OPTIONAL PURE / RECURRENCE RELATIONS

Modelling with Recurrence Relations

Worked solutions 10 questions · 120 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

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1. A vet begins a course of treatment for an animal when a blood sample taken from it is found to contain 600 000 bacteria of a particular type. The aim of the treatment is to reduce the number of bacteria in such a sample to below 10 000, the level at which the animal's immune system can control the infection without help. If, at any time after the treatment has started, a sample contains 10 000 or more bacteria, then the animal needs further treatment.
The progress of the infection is modelled by the recurrence relation

$$P_{n+1} = \frac{3P_n}{n+2} + \frac{4n}{P_n}$$

for $n \geq 0$, with $P_0 = 600$, where P_n is the number of bacteria, **in thousands**, in a sample taken n days after the treatment started.
The table below shows the values of P_n given by the model for some values of n, correct to 2 decimal places where appropriate.

n	0	1	2	3	4	5	6	7	8	9
P_n	600	900	900.00	675.01	405.03			32.87	11.81	6.25
n	10	15	20	40	60	80	100	200	300	400
P_n	7.46	8.37	9.51	13.04	15.81	18.17	20.25	28.46	34.79	40.13

(a) Find the values of P_5 and P_6, giving each correct to 2 decimal places. **[3]**

(b) Assuming that the model is valid, and using the values of P_0 to P_{10},

(i) describe how the number of bacteria changes during the first 10 days of the treatment, **[1]**

(ii) state the number of days after the treatment started when the animal's immune system is first expected to be able to control the infection for itself. **[1]**

(c) (i) For large values of n, the values in the table suggest that P_n is approximately $f(n)$, where $f(n) = an^k$ for constants a and k. Use the values of P_{100} and P_{400} to suggest suitable values of a and k, where k is a simple fraction. **[2]**

(ii) Use your function f to estimate the number of days after the treatment started when the animal may once again need treatment. **[1]**

(iii) Using the value of P_{20} in the table and the recurrence relation, verify or correct your estimate in part **(c)(ii)**. **[2]**

(d) Give one reason why the predictions of the model for large values of n may be unreliable. **[1]**

Solution

(a) Substituting $n = 4$ and $P_4 = 405.03$ into the recurrence relation gives

$$\begin{aligned} P_5 &= \frac{3 \times 405.03}{6} + \frac{4 \times 4}{405.03} \\ &\approx 202.515 + 0.0395 \\ &= 202.55 \qquad (2 \text{ d.p.}) \end{aligned}$$

Substituting $n = 5$ and the unrounded value of P_5 gives

$$\begin{aligned} P_6 &= \frac{3P_5}{7} + \frac{4 \times 5}{P_5} \\ &\approx 86.8091 + 0.0987 \\ &= 86.91 \qquad (2 \text{ d.p.}) \end{aligned}$$

(b) (i) The number of bacteria rises from 600 000 to 900 000 after 1 day and stays at about 900 000 on day 2. It then falls rapidly, reaching a minimum of about 6250 on day 9, and then rises slightly to about 7460 on day 10.

(ii) The first sample with fewer than 10 000 bacteria is when $P_n < 10$. From the table, $P_8 = 11.81 \geq 10$ and $P_9 = 6.25 < 10$, so the immune system is first expected to control the infection 9 days after the treatment started.

(c) (i) Dividing $f(400)$ by $f(100)$:

$$\frac{f(400)}{f(100)} = \frac{a \times 400^k}{a \times 100^k} = 4^k$$

Using P_{100} and P_{400} from the table:

$$4^k \approx \frac{40.13}{20.25} \approx 1.98 \approx 2$$

so, since $4^{\frac{1}{2}} = 2$, take $k = \frac{1}{2}$. Then $f(100) = a\sqrt{100} = 10a$, and

$$10a \approx 20.25 \quad \Longrightarrow \quad a \approx 2.025$$

so suitable values are $a = 2$ and $k = \frac{1}{2}$, giving $f(n) = 2\sqrt{n}$.

(ii) The animal needs further treatment when $P_n \geq 10$. Setting $f(n) = 10$:

$$2\sqrt{n} = 10 \quad \Longrightarrow \quad \sqrt{n} = 5 \quad \Longrightarrow \quad n = 25$$

Since f is increasing, the animal may once again need treatment about 25 days after the treatment started.

(iii) Using the recurrence relation with $n = 20$ and $P_{20} = 9.51$:

$$P_{21} = \frac{3 \times 9.51}{22} + \frac{4 \times 20}{9.51} \approx 1.2968 + 8.4122 = 9.7090$$

Then with $n = 21$ and $n = 22$, using unrounded values:

$$P_{22} = \frac{3P_{21}}{23} + \frac{4 \times 21}{P_{21}} \approx 1.2664 + 8.6518 \approx 9.9181$$

$$P_{23} = \frac{3P_{22}}{24} + \frac{4 \times 22}{P_{22}} \approx 1.2398 + 8.8726 = 10.1124$$

So $P_{22} < 10$ and $P_{23} \geq 10$: the estimate is corrected to 23 days after the treatment started.

(d) For large n the model predicts that the number of bacteria grows without limit, roughly like $2\sqrt{n}$ thousand. In reality the animal would be given further treatment once the level reached 10 000, and the model does not include this, so its predictions far beyond the start of the treatment may be unreliable.

