OPTIONAL PURE / RECURRENCE RELATIONS

Proof by Induction - Recurrence Relations

Worked solutions 12 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · CIE 9231

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1. The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2, \qquad u_{n+1} = u_n + n2^n$$

Prove by induction that, for all integers $n \geq 1$,

$$u_n = (n-2)2^n + 4$$ **[6]**

Solution

Base case: When $n = 1$, the formula gives

$$(1-2) \times 2^1 + 4 = -2 + 4 = 2$$

and $u_1 = 2$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$u_k = (k-2)2^k + 4$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+1} &= u_k + k2^k \\ &= (k-2)2^k + 4 + k2^k \\ &= (2k-2)2^k + 4 \\ &= (k-1) \times 2 \times 2^k + 4 \\ &= (k-1)2^{k+1} + 4 \\ &= \big((k+1)-2\big)2^{k+1} + 4 \end{aligned}$$

This is the formula with $n = k+1$, so the statement is true for $n = k+1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k+1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all integers $n \geq 1$.

2. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 4, \qquad u_{n+1} = 3u_n + 3^{n+1} - 2n + 1, \qquad n \geq 1$$

Prove by induction that $u_n = n\,(3^n + 1)$ for all positive integers n. **[6]**

Solution

Base case: When $n = 1$, the formula gives

$$1 \times \left(3^1 + 1\right) = 1 \times 4 = 4$$

and $u_1 = 4$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$u_k = k\left(3^k + 1\right)$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+1} &= 3u_k + 3^{k+1} - 2k + 1 \\ &= 3k\left(3^k + 1\right) + 3^{k+1} - 2k + 1 \\ &= k \times 3^{k+1} + 3k + 3^{k+1} - 2k + 1 \\ &= k \times 3^{k+1} + 3^{k+1} + k + 1 \\ &= (k+1) \times 3^{k+1} + (k+1) \\ &= (k+1)\left(3^{k+1} + 1\right) \end{aligned}$$

This is the formula with $n = k + 1$, so the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

3. A sequence of numbers is defined by

$$u_1 = 2, \qquad u_2 = 6$$

$$u_{n+2} = 5u_{n+1} - 6u_n + 2, \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 3^n - 2^n + 1$$ **[6]**

Solution

Base case: The recurrence relation involves two previous terms, so the statement is checked for $n = 1$ and $n = 2$. When $n = 1$:

$$3^1 - 2^1 + 1 = 3 - 2 + 1 = 2 = u_1$$

When $n = 2$:

$$3^2 - 2^2 + 1 = 9 - 4 + 1 = 6 = u_2$$

So the statement is true for $n = 1$ and $n = 2$.

Inductive hypothesis: Assume the statement is true for $n = k$ and $n = k + 1$, so that

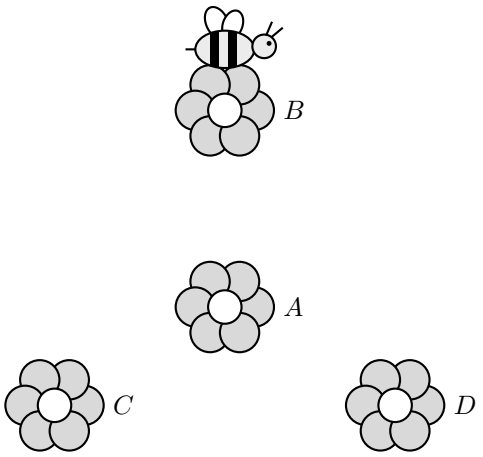
$$u_k = 3^k - 2^k + 1 \qquad \text{and} \qquad u_{k+1} = 3^{k+1} - 2^{k+1} + 1$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+2} &= 5u_{k+1} - 6u_k + 2 \\ &= 5\left(3^{k+1} - 2^{k+1} + 1\right) - 6\left(3^k - 2^k + 1\right) + 2 \\ &= 5 \times 3^{k+1} - 6 \times 3^k - 5 \times 2^{k+1} + 6 \times 2^k + 5 - 6 + 2 \\ &= 3^k(15 - 6) - 2^k(10 - 6) + 1 \\ &= 9 \times 3^k - 4 \times 2^k + 1 \\ &= 3^{k+2} - 2^{k+2} + 1 \end{aligned}$$

This is the formula with $n = k + 2$, so the statement is true for $n = k + 2$.

