OPTIONAL PURE / SURFACES & PARTIAL DIFFERENTIATION

Stationary Points of Surfaces

Mark scheme 12 questions · 128 marks

EXAM BOARDS OCR A Y545 · OCR MEI Extra Pure Y435

Questions

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	Total	**128**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

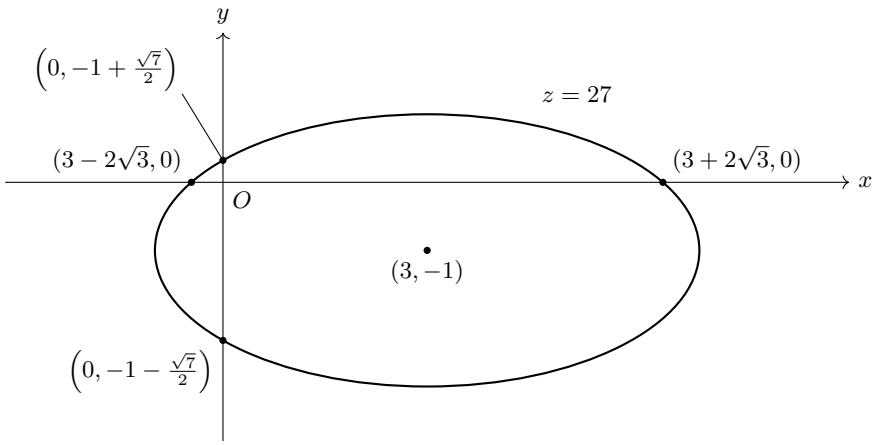
Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

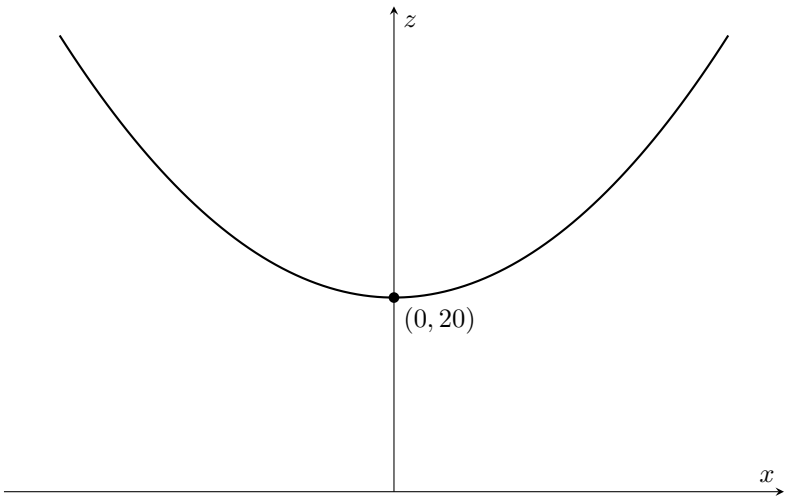
- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$\frac{\partial z}{\partial x} = 6xy$	**B1**	
	$\frac{\partial z}{\partial y} = 3x^2 + 3y^2 - 12$	**B1**	
	Sets both to zero: $6xy = 0 \Rightarrow x = 0$ $3y^2 - 12 = 0 \Rightarrow y = \ldots$	**M1**	Sets both their partial derivatives to zero and uses one case from $\frac{\partial z}{\partial x} = 0$ in the other equation to find the other coordinate. Condone sign slips.
	$(0, 2, -16)$, $(0, -2, 16)$	**A1**	Both points, with z-coordinates.
	$y = 0$: $3x^2 - 12 = 0 \Rightarrow x = \ldots$	**dM1**	Uses the remaining case in their $\frac{\partial z}{\partial y} = 0$ and solves for the other coordinate.
	$(2, 0, 0)$, $(-2, 0, 0)$	**A1**	All four points with z-coordinates, and no others.
	Total for Question 1	**6**	