2. In a conservation project in a nature reserve, scientists are modelling the population of one species of animal.
The initial population of the species, P_0, is 6000. After n years, the population is P_n. The scientists believe that the year-on-year change in the population can be modelled by a recurrence relation of the form

$$P_{n+1} = 2.5P_n(1 - kP_n), \qquad n \geq 0$$

where k is a constant.

(a) The initial aim of the project is to keep the population constant at 6000. Find the value of k for which this happens, according to this model. **[2]**

(b) A few years later, with the population still at 6000, the scientists propose adding 250 animals to the reserve at the end of each year. Using the value of k found in part **(a)**, the population in this scenario is modelled by the recurrence relation

$$P_0 = 6000, \qquad P_{n+1} = 2.5P_n(1 - 0.0001P_n) + 250, \qquad n \geq 0$$

Use your calculator to find the long-term behaviour of P_n predicted by this recurrence relation, describing the way in which P_n approaches its long-term value. **[2]**

(c) The scientists decide **not** to add any animals at the end of each year. Instead, further research predicts that certain factors will remove 1440 animals from the population each year.

(i) Write down a modified form of the recurrence relation given in part **(b)** that models the population when 1440 animals are removed each year and no animals are added. **[1]**

(ii) Starting from $P_0 = 6000$, use your calculator to describe the behaviour of P_n predicted by this modified recurrence relation over the next ten years. **[1]**

(iii) Find algebraically the two values of P for which $P_n = P$ gives $P_{n+1} = P$ in this modified recurrence relation. Explain which of these values is the long-term population predicted when $P_0 = 6000$. **[4]**

(iv) Determine the value that should replace 0.0001 in this modified recurrence relation so that P_n remains constant at 6000. **[2]**

Solution

(a) For the population to stay constant at 6000, $P_n = 6000$ must give $P_{n+1} = 6000$:

$$\begin{aligned} 6000 &= 2.5 \times 6000\,(1 - 6000k) \\ 1 - 6000k &= 0.4 \\ 6000k &= 0.6 \\ k &= 0.0001 \end{aligned}$$

(b) Iterating on the calculator from $P_0 = 6000$ gives

$$P_1 = 6250, \quad P_2 \approx 6109.4, \quad P_3 \approx 6192.3, \quad P_4 \approx 6144.6, \quad P_5 \approx 6172.5, \quad P_6 \approx 6156.3, \quad \ldots$$

The values lie alternately above and below a limiting value, with the size of each swing decreasing. So P_n converges to approximately 6162, oscillating about it with decreasing amplitude.

(c) (i) Replacing the +250 with −1440:

$$P_{n+1} = 2.5P_n(1 - 0.0001P_n) - 1440$$

(ii) Iterating on the calculator from $P_0 = 6000$ gives

$$P_1 = 4560, \quad P_2 = 4761.6, \quad P_3 \approx 4795.8, \quad P_4 \approx 4799.6, \quad P_5 \approx 4800.0, \quad \ldots$$

The population falls sharply to 4560 in the first year, then increases steadily, without oscillating, and settles at 4800 within about six years, staying there for the rest of the ten years.

(iii) Setting $P_{n+1} = P_n = P$:

$$\begin{aligned} P &= 2.5P(1 - 0.0001P) - 1440 \\ P &= 2.5P - 0.00025P^2 - 1440 \\ 0.00025P^2 - 1.5P + 1440 &= 0 \\ P^2 - 6000P + 5\,760\,000 &= 0 \\ (P - 1200)(P - 4800) &= 0 \end{aligned}$$

so $P = 1200$ or $P = 4800$.

Starting from $P_0 = 6000$, the iterates in part (c)(ii) move towards 4800 and stay there, never approaching 1200. So the long-term population predicted is 4800.

(iv) Let q be the value that replaces 0.0001. For $P_n = 6000$ to give $P_{n+1} = 6000$:

$$\begin{aligned} 6000 &= 2.5 \times 6000\,(1 - 6000q) - 1440 \\ 7440 &= 15\,000\,(1 - 6000q) \\ 1 - 6000q &= 0.496 \\ 6000q &= 0.504 \\ q &= 0.000084 \end{aligned}$$

3. An investment company offers customers a three-year investment scheme, in which a fixed bonus of £200 is added at the end of each year.
The scheme is modelled by the recurrence relation

$$I_0 = \alpha, \qquad I_{n+1} = 0.0008I_n^2 - 0.6I_n + 200, \qquad n \geq 0$$

where I_n is the value in pounds of the scheme n years after its start, and £α, where $\alpha > 0$, is the **integer** value of the initial sum invested.

(a) Determine the minimum initial sum invested that guarantees that the value of the scheme increases every year. Justify your answer. **[4]**

(b) The investment company decides to make the following change to the scheme. At the end of each year, the value of I_n is always rounded **down** to the nearest £10.