Conclusion: If the statement is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$. Since it is true for $n = 1$ and $n = 2$, by the principle of mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.



4. Four flowers, A, B, C and D, grow in a flower bed, with A in the middle and B, C and D around it, as shown in the diagram above. A bee flies repeatedly from one flower to another.
The bee starts on flower B, and in a model of its flights

- when the bee is on A, it flies to each of B, C and D with probability $\frac{1}{3}$
- when the bee is on one of B, C or D, it flies to A with probability $\frac{2}{5}$ and to each of the other two outer flowers with probability $\frac{3}{10}$.

Let p_n be the probability that the bee is on flower A after n flights.

(a) Show that $p_2 = \frac{6}{25}$. **[2]**

The probabilities $p_1, p_2, p_3, \ldots$ satisfy the recurrence relation

$$p_1 = \frac{2}{5}, \qquad p_{n+1} = \frac{2}{5}(1 - p_n), \qquad n \geq 1$$

(b) Prove by induction that, for $n \geq 1$,

$$p_n = \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^n$$ **[6]**

(c) Use the result in part (b) to find, with a reason, the long-term probability that the bee is on flower A. **[1]**

Solution

(a) The bee starts on B. After one flight it is on A with probability $\frac{2}{5}$, and on one of the outer flowers C or D with probability $1 - \frac{2}{5} = \frac{3}{5}$.

From A the bee always flies to an outer flower, so it cannot be on A after two flights if it was on A after one flight. From an outer flower it flies to A with probability $\frac{2}{5}$. Hence

$$p_2 = \frac{2}{5} \times 0 + \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$$

as required.

(b) **Base case:** When $n = 1$, the right-hand side is

$$\frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^1 = \frac{2}{7} + \frac{4}{35} = \frac{10}{35} + \frac{4}{35} = \frac{14}{35} = \frac{2}{5}$$

which equals p_1, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$p_k = \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^k$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} p_{k+1} &= \frac{2}{5}(1 - p_k) \\ &= \frac{2}{5}\left(1 - \frac{2}{7} + \frac{2}{7}\left(-\frac{2}{5}\right)^k\right) \\ &= \frac{2}{5}\left(\frac{5}{7} + \frac{2}{7}\left(-\frac{2}{5}\right)^k\right) \\ &= \frac{2}{7} + \frac{4}{35}\left(-\frac{2}{5}\right)^k \\ &= \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)^k \\ &= \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^{k+1} \end{aligned}$$

So the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all $n \geq 1$.

(c) Since $\left|-\frac{2}{5}\right| < 1$, $\left(-\frac{2}{5}\right)^n \to 0$ as $n \to \infty$. Using part (b):

$$p_n \to \frac{2}{7} - \frac{2}{7} \times 0 = \frac{2}{7}$$

The long-term probability that the bee is on flower A is $\frac{2}{7}$.

5. For positive integers n, the sequence of Lucas numbers, L_n, starts with the terms $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, ... and is given by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

(a) Show that $L_{4k+2} = 3L_{4k-1} + 2L_{4k-2}$ for every positive integer k. **[3]**

(b) Prove by induction that L_{4n-2} is divisible by 3 for all positive integers n. **[4]**

Solution

(a) For a positive integer k, the subscripts $4k$, $4k+1$ and $4k+2$ are all at least 3, so the recurrence relation applies to each of these terms. Working forwards from L_{4k-2} and L_{4k-1}:

$$\begin{aligned} L_{4k} &= L_{4k-1} + L_{4k-2} \\ L_{4k+1} &= L_{4k} + L_{4k-1} \\ &= (L_{4k-1} + L_{4k-2}) + L_{4k-1} \\ &= 2L_{4k-1} + L_{4k-2} \end{aligned}$$

Then

$$\begin{aligned} L_{4k+2} &= L_{4k+1} + L_{4k} \\ &= (2L_{4k-1} + L_{4k-2}) + (L_{4k-1} + L_{4k-2}) \\ &= 3L_{4k-1} + 2L_{4k-2} \end{aligned}$$

as required.