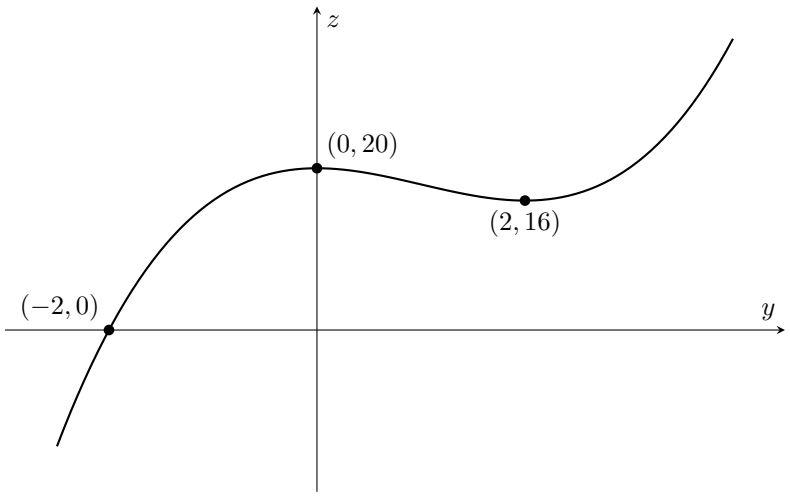
Question	Scheme	Marks	Notes
2(a)	$\frac{\partial z}{\partial x} = 6 - 2x = 0$ $\frac{\partial z}{\partial y} = -8 - 8y = 0 \Rightarrow x = \ldots,\ y = \ldots$	**M1**	Sets both partial derivatives to zero and solves for x and y. Condone one slip in a derivative.
	$x = 3,\ y = -1$ only $z = 30 + 18 + 8 - 9 - 4 = 43$, so $P(3, -1, 43)$ only	**A1 AG**	Both derivatives correct, each giving one value, and $z = 43$ from substitution, with a conclusion that P is the only stationary point.
		[2]	
2(b)(i)	$27 = 30 + 6x - 8y - x^2 - 4y^2$ $\Rightarrow (x-3)^2 + 4(y+1)^2 = 16$	**M1**	Sets $z = 27$ and completes the square in both variables, or substitutes $x = 0$ and $y = 0$ and solves the resulting quadratics. Condone arithmetic slips.
			
	Shape	**B1**	Closed ellipse centred at $(3, -1)$, longer in the x-direction, crossing each axis twice.
	Intercepts	**B1**	All four crossings exact: $(3 \pm 2\sqrt{3}, 0)$ and $\left(0, -1 \pm \frac{\sqrt{7}}{2}\right)$, on the sketch or stated.
		[3]	
2(b)(ii)	Contour $z = 27$ closed, with $(3, -1)$ inside it	**E1**	States that the contour is a closed curve and that $(3, -1)$, below P, lies inside it.
	$z = 27$ on the contour, $z = 43 > 27$ at P	**E1**	Compares the height 27 everywhere on the contour with the height 43 at P.

Question 2 continued

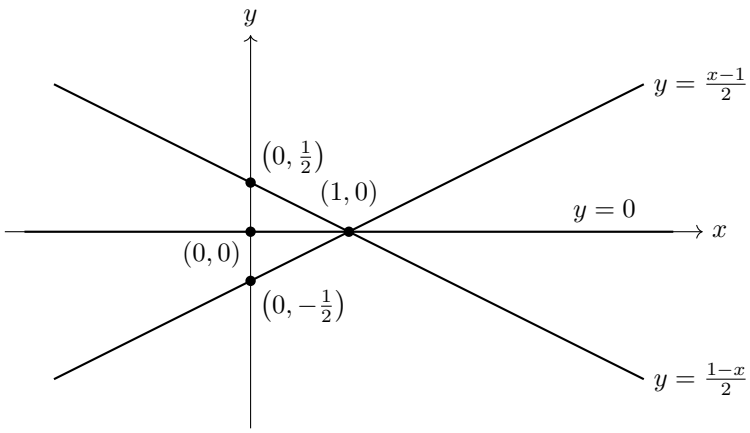
Question	Scheme	Marks	Notes
	Highest point over the enclosed region is inside the contour, so stationary Only stationary point is P, so P is a maximum	**E1**	Argues that the highest point over the enclosed region lies inside the contour, so is stationary, and uses part (a) to conclude it is P. Not for the conclusion alone.
		[3]	
	Total for Question 2	**8**	

Question	Scheme	Marks	Notes
3(a)	$\frac{\partial z}{\partial x} = -2xy + 6x$	**B1**	Any equivalent form.
	$\frac{\partial z}{\partial y} = 3y^2 - x^2 - 6y$	**B1**	
	$x = 0 \Rightarrow 3y^2 - 6y = 0 \Rightarrow y = \ldots$	**M1**	Sets both their partial derivatives to zero and substitutes one case from $\frac{\partial z}{\partial x} = 0$, such as $x = 0$, into the other, solving for the remaining variable.
	$(0, 2, 16)$	**A1**	All three coordinates. Condone $(0, 0, 20)$ also listed.
	$y = 3 \Rightarrow 27 - x^2 - 18 = 0 \Rightarrow x = \ldots$	**dM1**	Substitutes the other case, $y = 3$, into their $\frac{\partial z}{\partial y} = 0$ and solves for x.
	$(3, 3, 20)$, $(-3, 3, 20)$	**A1**	Both points, with all three coordinates.
		[6]	
3(b)			
	Shape	**B1**	U-shaped parabola with its minimum at $(0, 20)$, labelled or marked on the z-axis.