(i) Write down a modified recurrence relation that takes this change into account. **[1]**

(ii) A customer invests £1900 in the three-year investment scheme. Work out how much less the customer's investment would be worth after three years under the new scheme compared with the original scheme, giving your answer to the nearest penny. **[6]**

Solution

(a) The value increases in a given year when $I_{n+1} > I_n$, that is

$$0.0008I_n^2 - 0.6I_n + 200 > I_n$$

$$0.0008I_n^2 - 1.6I_n + 200 > 0$$

$$I_n^2 - 2000I_n + 250\,000 > 0$$

The roots of $I^2 - 2000I + 250\,000 = 0$ are

$$I = \frac{2000 \pm \sqrt{2000^2 - 4 \times 250\,000}}{2} = 1000 \pm 500\sqrt{3}$$

so the value increases in a given year exactly when

$$I_n < 1000 - 500\sqrt{3} \approx 133.97 \qquad \text{or} \qquad I_n > 1000 + 500\sqrt{3} \approx 1866.03$$

Completing the square gives

$$I_{n+1} = 0.0008(I_n - 375)^2 + 87.5$$

so I_{n+1} decreases as I_n increases for $0 < I_n < 375$. Also $I = 1000 - 500\sqrt{3}$ gives $I_{n+1} = I_n$, since it is a root of the equation above. Hence if $0 < \alpha < 1000 - 500\sqrt{3}$, then

$$1000 - 500\sqrt{3} < I_1 < 200$$

so I_1 lies between the two roots, and $I_2 < I_1$: the value falls in the second year. Initial sums in this range do not work.

If $\alpha > 1000 + 500\sqrt{3}$, then $I_1 > I_0 > 1000 + 500\sqrt{3}$, so I_1 is also above $1000 + 500\sqrt{3}$ and $I_2 > I_1$, and so on: the value increases every year.

Since α is an integer and $1000 + 500\sqrt{3} \approx 1866.03$, the minimum initial sum is £1867.

(b) (i) Rounding down to the nearest £10 means dividing by 10, taking the integer part, written INT, and multiplying by 10:

$$I_0 = \alpha, \qquad I_{n+1} = 10\,\mathrm{INT}\left(\frac{0.0008I_n^2 - 0.6I_n + 200}{10}\right), \qquad n \geq 0$$

(ii) Under the original scheme, with $I_0 = 1900$:

$$I_1 = 0.0008 \times 1900^2 - 0.6 \times 1900 + 200 = 1948$$

$$I_2 = 0.0008 \times 1948^2 - 0.6 \times 1948 + 200 = 2066.9632$$

$$I_3 = 0.0008 \times 2066.9632^2 - 0.6 \times 2066.9632 + 200 \approx 2377.6916$$

Under the new scheme, with $I_0 = 1900$:

$$0.0008 \times 1900^2 - 0.6 \times 1900 + 200 = 1948 \implies I_1 = 1940$$

$$0.0008 \times 1940^2 - 0.6 \times 1940 + 200 = 2046.88 \implies I_2 = 2040$$

$$0.0008 \times 2040^2 - 0.6 \times 2040 + 200 = 2305.28 \implies I_3 = 2300$$

The difference after three years is

$$2377.6916 - 2300 \approx 77.69 \qquad (2\text{ d.p.})$$

so the investment would be worth £77.69 less under the new scheme, to the nearest penny.

4. A streaming service models the number of subscribers that it will have at the end of each month after its launch. The service assumes that

- a constant proportion, r (where $0 < r < 1$), of the previous month's subscribers will cancel during the month
- a constant number, k, of new subscribers will join each month.

Let u_n (where $n \geq 0$) represent the number of subscribers that the service will have at the end of n months, so that u_0 is the number of subscribers at launch.
The service has 60 000 subscribers at launch.

(a) By setting up a first-order recurrence relation for u_{n+1} in terms of u_n, determine an expression for u_n in terms of n, r and k. **[6]**

The service now assumes that 4% of the previous month's subscribers will cancel each month and that 1200 new subscribers will join each month.
According to the model, the service will first have fewer than 32 000 subscribers at the end of the mth month.

(b) Using logarithms, determine the value of m, and explain why, according to the model, the number of subscribers will never fall below 30 000. **[4]**

Solution

(a) During month $n+1$, a proportion r of the u_n subscribers cancel, leaving $(1-r)u_n$, and then k new subscribers join. So the recurrence relation is

$$u_0 = 60\,000, \qquad u_{n+1} = (1-r)u_n + k$$

The complementary function solves $u_{n+1} = (1-r)u_n$, so it is

$$A(1-r)^n$$

For a particular solution, try a constant $u_n = p$ for all n. Substituting into the recurrence relation:

$$p = (1-r)p + k$$

$$rp = k$$

$$p = \frac{k}{r}$$

This is valid since $r \neq 0$. The general solution is therefore

$$u_n = A(1-r)^n + \frac{k}{r}$$

Using $u_0 = 60\,000$:

$$60\,000 = A + \frac{k}{r} \quad \Longrightarrow \quad A = 60\,000 - \frac{k}{r}$$

Hence

$$u_n = \left(60\,000 - \frac{k}{r}\right)(1-r)^n + \frac{k}{r}$$

(b) Here $r = 0.04$ and $k = 1200$, so $\frac{k}{r} = \frac{1200}{0.04} = 30\,000$ and $1 - r = 0.96$. From part (a):

$$u_n = 30\,000(0.96)^n + 30\,000$$

The service has fewer than 32 000 subscribers when

$$30\,000(0.96)^n + 30\,000 < 32\,000$$

$$30\,000(0.96)^n < 2000$$

$$(0.96)^n < \frac{1}{15}$$

Taking natural logarithms of both sides:

$$n \ln 0.96 < \ln\left(\frac{1}{15}\right)$$

Since $\ln 0.96 < 0$, dividing by it reverses the inequality:

$$n > \frac{\ln\left(\frac{1}{15}\right)}{\ln 0.96} \approx 66.3$$

The smallest integer satisfying this is

$$m = 67$$

For every n, $30\,000(0.96)^n > 0$, so $u_n = 30\,000 + 30\,000(0.96)^n > 30\,000$. As $n \to \infty$, $(0.96)^n \to 0$, so u_n decreases towards 30 000 but never reaches it. Hence, according to the model, the number of subscribers never falls below 30 000.