(b) **Base case:** When $n = 1$, $L_{4n-2} = L_2 = 3 = 3 \times 1$, which is divisible by 3. So the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$L_{4k-2} = 3m$$

for some integer m.

Inductive step: When $n = k + 1$, $L_{4(k+1)-2} = L_{4k+2}$. Using part (a) and then the inductive hypothesis:

$$\begin{aligned} L_{4k+2} &= 3L_{4k-1} + 2L_{4k-2} \\ &= 3L_{4k-1} + 2 \times 3m \\ &= 3(L_{4k-1} + 2m) \end{aligned}$$

Every Lucas number is an integer, since L_1 and L_2 are integers and each later term is the sum of the two before it. So $L_{4k-1} + 2m$ is an integer, and $L_{4(k+1)-2}$ is divisible by 3.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

6. The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$ and, for $n \geq 1$,

$$u_{n+1} = \frac{5u_n + 6}{u_n + 4}$$

(a) By considering $u_{n+1} - 3$, or otherwise, prove by mathematical induction that $0 < u_n < 3$ for all positive integers n. **[6]**

(b) Show that $u_{n+1} > u_n$ for all positive integers n. **[3]**

Solution

(a) **Base case:** When $n = 1$, $u_1 = 1$, and $0 < 1 < 3$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$0 < u_k < 3$$

Inductive step: Consider $u_{k+1} - 3$:

$$\begin{aligned} u_{k+1} - 3 &= \frac{5u_k + 6}{u_k + 4} - 3 \\ &= \frac{5u_k + 6 - 3(u_k + 4)}{u_k + 4} \\ &= \frac{2u_k - 6}{u_k + 4} \\ &= \frac{2(u_k - 3)}{u_k + 4} \end{aligned}$$

By the inductive hypothesis, $u_k < 3$, so the numerator $2(u_k - 3)$ is negative, and $u_k > 0$, so the denominator $u_k + 4$ is positive. Hence $u_{k+1} - 3 < 0$, that is, $u_{k+1} < 3$.

Also, since $u_k > 0$, both $5u_k + 6 > 0$ and $u_k + 4 > 0$, so

$$u_{k+1} = \frac{5u_k + 6}{u_k + 4} > 0$$

Therefore $0 < u_{k+1} < 3$, so the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

(b) Consider $u_{n+1} - u_n$:

$$\begin{aligned} u_{n+1} - u_n &= \frac{5u_n + 6}{u_n + 4} - u_n \\ &= \frac{5u_n + 6 - u_n(u_n + 4)}{u_n + 4} \\ &= \frac{6 + u_n - u_n^2}{u_n + 4} \\ &= \frac{(3 - u_n)(2 + u_n)}{u_n + 4} \end{aligned}$$

By part (a), $0 < u_n < 3$ for all positive integers n, so $3 - u_n > 0$, $2 + u_n > 0$ and $u_n + 4 > 0$. Hence

$$u_{n+1} - u_n > 0$$

so $u_{n+1} > u_n$ for all positive integers n, as required.

7. A sequence of integers is defined by $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$.
Prove by induction that, for each positive integer n,

$$\left(1+\sqrt{2}\right)^n = u_n + u_{n-1} + u_n\sqrt{2}$$

[6]

Solution

Base case: When $n = 1$, the left-hand side is $\left(1+\sqrt{2}\right)^1 = 1+\sqrt{2}$, and the right-hand side is

$$u_1 + u_0 + u_1\sqrt{2} = 1 + 0 + 1 \times \sqrt{2} = 1+\sqrt{2}$$

so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$\left(1+\sqrt{2}\right)^k = u_k + u_{k-1} + u_k\sqrt{2}$$

Inductive step: Using the inductive hypothesis:

$$\begin{aligned}\left(1+\sqrt{2}\right)^{k+1} &= \left(1+\sqrt{2}\right)^k\left(1+\sqrt{2}\right)\\ &= \left(u_k + u_{k-1} + u_k\sqrt{2}\right)\left(1+\sqrt{2}\right)\\ &= u_k + u_{k-1} + (u_k + u_{k-1})\sqrt{2} + u_k\sqrt{2} + 2u_k\\ &= (3u_k + u_{k-1}) + (2u_k + u_{k-1})\sqrt{2}\end{aligned}$$

Since $k \geq 1$, the recurrence relation gives $u_{k+1} = 2u_k + u_{k-1}$, and so

$$3u_k + u_{k-1} = (2u_k + u_{k-1}) + u_k = u_{k+1} + u_k$$

Therefore

$$\left(1+\sqrt{2}\right)^{k+1} = u_{k+1} + u_k + u_{k+1}\sqrt{2}$$

which is the statement for $n = k+1$, since $u_{(k+1)-1} = u_k$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k+1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for each positive integer n.