Question 3 continued

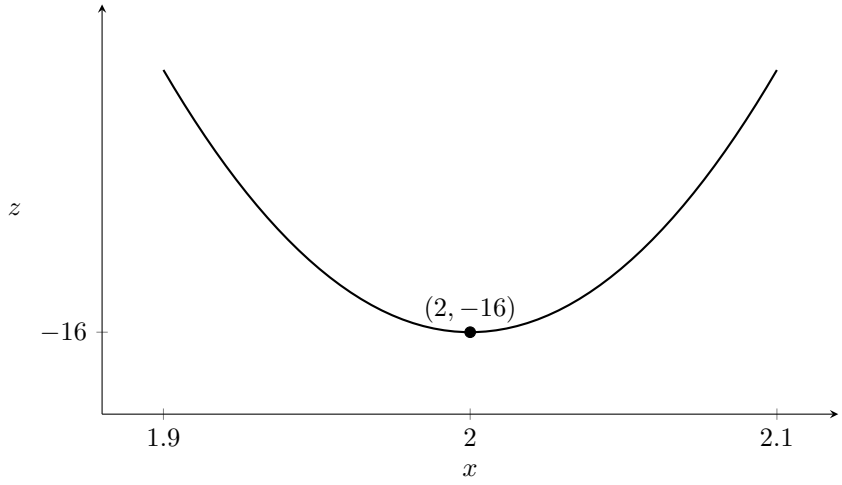
Question	Scheme	Marks	Notes
			
	Shape	**B1**	Positive cubic with a maximum at $(0, 20)$, labelled or marked, and a minimum for $y > 0$. Intercepts not needed.
	Minimum on $z = f(x, 0)$, maximum on $z = f(0, y)$ $\Rightarrow$ saddle point	**E1**	States that $(0, 0, 20)$ is a minimum on one section and a maximum on the other, so a saddle point. Needs both natures correct.
		[3]	
	Total for Question 3	**9**	

Question	Scheme	Marks	Notes
4(a)	$\frac{\partial z}{\partial x} = -18x + 12y + 15$	**B1**	
	$\frac{\partial z}{\partial y} = 12x - 8y - 10$	**B1**	
		[2]	
4(b)	$-18x + 12y + 15 = 0, \quad 12x - 8y - 10 = 0$	**M1**	Sets both their partial derivatives equal to zero.
	Both $\Rightarrow 6x - 4y = 5$, so every point on the line is stationary: infinitely many	**A1**	Both equations shown equivalent to $6x - 4y = 5$ oe, with a conclusion that every point on the line is stationary, so infinitely many.
	$z = -(3x - 2y)^2 + 5(3x - 2y) + k$ $3x - 2y = \frac{5}{2} \Rightarrow z = \ldots$	**M1**	Substitutes the line into z, via $3x - 2y = \frac{5}{2}$ or by eliminating one variable, to reach z in terms of k only. Condone arithmetic slips.
	$z = k + \frac{25}{4}$	**A1**	oe, such as $k + 6.25$.
		[4]	
4(c)	$z = -\left[\left(3x - 2y - \frac{5}{2}\right)^2 - \frac{25}{4}\right] + k$	**M1**	Completes the square in $3x - 2y$, oe such as treating z as a quadratic in $3x - 2y$ with negative leading coefficient. Condone sign slips.
	$z = k + \frac{25}{4} - \left(3x - 2y - \frac{5}{2}\right)^2 \leq k + \frac{25}{4}$	**A1**	Greatest value $k + \frac{25}{4}$, justified by the square being non-negative or the quadratic's maximum.
	$k + \frac{25}{4} < 0 \Rightarrow k < \ldots$	**M1**	Sets their greatest value of z less than zero and solves for k. Condone $\leq$.
	$k = -7$	**A1**	cao.
		[4]	
	Total for Question 4	**10**	