5. A free online coding course is launched, and 40 000 learners enrol on its first run, which lasts 8 weeks. Data from similar courses suggest that, of every 50 000 learners who enrol, only 5006 are still taking part at the end of the 8 weeks, when they complete the course.
It is assumed that the steady decline in the number of learners taking part, from all causes, can be explained by a weekly drop-out factor, r, which is constant across each week of the course.
Let T_k denote the number of learners still taking part at the end of k weeks after the start of the course.

(a) (i) Write down a recurrence relation for T_k, stating the values of k for which it is valid, and hence show that $T_k = 40\,000(1-r)^k$ for $0 \leq k \leq 8$. **[3]**

(ii) Show that $r = 0.250$, correct to 3 significant figures. **[2]**

Learners who complete the course join an online community, and a proportion p of them, where $0 < p < 1$, act as volunteer tutors for future runs of the course. In a simple model of the community, the following assumptions are made.

- From the end of the course, the weekly drop-out factor for members of the community is $\frac{1}{5}r$, and it is the same for tutors and for other members.
- The course can continue to run provided there are at least **200** volunteer tutors still active 26 weeks after the start of the first run.

(b) (i) Find the smallest value of p for which the course can continue to run according to the model. **[4]**

(ii) Write down one assumption that has been made in order to obtain this result. **[1]**

Each volunteer tutor still active 26 weeks after the start of the first run then recruits 150 new learners, who enrol on the second run of the course.

(c) In the case where exactly 200 volunteer tutors are active 26 weeks after the start of the first run, find the total number of learners who enrol on the second run. By comparing this with the first run, give one criticism of the assumption that the course can continue to run provided there are at least **200** volunteer tutors active at that time, and find the least number of volunteer tutors needed for the second run to have at least as many learners as the first. **[3]**

Solution

(a) (i) Each week a fraction r of the learners taking part drop out, so a fraction $1-r$ remain. The recurrence relation is

$$T_0 = 40\,000, \qquad T_{k+1} = (1-r)T_k, \qquad 0 \leq k \leq 7$$

where k is an integer, since the course lasts 8 weeks. Applying the recurrence repeatedly:

$$T_1 = 40\,000(1-r)$$

$$T_2 = (1-r)T_1 = 40\,000(1-r)^2$$

$$T_3 = (1-r)T_2 = 40\,000(1-r)^3$$

and so on: each term is $(1-r)$ times the previous one, so T_k is a geometric sequence with first term 40 000 and common ratio $1-r$. Hence

$$T_k = 40\,000(1-r)^k \quad \text{for } 0 \leq k \leq 8$$

as required.

(ii) The proportion of enrolled learners who complete the 8 weeks is

$$\frac{T_8}{T_0} = (1-r)^8 = \frac{5006}{50\,000} = 0.10012$$

so

$$1 - r = 0.10012^{\frac{1}{8}} \approx 0.7500066$$

$$r \approx 0.2499934 = 0.250 \qquad (3 \text{ s.f.})$$

as required.

(b) (i) The number of learners completing the course is

$$T_8 = 40\,000 \times 0.10012 = 4004.8$$

so $4004.8p$ of them act as volunteer tutors at the end of week 8. With $r = 0.25$, the weekly drop-out factor in the community is $\frac{1}{5}r = 0.05$, so a fraction 0.95 of the tutors remain each week. Let V_n be the number of tutors still active n weeks after the end of the course. Then

$$V_0 = 4004.8p, \qquad V_{n+1} = 0.95V_n$$

so $V_n = 4004.8p \times 0.95^n$. Week 26 after the start of the first run is $26 - 8 = 18$ weeks after the end of the course, so

$$V_{18} = 4004.8 \times 0.95^{18} \times p \approx 1590.76p$$

The course can continue to run provided $V_{18} \geq 200$:

$$4004.8 \times 0.95^{18} \times p \geq 200$$

$$p \geq \frac{200}{4004.8 \times 0.95^{18}} \approx 0.12573$$

The smallest value of p is 0.126 (3 s.f.).

(ii) The weekly drop-out factor for the community stays constant at $\frac{1}{5}r$ throughout the 18 weeks after the course ends.

(c) The number of learners who enrol on the second run is

$$200 \times 150 = 30\,000$$

This is fewer than the 40 000 who enrolled on the first run, so with only 200 tutors the course shrinks from one run to the next: 200 tutors is enough for the course to run, but not to sustain it at its original size.

For at least as many learners as the first run, with N tutors:

$$150N \geq 40\,000 \quad \Longrightarrow \quad N \geq \frac{800}{3} = 266\tfrac{2}{3}$$

so the least number of volunteer tutors needed is 267.