8. The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 16$ and

$$u_{n+1} = 5u_n + 16(n+1), \qquad n \geq 1$$

(a) Prove by induction that $u_n = 5^{n+1} - 4n - 5$ for all positive integers n. **[6]**

(b) Using the recurrence relation, deduce that $5^{n+1} - 4n - 5$ is divisible by 16 for all positive integers n. **[4]**

Solution

(a) **Base case:** When $n = 1$, the formula gives

$$5^2 - 4 \times 1 - 5 = 25 - 4 - 5 = 16$$

and $u_1 = 16$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$u_k = 5^{k+1} - 4k - 5$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+1} &= 5u_k + 16(k+1) \\ &= 5\big(5^{k+1} - 4k - 5\big) + 16k + 16 \\ &= 5^{k+2} - 20k - 25 + 16k + 16 \\ &= 5^{k+2} - 4k - 9 \\ &= 5^{(k+1)+1} - 4(k+1) - 5 \end{aligned}$$

so the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

(b) By part (a), $u_n = 5^{n+1} - 4n - 5$ for all positive integers n, so it is enough to prove the statement that u_n is divisible by 16 for all positive integers n.

Base case: $u_1 = 16 = 16 \times 1$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$u_k = 16m$$

for some integer m.

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+1} &= 5u_k + 16(k+1) \\ &= 5 \times 16m + 16(k+1) \\ &= 16(5m + k + 1) \end{aligned}$$

Since $5m + k + 1$ is an integer, u_{k+1} is divisible by 16, so the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n, and so $5^{n+1} - 4n - 5$ is divisible by 16 for all positive integers n.

9. A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$, $u_2 = 10$ and

$$u_{n+2} = 3(n+2)u_{n+1} - 2(n+1)(n+2)u_n, \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = n!\,(2^n + 1)$$

[6]

Solution

Base case: The recurrence relation involves two previous terms, so the statement is checked for $n = 1$ and $n = 2$. When $n = 1$:

$$1!\big(2^1 + 1\big) = 1 \times 3 = 3 = u_1$$

When $n = 2$:

$$2!\big(2^2 + 1\big) = 2 \times 5 = 10 = u_2$$

So the statement is true for $n = 1$ and $n = 2$.

Inductive hypothesis: Assume the statement is true for $n = k$ and $n = k + 1$, so that

$$u_k = k!\big(2^k + 1\big) \qquad \text{and} \qquad u_{k+1} = (k+1)!\big(2^{k+1} + 1\big)$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+2} &= 3(k+2)u_{k+1} - 2(k+1)(k+2)u_k \\ &= 3(k+2)(k+1)!\big(2^{k+1} + 1\big) - 2(k+1)(k+2)k!\big(2^k + 1\big) \end{aligned}$$

Since $(k+2)(k+1)! = (k+2)!$ and $(k+1)(k+2)k! = (k+2)!$, the factor $(k+2)!$ can be taken out:

$$\begin{aligned} u_{k+2} &= (k+2)!\big[3\big(2^{k+1} + 1\big) - 2\big(2^k + 1\big)\big] \\ &= (k+2)!\big[3 \times 2^{k+1} + 3 - 2^{k+1} - 2\big] \\ &= (k+2)!\big[2 \times 2^{k+1} + 1\big] \\ &= (k+2)!\big(2^{k+2} + 1\big) \end{aligned}$$

This is the formula with $n = k + 2$, so the statement is true for $n = k + 2$.

Conclusion: If the statement is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$. Since it is true for $n = 1$ and $n = 2$, by the principle of mathematical induction the statement is true for all $n \in \mathbb{N}$.

10. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 197, \qquad u_2 = 409, \qquad u_{n+2} = 6u_n - u_{n+1}, \qquad n \geq 1$$

(a) Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = 100(2^n) + (-3)^n$$ **[6]**

(b) Hence find the value of the first negative term of the sequence. **[4]**

Solution

(a) **Base case:** The recurrence relation involves two previous terms, so the statement is checked for $n = 1$ and $n = 2$. When $n = 1$:

$$100(2^1) + (-3)^1 = 200 - 3 = 197 = u_1$$

When $n = 2$:

$$100(2^2) + (-3)^2 = 400 + 9 = 409 = u_2$$

So the statement is true for $n = 1$ and $n = 2$.

Inductive hypothesis: Assume the statement is true for $n = k$ and $n = k + 1$, so that

$$u_k = 100(2^k) + (-3)^k \qquad \text{and} \qquad u_{k+1} = 100(2^{k+1}) + (-3)^{k+1}$$

Inductive step: Using the recurrence relation with $n = k$, and then the inductive hypothesis:

$$\begin{aligned} u_{k+2} &= 6u_k - u_{k+1} \\ &= 6\left[100(2^k) + (-3)^k\right] - \left[100(2^{k+1}) + (-3)^{k+1}\right] \\ &= 600(2^k) + 6(-3)^k - 200(2^k) + 3(-3)^k \\ &= 400(2^k) + 9(-3)^k \\ &= 100(2^2 \times 2^k) + (-3)^2 \times (-3)^k \\ &= 100(2^{k+2}) + (-3)^{k+2} \end{aligned}$$

So the statement is true for $n = k + 2$.

Conclusion: If the statement is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$. Since it is true for $n = 1$ and $n = 2$, by the principle of mathematical induction the statement is true for all $n \in \mathbb{N}$.

(b) Using part (a), $u_n = 100(2^n) + (-3)^n$.

If n is even, then $(-3)^n = 3^n > 0$, so $u_n > 0$. Hence any negative term has n odd, and then $(-3)^n = -3^n$, so

$$u_n = 100(2^n) - 3^n$$

This is negative when

$$3^n > 100(2^n)$$

$$\left(\frac{3}{2}\right)^n > 100$$

$$n \ln 1.5 > \ln 100$$

$$n > \frac{\ln 100}{\ln 1.5} \approx 11.36$$

So $u_n < 0$ exactly when n is odd and $n \geq 12$. The smallest such n is $n = 13$, and the first negative term is

$$\begin{aligned} u_{13} &= 100(2^{13}) - 3^{13} \\ &= 819\,200 - 1\,594\,323 \\ &= -775\,123 \end{aligned}$$

11. The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$ and

$$u_{n+1} = \frac{u_n^2 + 5}{2u_n}, \qquad n \geq 1$$

(a) Prove by induction that $u_n > \sqrt{5}$ for all positive integers n. **[6]**

(b) By considering $u_{n+1} - u_n$, use the result of part (a) to show that $u_{n+1} < u_n$ for all positive integers n. **[3]**

(c) Show that $u_{n+1} - \sqrt{5} < \frac{1}{2}\left(u_n - \sqrt{5}\right)$ for all positive integers n. **[2]**

Solution

(a) **Base case:** When $n = 1$, $u_1 = 3$. Since $3^2 = 9 > 5$ and both 3 and $\sqrt{5}$ are positive, $3 > \sqrt{5}$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$u_k > \sqrt{5}$$

In particular, $u_k > 0$.

Inductive step: Using the recurrence relation:

$$\begin{aligned} u_{k+1} - \sqrt{5} &= \frac{u_k^2 + 5}{2u_k} - \sqrt{5} \\ &= \frac{u_k^2 - 2\sqrt{5}\,u_k + 5}{2u_k} \\ &= \frac{\left(u_k - \sqrt{5}\right)^2}{2u_k} \end{aligned}$$