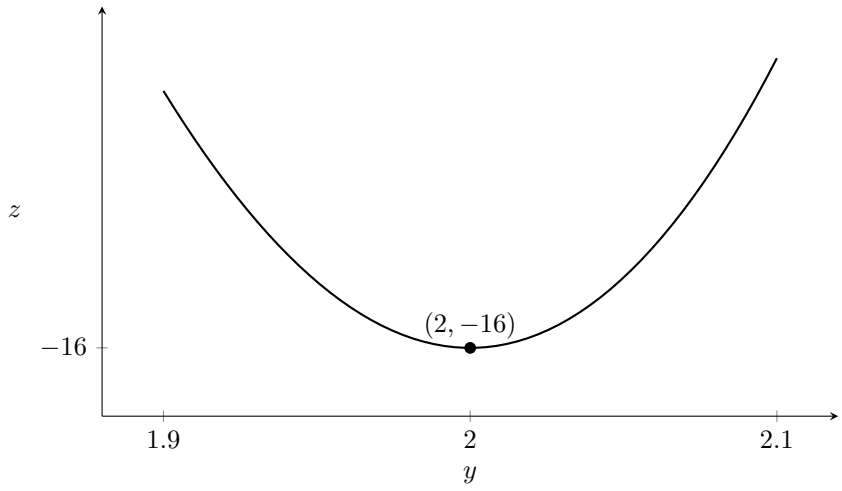
Question	Scheme	Marks	Notes
5(a)	$\frac{\partial f}{\partial x} = 2xy - 2y$	**B1**	
	$\frac{\partial f}{\partial y} = x^2 - 2x + 1 - 12y^2$	**B1**	
	$2y(x-1) = 0 \Rightarrow y = 0$ or $x = 1$ $y = 0$: $(x-1)^2 = 0 \Rightarrow x = \ldots$	**M1**	Sets both partial derivatives to zero and substitutes one case of their $\frac{\partial f}{\partial x} = 0$, usually $y = 0$, into the other equation. Condone slips in their derivatives.
	$x = 1$	**A1**	
	$x = 1$: $-12y^2 = 0 \Rightarrow y = \ldots$	**dM1**	Substitutes the other case, $x = 1$, into $\frac{\partial f}{\partial y} = 0$ and solves for y.
	$y = 0$, so the only stationary point is $x = 1$, $y = 0$	**A1 AG**	Both cases considered, leading without error to $x = 1$, $y = 0$, with a conclusion that there is only one stationary point.
		[6]	
5(b)	$s = 3$	**B1**	
		[1]	
5(c)	$f(x,y) - 3 = y\left((x-1)^2 - (2y)^2\right) = \ldots$	**M1**	Takes out the factor y and factorises the quadratic factor, as a difference of two squares or otherwise. Condone sign slips.
	$f(x,y) - 3 = y(x - 1 - 2y)(x - 1 + 2y)$	**A1**	oe, such as $-y(2y - x + 1)(x - 1 + 2y)$.
			
	Shape	**B1**	Three straight lines through $(1, 0)$: the x-axis and two lines crossing the y-axis at $\left(0, \pm\frac{1}{2}\right)$, with these points shown.
		[3]	

Question 5 continued

Question	Scheme	Marks	Notes
5(d)	Considers the sign on $x = 1$: $f(1, y) - 3 = -4y^3$	**M1**	Finds the sign of $f(x, y) - 3$ in at least two regions meeting at $(1, 0)$, by a line through $(1, 0)$, test points or the signs of the factors.
	Signs alternate round $(1, 0)$: $z > 3$ and $z < 3$ in regions meeting there $\Rightarrow$ saddle point	**A1**	Saddle point, with the reason that $f(x, y) - 3$ takes both signs in regions meeting at $(1, 0)$. Not for "saddle point" alone.
		[2]	
	Total for Question 5	**12**	

Question	Scheme	Marks	Notes
6(a)	$f_x = 16xy - 16y^2$	**B1**	oe, such as $16y(x-y)$.
	$f_y = 8x^2 - 32xy + 12y^3$	**B1**	
	$16y(x-y) = 0$ $y = 0$: $8x^2 = 0 \Rightarrow x = \ldots$	**M1**	Sets both partial derivatives to zero and factorises their f_x into the cases $y = 0$ and $x = y$.
	$x = 0$, the origin O	**A1**	$y = 0$ gives only O, by substitution or from the stem.
	$x = y$: $12y^3 - 24y^2 = 0 \Rightarrow y = \ldots$	**M1**	Substitutes $x = y$ into their f_y and solves the cubic for a nonzero root by any method. Condone slips.
	$a = 2,\ b = 2,\ c = f(2,2) = -16$	**A1**	All three values, including c.
		[6]	
6(b)(i)	$f_x(1.9, 2) = 16 \times 2 \times (1.9 - 2) = \ldots$ $f_x(2.1, 2) = 16 \times 2 \times (2.1 - 2) = \ldots$	**M1**	Substitutes both points into their f_x from (a).
	-3.2 and 3.2	**A1**	Both values.
		[2]	
6(b)(ii)	$f_y(2, 1.9) = 32 - 64(1.9) + 12(1.9)^3 = \ldots$ $f_y(2, 2.1) = 32 - 64(2.1) + 12(2.1)^3 = \ldots$	**M1**	Substitutes both points into their f_y from (a).
	-7.292 and 8.732	**A1**	Both values, or awrt -7.29 and 8.73.
		[2]	
6(b)(iii)			
	Shape	**B1ft**	Section $z = f(x, b)$ decreasing then increasing on $1.9 \leq x \leq 2.1$, minimum marked at $(2, -16)$. ft the signs of their (b)(i) gradients and their a, c.