6. Priya borrows £12 000 from a bank to buy a car.
At the start of each year, Priya pays the bank a fixed amount, £F, or the whole of the amount owed if this is less than £F. At the end of each year, the bank adds 4% interest to the amount still owed.
Let £A_n be the amount owed at the start of year n, before Priya's payment for that year, so that $A_1 = 12\,000$.

(a) Write down an expression for A_{n+1} in terms of A_n and F, valid when $A_n \geq F$. **[1]**

(b) Prove by induction that, for each $n \geq 1$ such that all of Priya's payments before year n are the full £F,

$$A_n = (12\,000 - 26F)\,1.04^{n-1} + 26F$$ **[6]**

(c) Given that $F = 1000$, find the number of payments that Priya makes and the amount of her final payment, giving this amount to the nearest penny. **[5]**

Solution

(a) After paying £F, Priya owes £$(A_n - F)$, and the bank then adds 4% interest, so

$$A_{n+1} = 1.04(A_n - F)$$

(b) **Base case:** When $n = 1$, there are no payments before year 1, and

$$(12\,000 - 26F) \times 1.04^0 + 26F = 12\,000 - 26F + 26F = 12\,000 = A_1$$

so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$: if all of Priya's payments before year k are the full £F, then

$$A_k = (12\,000 - 26F)\,1.04^{k-1} + 26F$$

Inductive step: Suppose that all of Priya's payments before year $k+1$ are the full £F. Then all of her payments before year k are full, so the inductive hypothesis gives A_k. Her payment in year k is also the full £F, so $A_k \geq F$ and the relation from part (a) applies:

$$\begin{aligned} A_{k+1} &= 1.04(A_k - F) \\ &= 1.04\big((12\,000 - 26F)\,1.04^{k-1} + 26F - F\big) \\ &= (12\,000 - 26F)\,1.04^{k} + 1.04 \times 25F \\ &= (12\,000 - 26F)\,1.04^{(k+1)-1} + 26F \end{aligned}$$

This is the statement for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all $n \geq 1$ such that all of Priya's payments before year n are the full £F.

(c) With $F = 1000$, $26F = 26\,000$ and $12\,000 - 26F = -14\,000$, so part (b) gives

$$A_n = 26\,000 - 14\,000 \times 1.04^{n-1}$$

while all payments before year n are full. This expression decreases as n increases, so Priya pays the

full £1000 each year until the first year in which $A_n < 1000$, and that year's payment is her last. The condition $A_n < 1000$ becomes

$$26\,000 - 14\,000 \times 1.04^{n-1} < 1000$$

$$1.04^{n-1} > \frac{25}{14}$$

$$n - 1 > \frac{\ln\left(\frac{25}{14}\right)}{\ln 1.04} \approx 14.78$$

so the first such year is $n = 16$. The values on either side are

$$A_{15} = 26\,000 - 14\,000 \times 1.04^{14} \approx 1756.53 \geq 1000$$

$$A_{16} = 26\,000 - 14\,000 \times 1.04^{15} \approx 786.79 < 1000 \qquad (2 \text{ d.p.})$$

So Priya makes 15 full payments of £1000 and then pays off the remaining amount owed in year 16. She makes 16 payments, and her final payment is £786.79 to the nearest penny.

7. Aisha opens a bank account to help her save for a car. The account is empty when she opens it.
In the first month she pays £k into the account, and in each month after that she pays in £20 more than she paid in the month before. In each month after the first, she also allows herself to spend one fifth of what was in the account at the end of the previous month.
Let u_n, where $n \geq 1$, be the amount, in pounds, in the account at the end of month n, so that $u_1 = k$.

(a) By setting up a first-order recurrence relation for u_{n+1} in terms of u_n and n, determine an expression for u_n in terms of n and k. **[8]**

At the end of the 12th month, Aisha needs to have at least £2500 in the account to pay for the car.

(b) Determine, to the nearest penny, the minimum amount that Aisha should pay into the account in the first month. **[2]**

Solution

(a) In month $n+1$, Aisha pays in $k+20n$ pounds, and she spends $\frac{1}{5}u_n$ pounds. So the recurrence relation is

$$u_1 = k, \qquad u_{n+1} = u_n - \frac{1}{5}u_n + k + 20n = \frac{4}{5}u_n + 20n + k$$

The complementary function solves $u_{n+1} = \frac{4}{5}u_n$, so it is

$$A\left(\frac{4}{5}\right)^n$$

The non-homogeneous part is linear in n, so for a particular solution try $u_n = an + b$:

$$a(n+1) + b = \frac{4}{5}(an + b) + 20n + k$$

$$an + a + b = \frac{4}{5}an + \frac{4}{5}b + 20n + k$$

Equating coefficients of n:

$$a = \frac{4}{5}a + 20 \quad \Longrightarrow \quad \frac{1}{5}a = 20 \quad \Longrightarrow \quad a = 100$$

Equating constant terms:

$$100 + b = \frac{4}{5}b + k \quad \Longrightarrow \quad \frac{1}{5}b = k - 100 \quad \Longrightarrow \quad b = 5k - 500$$