By the inductive hypothesis, $u_k \neq \sqrt{5}$, so $\left(u_k - \sqrt{5}\right)^2 > 0$, and $u_k > 0$, so $2u_k > 0$. Hence

$$u_{k+1} - \sqrt{5} > 0 \quad \Longrightarrow \quad u_{k+1} > \sqrt{5}$$

so the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

(b) Using the recurrence relation:

$$\begin{aligned} u_{n+1} - u_n &= \frac{u_n^2 + 5}{2u_n} - u_n \\ &= \frac{u_n^2 + 5 - 2u_n^2}{2u_n} \\ &= \frac{5 - u_n^2}{2u_n} \end{aligned}$$

By part (a), $u_n > \sqrt{5} > 0$, so $u_n^2 > 5$ and the numerator $5 - u_n^2$ is negative, while the denominator $2u_n$ is positive. Therefore $u_{n+1} - u_n < 0$, so $u_{n+1} < u_n$ for all positive integers n, as required.

(c) As in part (a), the recurrence relation gives

$$u_{n+1} - \sqrt{5} = \frac{\left(u_n - \sqrt{5}\right)^2}{2u_n}$$

Consider the difference:

$$\begin{aligned}
\tfrac{1}{2}\left(u_n - \sqrt{5}\right) - \left(u_{n+1} - \sqrt{5}\right) &= \frac{u_n\left(u_n - \sqrt{5}\right) - \left(u_n - \sqrt{5}\right)^2}{2u_n} \\
&= \frac{\left(u_n - \sqrt{5}\right)\left(u_n - u_n + \sqrt{5}\right)}{2u_n} \\
&= \frac{\sqrt{5}\left(u_n - \sqrt{5}\right)}{2u_n}
\end{aligned}$$

By part (a), $u_n - \sqrt{5} > 0$ and $u_n > 0$, so this difference is positive. Hence $u_{n+1} - \sqrt{5} < \frac{1}{2}\left(u_n - \sqrt{5}\right)$ for all positive integers n, as required.

12. The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and, for all integers $n \geq 1$,

$$a_{n+1} = 2a_n^2 + 1$$

(a) Prove by mathematical induction that $\ln a_n \leq \left(2^{n-1} - 1\right) \ln 3$ for all integers $n \geq 1$. **[6]**

(b) Hence show that $\ln a_{n+1} - \ln a_n \leq 2^{n-1} \ln 3$ for all integers $n \geq 1$. **[2]**

Solution

(a) Since $a_n^2 \geq 0$, every term after the first satisfies $a_{n+1} = 2a_n^2 + 1 \geq 1$, and $a_1 = 1$. So $a_n \geq 1$ for all integers $n \geq 1$: in particular $\ln a_n$ is defined and $a_n^2 \geq 1$.

Base case: When $n = 1$, the two sides are

$$\ln a_1 = \ln 1 = 0 \qquad \text{and} \qquad \left(2^0 - 1\right) \ln 3 = 0$$

so $\ln a_1 \leq \left(2^0 - 1\right) \ln 3$ and the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$\ln a_k \leq \left(2^{k-1} - 1\right) \ln 3$$

Inductive step: Since $a_k^2 \geq 1$, the recurrence relation gives

$$a_{k+1} = 2a_k^2 + 1 \leq 2a_k^2 + a_k^2 = 3a_k^2$$

The function ln is increasing, so, using the inductive hypothesis:

$$\begin{aligned} \ln a_{k+1} &\leq \ln\left(3a_k^2\right) \\ &= \ln 3 + 2 \ln a_k \\ &\leq \ln 3 + 2\left(2^{k-1} - 1\right) \ln 3 \\ &= \left(1 + 2^k - 2\right) \ln 3 \\ &= \left(2^{(k+1)-1} - 1\right) \ln 3 \end{aligned}$$

So the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all integers $n \geq 1$.

(b) As in part (a), $a_n^2 \geq 1$, so $a_{n+1} = 2a_n^2 + 1 \leq 3a_n^2$ and

$$\ln a_{n+1} \leq \ln 3 + 2 \ln a_n$$

Subtracting $\ln a_n$ and then using part (a):

$$\begin{aligned} \ln a_{n+1} - \ln a_n &\leq \ln 3 + \ln a_n \\ &\leq \ln 3 + \left(2^{n-1} - 1\right) \ln 3 \\ &= 2^{n-1} \ln 3 \end{aligned}$$

for all integers $n \geq 1$, as required.