Question 6 continued

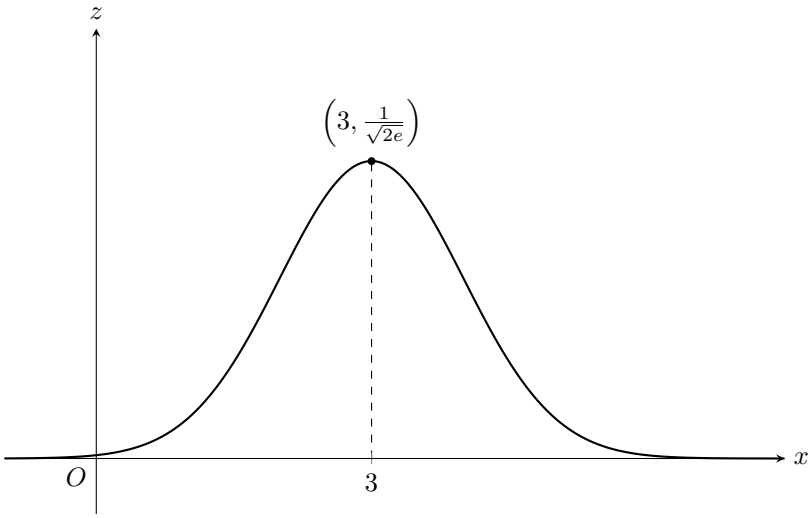
Question	Scheme	Marks	Notes
			
	Shape	**B1ft**	Section $z = f(a, y)$ decreasing then increasing on $1.9 \leq y \leq 2.1$, minimum marked at $(2, -16)$. ft the signs of their (b)(ii) gradients and their b, c.
		[2]	
6(b)(iv)	$z = f(x, 2)$ and $z = f(2, y)$ show only directions parallel to the axes S could decrease from P in another direction	**E1**	States that the sections show only the two axis directions and that S may decrease from P in another direction, oe saddle. Not for naming another test alone.
		[1]	
	Total for Question 6	**13**	

Question	Scheme	Marks	Notes
7(a)	$f_x = 4xy + y^2$	**B1**	
	$f_y = 2x^2 + 2xy + 9$	**B1**	
	$y(4x + y) = 0 \Rightarrow y = 0$ or $y = -4x$ $y = 0 \Rightarrow 2x^2 + 9 = 0$	**M1**	Sets both partial derivatives to zero, finds the case $y = 0$ and substitutes it into their $f_y = 0$.
	$2x^2 + 9 \geq 9$, so no real solutions	**A1**	No real solutions, with a reason such as $2x^2 + 9 > 0$ or $x^2 = -\frac{9}{2}$.
	$y = -4x \Rightarrow 2x^2 - 8x^2 + 9 = 0$ $\Rightarrow x = \ldots \left[= \pm\frac{\sqrt{6}}{2}\right]$	**M1**	Substitutes $y = -4x$ into their $f_y = 0$ and solves for two values of x. Condone sign slips.
	Exactly two stationary points $Q\left(-\frac{\sqrt{6}}{2}, 2\sqrt{6}, 12\sqrt{6}\right)$	**A1**	Exact coordinates of Q, oe such as $x = -\sqrt{\frac{3}{2}}$, with exactly two stationary points concluded from both cases; A0 if the $y = 0$ case is missing.
		[6]	
7(b)	$f_{xx} = 4y$ $f_{xy} = 4x + 2y$ $f_{yy} = 2x$	**M1 A1**	**M1** Differentiates their first partial derivatives again to obtain all three second partial derivatives. Condone slips. **A1** All three correct.
	$\det(\mathbf{H}) = \left(8\sqrt{6}\right)\left(-\sqrt{6}\right) - \left(2\sqrt{6}\right)^2$	**dM1**	Evaluates their second derivatives at Q and forms the determinant of their **H**, in either order. Condone a sign slip.
	$\det(\mathbf{H}) = -72$	**A1**	cao.
		[4]	
7(c)	$\det(\mathbf{H}) < 0$, so Q is a saddle point	**B1ft**	Saddle point from the sign of $\det(\mathbf{H})$. ft their $\det(\mathbf{H})$; if positive, maximum or minimum by the sign of their f_{xx}.
		[1]	
	Total for Question 7	**11**	