So the general solution is

$$u_n = A\left(\frac{4}{5}\right)^n + 100n + 5k - 500$$

Using $u_1 = k$:

$$\frac{4}{5}A + 100 + 5k - 500 = k$$

$$\frac{4}{5}A = 400 - 4k$$

$$A = 500 - 5k$$

Hence

$$u_n = (500 - 5k)\left(\frac{4}{5}\right)^n + 100n + 5k - 500$$

(b) Using part (a) with $n = 12$, Aisha needs $u_{12} \geq 2500$:

$$(500 - 5k)\left(\frac{4}{5}\right)^{12} + 1200 + 5k - 500 \geq 2500$$

$$5k\left(1 - \left(\frac{4}{5}\right)^{12}\right) \geq 1800 - 500\left(\frac{4}{5}\right)^{12}$$

$$k \geq \frac{1800 - 500\left(\frac{4}{5}\right)^{12}}{5\left(1 - \left(\frac{4}{5}\right)^{12}\right)} \approx 379.185$$

Paying £379.18 would leave her just short of £2500, so the amount is rounded up. The minimum first payment is £379.19.

8. A colony of seabirds is established on a small island. The number of birds in the colony t years after it is established is denoted by N_t, and the initial number of birds is N_0.
When the colony is small, few of its chicks survive, so the following discrete population model is proposed for N_t.

$$\text{Model I: } N_{t+1} = \frac{200N_t^2}{6400 + N_t^2}$$

(a) (i) Find the three steady state values of the number of birds for Model I. **[4]**

(ii) Show that $0 < N_{t+1} - N_t < 160 - N_t$ when $40 < N_t < 160$. **[4]**

(iii) Hence describe the long-term behaviour of the number of birds predicted by Model I when $40 < N_0 < 160$. **[2]**

An alternative discrete population model is proposed for N_t, where $\text{INT}(x)$ denotes the integer part of x.

$$\text{Model II: } N_{t+1} = \text{INT}\left(\frac{200N_t^2}{6400 + N_t^2}\right)$$

(b) (i) Given that $N_0 = 80$, find the value of N_3 for each of the two models, giving the value for Model I correct to 1 decimal place. **[2]**

(ii) Show that 159 is a steady state value of the number of birds for Model II. Given also that, for Model II, $N_{t+1} > N_t$ whenever N_t is an integer with $80 \leq N_t \leq 158$, state how the long-term behaviour predicted by Model II when $N_0 = 80$ differs from that predicted by Model I. **[2]**

Solution

(a) (i) A steady state value N satisfies $N_{t+1} = N_t = N$, so

$$\begin{aligned} N &= \frac{200N^2}{6400 + N^2} \\ N\left(6400 + N^2\right) &= 200N^2 \\ N^3 - 200N^2 + 6400N &= 0 \\ N\left(N^2 - 200N + 6400\right) &= 0 \\ N(N-40)(N-160) &= 0 \end{aligned}$$

The three steady state values are $N = 0$, $N = 40$ and $N = 160$.

(ii) Writing the difference over a common denominator:

$$\begin{aligned} N_{t+1} - N_t &= \frac{200N_t^2 - N_t\left(6400 + N_t^2\right)}{6400 + N_t^2} \\ &= \frac{-N_t\left(N_t^2 - 200N_t + 6400\right)}{6400 + N_t^2} \\ &= -\frac{N_t(N_t - 40)(N_t - 160)}{6400 + N_t^2} \end{aligned}$$

When $40 < N_t < 160$, the factors N_t and $N_t - 40$ are positive and $N_t - 160$ is negative, so $N_t(N_t - 40)(N_t - 160) < 0$. The denominator $6400 + N_t^2$ is positive, so $N_t < N_{t+1}$, that is, $N_{t+1} - N_t > 0$.

Writing $160 - N_{t+1}$ over the same denominator:

$$\begin{aligned} 160 - N_{t+1} &= \frac{160\left(6400 + N_t^2\right) - 200N_t^2}{6400 + N_t^2} \\ &= \frac{1\,024\,000 - 40N_t^2}{6400 + N_t^2} \\ &= \frac{40\left(25\,600 - N_t^2\right)}{6400 + N_t^2} \\ &= \frac{40(160 - N_t)(160 + N_t)}{6400 + N_t^2} \end{aligned}$$

When $40 < N_t < 160$, both $160 - N_t$ and $160 + N_t$ are positive, so $160 - N_{t+1} > 0$, that is, $N_{t+1} < 160$. Subtracting N_t from both sides gives $N_{t+1} - N_t < 160 - N_t$. Combining the two results:

$$0 < N_{t+1} - N_t < 160 - N_t$$

as required.

(iii) By part (a)(ii), if $40 < N_t < 160$ then $N_t < N_{t+1} < 160$, so N_{t+1} also lies between 40 and 160. Starting from $40 < N_0 < 160$, this applies every year. Hence the number of birds increases every year but always stays below 160. The number of birds increases and tends to the steady state value 160, without ever reaching it.

(b) (i) For Model I, with $N_0 = 80$:

$$N_1 = \frac{200 \times 80^2}{6400 + 80^2} = 100$$

$$N_2 = \frac{200 \times 100^2}{6400 + 100^2} = \frac{5000}{41} \approx 121.95$$

$$N_3 = \frac{200 \times \left(\frac{5000}{41}\right)^2}{6400 + \left(\frac{5000}{41}\right)^2} = 139.8 \qquad (1 \text{ d.p.})$$

For Model II, with $N_0 = 80$:

$$N_1 = \text{INT}(100) = 100$$

$$N_2 = \text{INT}\left(\frac{5000}{41}\right) = 121$$

$$N_3 = \text{INT}\left(\frac{200 \times 121^2}{6400 + 121^2}\right) = \text{INT}\left(\frac{2\,928\,200}{21\,041}\right) = 139$$

(ii) With $N_t = 159$:

$$N_{t+1} = \text{INT}\left(\frac{200 \times 159^2}{6400 + 159^2}\right) = \text{INT}\left(\frac{5\,056\,200}{31\,681}\right) = 159$$

so 159 is a steady state value for Model II.

