for all real x and y. Find the stationary point of D, explain why the test you used in part (b) cannot determine its nature, and use another method to determine its nature. **[3]**

12. The surface S has equation

$$x^2 + 2y^2 + z^3 = 3xyz + 8$$

(a) Show that

$$3\left(z^2 - xy\right)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 2\left(z^3 - 8\right) \quad \textbf{[6]}$$

(b) Hence show that S has exactly one stationary point, and find its coordinates. **[3]**