Question	Scheme	Marks	Notes
8(a)	$\frac{\partial z}{\partial x} = \frac{y}{x} - \sin y$	**B1**	
	$\frac{\partial z}{\partial y} = \ln x - x\cos y$	**B1**	
	$\frac{\partial^2 z}{\partial x^2} = -\frac{y}{x^2}, \quad \frac{\partial^2 z}{\partial y^2} = x\sin y$ $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1}{x} - \cos y$	**M1 A1**	**M1** Differentiates their first partial derivatives to obtain all three second partial derivatives. Condone slips. **A1** All three correct.
	$\det(\mathbf{H}) = \left(-\frac{y}{x^2}\right)(x\sin y) - \left(\frac{1}{x} - \cos y\right)^2$	**dM1**	Forms the product of their two unmixed second derivatives minus the square of their mixed derivative.
	$\det(\mathbf{H}) = -\frac{y\sin y}{x} - \left(\frac{1}{x} - \cos y\right)^2$	**A1**	oe, such as expanded or over a common denominator.
		[6]	
8(b)	Considers the sign of each term: $\frac{y\sin y}{x} > 0$ and $\left(\frac{1}{x} - \cos y\right)^2 \geq 0$	**M1**	Uses $x > 0$ and $0 < y < \pi$ to find the sign of each term of their determinant.
	$\det(\mathbf{H}) < 0$ everywhere, so P is a saddle point	**A1**	cso, from a correct determinant. Needs $\det(\mathbf{H}) < 0$ and the conclusion saddle point.
		[2]	
8(c)	$\frac{\beta}{\alpha} - \sin\beta = 0 \Rightarrow \alpha = \frac{\beta}{\sin\beta}$ $\ln\alpha - \alpha\cos\beta = 0 \Rightarrow \ln\left(\frac{\beta}{\sin\beta}\right) = \ldots$	**M1**	Sets both first partial derivatives to zero at P and eliminates α. Condone sign slips.
	$\ln\left(\frac{\beta}{\sin\beta}\right) = \frac{\beta\cos\beta}{\sin\beta} = \beta\cot\beta$	**A1**	Printed result reached from correct equations without error.
	$\gamma = \beta\ln\alpha - \alpha\sin\beta$ $= \beta(\beta\cot\beta) - \beta$	**M1**	Substitutes α, β into the equation of S and uses both stationary-point relations to eliminate α.
	$\gamma = \beta(\beta\cot\beta - 1)$	**A1 AG**	Correct substituted equation leading without error to the printed result; combined steps accepted.
		[4]	
	Total for Question 8	**12**	

Question	Scheme	Marks	Notes
9(a)(i)	$\frac{\partial f}{\partial x} = -2(x-3)y^3 e^{-(x^2-6x+10)y^2}$	**B1**	oe; need not be simplified.
		[1]	
9(a)(ii)	$\frac{\partial f}{\partial y} = e^{-(x^2-6x+10)y^2}$ $- 2(x^2-6x+10)y^2 e^{-(x^2-6x+10)y^2}$ $= \left(1-2(x^2-6x+10)y^2\right) e^{-(x^2-6x+10)y^2}$	**B1**	Product rule with the chain rule on the exponential, both terms seen, reaching the printed result without error.
		[1]	
9(a)(iii)	$\frac{\partial f}{\partial x} = 0 \Rightarrow (x-3)y^3 = 0$ $\Rightarrow x = 3$ or $y = 0$	**M1**	Sets their $\frac{\partial f}{\partial x}$ equal to zero and solves, using the exponential factor nonzero.
	$y = 0 \Rightarrow \frac{\partial f}{\partial y} = 1 \neq 0$, so $x = 3$	**A1**	Rejects $y = 0$ with a reason, such as $\frac{\partial f}{\partial y} = 1$ there.
	$x = 3 \Rightarrow (1-2y^2)e^{-y^2} = 0 \Rightarrow y = \ldots$	**dM1**	Substitutes $x = 3$ into $\frac{\partial f}{\partial y} = 0$ and solves for y. Condone slips evaluating $x^2 - 6x + 10$.
	$\left(3, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$, $\left(3, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$	**A1**	Both points, exact; any exact z, such as $\pm\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$.
		[4]	
9(b)	$f\left(x, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}\left((x-3)^2+1\right)}$ $= \frac{1}{\sqrt{2e}}e^{-\left(\frac{x-3}{\sqrt{2}}\right)^2}$	**M1**	Substitutes $y = \frac{1}{\sqrt{2}}$ and completes the square, reaching the form $ke^{-p(x-3)^2}$. Condone slips in k and p.
	Stretch parallel to the x-axis, scale factor $\sqrt{2}$ Translation of 3 in the positive x-direction	**A1**	Both, in a valid order; a translation before the stretch must be $\frac{3}{\sqrt{2}}$. Accept awrt 1.41.
	Stretch parallel to the z-axis, scale factor $\frac{1}{\sqrt{2e}}$	**A1**	oe, such as $\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$, or awrt 0.429, with the whole sequence valid.
		[3]	