For Model II with $N_0 = 80$, every value of N_t is an integer. While $80 \leq N_t \leq 158$, the number of birds increases by at least one each year, and by part (a)(ii) the value inside the integer part is less than 160, so $N_{t+1} \leq 159$. Hence after a finite number of years Model II reaches $N_t = 159$ and then stays at 159 for ever. Model I instead predicts that the number of birds increases towards 160 without ever reaching it, so it never becomes constant.

9. A young tree is planted in a garden, and its height is measured on the first day of each year.
Let H_n metres be the height of the tree on the first day of year n. The measurements give $H_1 = 1.2$, $H_2 = 1.8$ and $H_3 = 2.2$.
A gardener models the growth of the tree by assuming that the increase in its height during each year is a fixed fraction p of the increase in its height during the previous year, where $0 < p < 1$.

(a) Explain why the model gives

$$H_{n+2} = (1+p)H_{n+1} - pH_n, \qquad n \geq 1$$

and use the measurements to show that $p = \frac{2}{3}$. **[3]**

(b) Prove by induction that

$$H_n = 3 - 1.8\left(\frac{2}{3}\right)^{n-1}, \qquad n \geq 1$$ **[6]**

(c) Explain why, according to the model, the height of the tree is always less than 3 metres, and find the height of the tree in the long term. **[2]**

(d) A second tree of the same species is planted in richer soil. Its heights on the first days of years 1 and 2 are also 1.2 metres and 1.8 metres, and its growth follows the same model with a different value of p. The gardener predicts that its height in the long term is 4 metres.
Find, as a percentage to 3 significant figures, the value of p for the second tree. **[2]**

Solution

(a) The increase in height during year $n+1$ is $H_{n+2} - H_{n+1}$, and the increase during the previous year, year n, is $H_{n+1} - H_n$. The model says the first is p times the second, so

$$\begin{aligned} H_{n+2} - H_{n+1} &= p(H_{n+1} - H_n) \\ H_{n+2} &= H_{n+1} + pH_{n+1} - pH_n \\ H_{n+2} &= (1+p)H_{n+1} - pH_n \end{aligned}$$

Putting $n = 1$ into the recurrence and using the measurements:

$$\begin{aligned} H_3 &= (1+p)H_2 - pH_1 \\ 2.2 &= 1.8(1+p) - 1.2p \\ 2.2 &= 1.8 + 0.6p \\ 0.6p &= 0.4 \\ p &= \frac{2}{3} \end{aligned}$$

as required.

(b) With $p = \frac{2}{3}$, the recurrence is

$$H_{n+2} = \frac{5}{3}H_{n+1} - \frac{2}{3}H_n$$

Base case: When $n = 1$, $3 - 1.8\left(\frac{2}{3}\right)^0 = 3 - 1.8 = 1.2 = H_1$. When $n = 2$, $3 - 1.8\left(\frac{2}{3}\right)^1 = 3 - 1.2 = 1.8 = H_2$. So the statement is true for $n = 1$ and $n = 2$.

Inductive hypothesis: Assume the statement is true for $n = k$ and $n = k + 1$, so that

$$H_k = 3 - 1.8\left(\frac{2}{3}\right)^{k-1}, \qquad H_{k+1} = 3 - 1.8\left(\frac{2}{3}\right)^{k}$$

Inductive step: Using the recurrence and the inductive hypothesis:

$$\begin{aligned}
H_{k+2} &= \frac{5}{3}H_{k+1} - \frac{2}{3}H_k \\
&= \frac{5}{3}\left(3 - 1.8\left(\frac{2}{3}\right)^{k}\right) - \frac{2}{3}\left(3 - 1.8\left(\frac{2}{3}\right)^{k-1}\right) \\
&= 5 - 3\left(\frac{2}{3}\right)^{k} - 2 + 1.2\left(\frac{2}{3}\right)^{k-1} \\
&= 3 - \left(\frac{2}{3}\right)^{k-1}\left(3 \times \frac{2}{3} - 1.2\right) \\
&= 3 - 0.8\left(\frac{2}{3}\right)^{k-1} \\
&= 3 - 1.8 \times \frac{4}{9}\left(\frac{2}{3}\right)^{k-1} \\
&= 3 - 1.8\left(\frac{2}{3}\right)^{k+1}
\end{aligned}$$

This is the statement for $n = k + 2$, since $(k + 2) - 1 = k + 1$.

Conclusion: If the statement is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$. Since it is true for $n = 1$ and $n = 2$, by the principle of mathematical induction the statement is true for all $n \geq 1$.

(c) Since $\frac{2}{3} > 0$, $\left(\frac{2}{3}\right)^{n-1} > 0$ for every n, so $1.8\left(\frac{2}{3}\right)^{n-1} > 0$ and

$$H_n = 3 - 1.8\left(\frac{2}{3}\right)^{n-1} < 3$$

So, according to the model, the height of the tree is always less than 3 metres.