Question 9 continued

Question	Scheme	Marks	Notes
9(c)			
	Shape	**B1**	Bell-shaped curve above the x-axis, symmetrical about $x = 3$, maximum $\left(3, \frac{1}{\sqrt{2e}}\right)$ or awrt 0.429 shown, tending to the x-axis both ways.
		[1]	
9(d)	Closed loops $z = \pm 0.3$ round $\left(3, \pm\frac{1}{\sqrt{2}}\right)$, so neither is a saddle	**E1**	States that each point lies inside a closed contour loop, so neither is a saddle point.
	Section in (c) has a maximum at $x = 3$ $\Rightarrow \left(3, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ is a local maximum	**E1**	Uses $\frac{1}{\sqrt{2e}} > 0.3$ inside the $z = 0.3$ loop, or the section's maximum at $x = 3$ with the first E1, to conclude the point with positive z is a local maximum.
	$f(x, -y) = -f(x, y)$ $\Rightarrow \left(3, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ is a local minimum	**E1**	Uses $f(x, -y) = -f(x, y)$, or the $z = -0.3$ loop with the reflected section, to conclude the other point is a local minimum.
		[3]	
	Total for Question 9	**13**	

Question	Scheme	Marks	Notes
10(a)	$\frac{\partial z}{\partial x} = 0.4y - \frac{7.2}{x^2}$	**B1**	
	$\frac{\partial z}{\partial y} = 0.4x - 0.6y$	**B1**	
	$0.4x = 0.6y \Rightarrow y = \frac{2x}{3}$ $0.4 \times \frac{2x}{3} = \frac{7.2}{x^2} \Rightarrow x = \ldots$	**M1**	Sets both partial derivatives to zero and eliminates one variable to reach an equation in one variable. Condone slips in their derivatives.
	$x = 3,\ y = 2$	**A1**	Both values, and no others.
	$z = 0.4 \times 3 \times 2 + \frac{7.2}{3} - 0.3 \times 2^2$	**dM1**	Substitutes their x and y into the formula for z.
	$z = 3.6$ or £36 000	**A1**	3.6, or £36 000.
		[6]	
10(b)	Fixes $y = 2$: $z = 0.8x + \frac{7.2}{x} - 1.2$ $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{14.4}{x^3}$	**M1**	Uses the section $z = 0.8x + \frac{7.2}{x} - 1.2$ and tests the nature in x: second derivative of the section, $\frac{\partial^2 z}{\partial x^2}$, or values either side of $x = 3$. Condone slips.
	$x = 3$: $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{14.4}{27} > 0$, minimum $\Rightarrow$ changing x alone raises the cost	**A1**	Correct positive value or comparison, concluding a minimum so changing x alone raises the cost.
	Fixes $x = 3$: $z = 1.2y + 2.4 - 0.3y^2$ $= 2.4 - 0.3\left[(y-2)^2 - 4\right]$	**M1**	Uses the section $z = 1.2y + 2.4 - 0.3y^2$ and completes the square in y, or uses $\frac{\partial^2 z}{\partial y^2}$ or values either side of $y = 2$. Condone sign slips.
	$z = 3.6 - 0.3(y-2)^2$, maximum at $y = 2$	**A1**	Correct form, or $\frac{\partial^2 z}{\partial y^2} = -0.6 < 0$, with a maximum at $y = 2$ concluded.
	Change y: $y = 2 \pm h \Rightarrow z = 3.6 - 0.3h^2$ Either direction reduces the cost equally	**E1**	States that y should be changed and that increasing or decreasing it both reduce the cost, so the direction does not matter. Needs both previous A marks.
		[5]	
	Total for Question 10	**11**	

Question	Scheme	Marks	Notes
11(a)	$f_x = 4x^3 + 2ay$	**B1**	
	$f_y = 2y + 2ax$	**B1**	
	$2y + 2ax = 0 \Rightarrow y = -ax$ $4x^3 + 2a(-ax) = 0$	**M1**	Sets both partial derivatives to zero and eliminates one variable to reach an equation in one variable. Condone sign slips.
	$2x(2x^2 - a^2) = 0 \Rightarrow x = 0,\ \pm\frac{a}{\sqrt{2}}$ $a \neq 0$: three distinct x, each with one y $x = 0 \Rightarrow (0, 0, 0)$	**A1**	All three roots, with $a \neq 0$ used for distinctness, exactly three points concluded and the origin identified.
	$x = \pm\frac{a}{\sqrt{2}} \Rightarrow y = \mp\frac{a^2}{\sqrt{2}},\ z = \ldots$	**dM1**	Substitutes a non-zero root into their relation for y, then both into the equation of C for z.
	$\left(\frac{a}{\sqrt{2}}, -\frac{a^2}{\sqrt{2}}, -\frac{a^4}{4}\right)$ $\left(-\frac{a}{\sqrt{2}}, \frac{a^2}{\sqrt{2}}, -\frac{a^4}{4}\right)$	**A1**	Both points, oe such as $\frac{\sqrt{2}}{2}a$.
		[6]	
11(b)	$f_{xx} = 12x^2,\ f_{yy} = 2$ $f_{xy} = 2a$	**M1 A1**	**M1** Differentiates their first partial derivatives again for all three second partial derivatives, the power of at least one term decreased by 1. **A1** All three correct.
	$\det(\mathbf{H}) = 12x^2 \times 2 - (2a)^2$ $= 24x^2 - 4a^2$	**dM1**	Forms $\det(\mathbf{H})$ from their second partial derivatives, with the mixed derivative squared and subtracted.
	Origin: $\det(\mathbf{H}) = -4a^2 < 0$, saddle point	**A1**	Negative determinant at the origin, using $a \neq 0$, and saddle point stated.
	$x^2 = \frac{a^2}{2} \Rightarrow \det(\mathbf{H}) = 8a^2 > 0$ $f_{xx} = 6a^2 > 0$, both minimum points	**A1**	Both values shown positive and both points stated as minima. cso.
		[5]	
11(c)	$4x^3 = 0,\ 2y = 0 \Rightarrow (0, 0, 0)$	**B1**	
	$\det(\mathbf{H}) = 0 \times 2 - 0^2 = 0$, test inconclusive	**B1**	Zero determinant at the origin, with the test stated as inconclusive.

Question 11 continued

Question	Scheme	Marks	Notes
	$z = x^4 + y^2 \geq 0$, $z = 0$ only at $x = y = 0$ Minimum point	**E1**	States $z \geq 0$ for all x, y, or $z > 0$ at every other point, and concludes a minimum. Sections through the origin alone are not enough.
		[3]	
	Total for Question 11	**14**	

Question	Scheme	Marks	Notes
12(a)	$2x + 3z^2\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x}$	**M1 A1**	**M1** Differentiates implicitly in x, with a $z^2\frac{\partial z}{\partial x}$ term from z^3 and the product rule on $3xyz$. Condone sign and coefficient slips. **A1** Correct equation in any arrangement.
	$4y + 3z^2\frac{\partial z}{\partial y} = 3xz + 3xy\frac{\partial z}{\partial y}$	**M1 A1**	**M1** Differentiates implicitly in y, with a $z^2\frac{\partial z}{\partial y}$ term from z^3 and the product rule on $3xyz$. Condone sign and coefficient slips. **A1** Correct equation in any arrangement.
	$3\left(z^2 - xy\right)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$ $= x(3yz - 2x) + y(3xz - 4y)$ $= -2\left(x^2 + 2y^2 - 3xyz\right) = -2\left(8 - z^3\right)$	**M1**	Multiplies their equations by x and y, adds, and uses the equation of S to replace $x^2 + 2y^2 - 3xyz$ or equivalent. Condone sign slips.
	$= 2\left(z^3 - 8\right)$	**A1 AG**	Both correct equations combined and the equation of S used, reaching the printed result without error; combined steps accepted.
		[6]	
12(b)	$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \Rightarrow 2\left(z^3 - 8\right) = 0$	**M1**	Sets both partial derivatives to zero in the result of (a) to obtain an equation in z alone.
	$z = 2$	**A1**	$z = 2$ as the only value.
	$6y - 2x = 0,\ 6x - 4y = 0 \Rightarrow x = y = 0$ $(0, 0, 2)$ only	**A1**	cso. $(0, 0, 2)$ from both partial derivatives zero with $z = 2$, and a statement that it is the only stationary point.
		[3]	
	Total for Question 12	**9**	