As $n \to \infty$, $\left(\frac{2}{3}\right)^{n-1} \to 0$ because $0 < \frac{2}{3} < 1$, so $H_n \to 3$. The height of the tree in the long term is 3 metres.

(d) The increase during year 1 is $H_2 - H_1 = 1.8 - 1.2 = 0.6$, and each year's increase is p times the previous year's, so the increases during years $1, 2, 3, \ldots$ are $0.6, 0.6p, 0.6p^2, \ldots$, a geometric series with $0 < p < 1$. The sum to infinity is

$$S_\infty = \frac{a}{1 - r} \qquad (|r| < 1)$$

so the height in the long term is

$$1.2 + \frac{0.6}{1 - p}$$

Setting this equal to 4:

$$\frac{0.6}{1-p} = 2.8$$

$$1 - p = \frac{0.6}{2.8} = \frac{3}{14}$$

$$p = \frac{11}{14}$$

As a percentage:

$$p = \frac{11}{14} \times 100\% = 78.6\% \qquad (3 \text{ s.f.})$$

10. A sequence u_n, where $n \geq 0$, satisfies the first-order recurrence relation

$$u_{n+1} = \alpha u_n - \beta\alpha^n \qquad (*)$$

where α and β are non-zero constants.

(a) Find, in terms of α, β and n, the general solution of $(*)$. **[4]**

(b) Given that $u_0 = L$ and $u_T = 0$, where T is a positive integer, show that

$$u_n = L\alpha^n\left(1 - \frac{n}{T}\right) \quad \textbf{[4]}$$

Jordan borrows a sum of money, L, to buy a flat.
On each anniversary of the loan being taken out, 6% interest is charged on the outstanding balance and Jordan then makes that year's repayment.
Each annual repayment is 6% larger than the one before, and the whole debt is cleared in exactly 25 years.
The recurrence relation $(*)$ can be used to model this situation, where u_n is the outstanding debt n years after the loan is taken out, immediately after that year's repayment.

(c) (i) State the value of α. **[1]**

(ii) Explain what β represents in the context of the loan. **[1]**

(iii) Find, as a multiple of L, the amount of Jordan's first annual repayment. **[1]**

(d) Find, according to the model, the number of complete years after which the outstanding debt first falls below the original loan L. **[3]**

Solution

(a) The complementary function solves $u_{n+1} = \alpha u_n$, so it is

$$A\alpha^n$$

The term $\beta\alpha^n$ is a multiple of α^n, which already appears in the complementary function, so try a particular solution of the form $u_n = an\alpha^n$. Substituting into $(*)$:

$$a(n+1)\alpha^{n+1} = \alpha \times an\alpha^n - \beta\alpha^n$$

$$an\alpha^{n+1} + a\alpha^{n+1} = an\alpha^{n+1} - \beta\alpha^n$$

$$a\alpha^{n+1} = -\beta\alpha^n$$

$$a = -\frac{\beta}{\alpha}$$

So a particular solution is $-\frac{\beta}{\alpha}n\alpha^n = -\beta n\alpha^{n-1}$, and the general solution is

$$u_n = A\alpha^n - \beta n\alpha^{n-1}$$

(b) Using $u_0 = L$ in the general solution:

$$L = A\alpha^0 - 0 = A$$

Using $u_T = 0$:

$$L\alpha^T - \beta T\alpha^{T-1} = 0$$

$$\alpha^{T-1}(L\alpha - \beta T) = 0$$

Since $\alpha \neq 0$, this gives $\beta T = L\alpha$, so

$$\beta = \frac{L\alpha}{T}$$

Substituting $A = L$ and this value of β into the general solution:

$$\begin{aligned} u_n &= L\alpha^n - \frac{L\alpha}{T}\, n\alpha^{n-1} \\ &= L\alpha^n - \frac{Ln}{T}\alpha^n \\ &= L\alpha^n\left(1 - \frac{n}{T}\right) \end{aligned}$$

as required.

(c) (i) Each year the outstanding balance is increased by 6% before the repayment, so it is multiplied by 1.06:

$$\alpha = 1.06$$

(ii) The repayment made at the end of year $n+1$ is $\beta\alpha^n = \beta(1.06)^n$, which is β when $n = 0$. So β is the amount of Jordan's first annual repayment, made on the first anniversary of the loan.

(iii) The debt is cleared in exactly 25 years, so $T = 25$. From part (b), $\beta = \frac{L\alpha}{T}$, so

$$\beta = \frac{1.06L}{25} = 0.0424L$$

The first annual repayment is $0.0424L$.

(d) Using part (b) with $\alpha = 1.06$ and $T = 25$:

$$u_n = L(1.06)^n\left(1 - \frac{n}{25}\right)$$

Since $L > 0$, the debt is below L when

$$(1.06)^n\left(1 - \frac{n}{25}\right) < 1$$

Tabulating the left-hand side on a calculator, it is greater than 1 for every n from 1 to 13, and

$$n = 13: \quad (1.06)^{13} \times \frac{12}{25} \approx 1.024 > 1$$

$$n = 14: \quad (1.06)^{14} \times \frac{11}{25} \approx 0.995 < 1$$

So the outstanding debt first falls below L after 14 complete years.