OPTIONAL PURE / RECURRENCE RELATIONS

Behaviour and Limits of Sequences

Worked solutions 11 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

Question 1	*5 marks*	2
Question 2	*8 marks*	3
Question 3	*6 marks*	5
Question 4	*5 marks*	6
Question 5	*10 marks*	7
Question 6	*10 marks*	9
Question 7	*8 marks*	11
Question 8	*11 marks*	13
Question 9	*14 marks*	15
Question 10	*9 marks*	17
Question 11	*8 marks*	19

1. The sequence u_n is defined for $n \geq 1$ by

$$u_n = 3 + 8\left(-\frac{3}{4}\right)^n$$

(a) State the limit, α, of the sequence, and describe the behaviour of the sequence as n increases. **[2]**

(b) Using logarithms, find the smallest integer N such that $|u_n - \alpha| < 0.01$ for all $n \geq N$. **[3]**

Solution

(a) Since $\left|-\frac{3}{4}\right| < 1$, $\left(-\frac{3}{4}\right)^n \to 0$ as $n \to \infty$, so $u_n \to 3$ and

$$\alpha = 3$$

The term $8\left(-\frac{3}{4}\right)^n$ is negative when n is odd and positive when n is even, and its size decreases as n increases. The sequence therefore oscillates, with terms alternately below and above 3, and converges to 3.

(b) Using part (a):

$$|u_n - \alpha| = \left|8\left(-\frac{3}{4}\right)^n\right| = 8\left(\frac{3}{4}\right)^n$$

The condition $|u_n - \alpha| < 0.01$ becomes

$$8\left(\frac{3}{4}\right)^n < 0.01 \quad \Longrightarrow \quad \left(\frac{3}{4}\right)^n < 0.00125$$

Taking logarithms of both sides:

$$n \ln\left(\frac{3}{4}\right) < \ln(0.00125)$$

Since $\ln\left(\frac{3}{4}\right) < 0$, dividing by it reverses the inequality:

$$n > \frac{\ln(0.00125)}{\ln\left(\frac{3}{4}\right)} \approx 23.2$$

The smallest integer satisfying this is $n = 24$. Since $8\left(\frac{3}{4}\right)^n$ decreases as n increases, the condition holds for every $n \geq 24$ and fails for $n = 23$, so

$$N = 24$$

2. Three sequences, a_n, b_n and c_n, are defined for $n \geq 1$ by the following recurrence relations.

$$a_{n+1}(1 - a_n) = 1 \qquad \text{with } a_1 = 2$$
$$b_{n+1} = \sqrt{b_n + 3} \qquad \text{with } b_1 = 1$$
$$c_{n+1} + nc_n = 1 \qquad \text{with } c_1 = 2$$

The output from a spreadsheet which presents the first 10 terms of a_n, b_n and c_n is shown below.

	A	B	C	D
1	n	a_n	b_n	c_n
2	1	2	1	2
3	2	−1	2	−1
4	3	0.5	2.23607	3
5	4	2	2.28825	−8
6	5	−1	2.29962	33
7	6	0.5	2.30209	−164
8	7	2	2.30263	985
9	8	−1	2.30274	−6894
10	9	0.5	2.30277	55 153
11	10	2	2.30277	−496 376

(a) **Without** attempting to solve any recurrence relations, describe the apparent behaviour, including as $n \to \infty$, of each of the three sequences. **[6]**

(b) Given that b_n converges to a limit L, use the recurrence relation for b_n to find the exact value of L. **[2]**

Solution

(a) **The sequence a_n:** the terms repeat the three values 2, −1, 0.5 in the same order, so a_n appears to be periodic with period 3. The terms stay between −1 and 2 but never settle towards a single value, so as $n \to \infty$ the sequence does not converge to a limit.

The sequence b_n: each term is larger than the one before, but the increases get smaller and smaller, with the terms agreeing to five decimal places by the end of the table. So b_n appears to be increasing and convergent, with $b_n \to 2.303$ (to 3 decimal places) from below as $n \to \infty$.

The sequence c_n: the terms alternate in sign, so c_n oscillates, and from $n = 2$ onwards the size of the terms increases more and more rapidly. As $n \to \infty$, the size of c_n appears to increase without bound, so the sequence is divergent: it oscillates between larger and larger positive and negative values and has no limit.

(b) As $n \to \infty$, both $b_n \to L$ and $b_{n+1} \to L$, so the recurrence relation $b_{n+1} = \sqrt{b_n + 3}$ gives

$$L = \sqrt{L + 3}$$
$$L^2 = L + 3$$
$$L^2 - L - 3 = 0$$

Using the quadratic formula:

$$L = \frac{1 \pm \sqrt{13}}{2}$$

Every term of b_n is positive, since $b_1 = 1$ and each later term is a positive square root, so L cannot be negative. Rejecting $\frac{1-\sqrt{13}}{2} < 0$ gives

$$L = \frac{1+\sqrt{13}}{2}$$

3. **(a)** The convergent sequence a_n is defined by $a_0 = 6$ and

$$a_{n+1} = \frac{3a_n}{4} + \frac{5}{a_n}$$

for $n \geq 0$.
Find the exact value of the limit of the sequence. **[2]**

(b) The convergent sequence b_n is defined by $b_0 = 5$ and

$$b_{n+1} = \frac{b_n}{2} + \frac{k}{b_n}$$

for $n \geq 0$, where k is a positive constant.
Given that the limit of the sequence is $k - 4$, determine the value of k. **[4]**

Solution

(a) Let the limit of the sequence be L. As $n \to \infty$, both $a_n \to L$ and $a_{n+1} \to L$, so

$$L = \frac{3L}{4} + \frac{5}{L}$$

Rearranging:

$$\frac{L}{4} = \frac{5}{L}$$

$$L^2 = 20$$

$$L = \pm 2\sqrt{5}$$

Since $a_0 = 6 > 0$, and $a_{n+1} > 0$ whenever $a_n > 0$, every term of the sequence is positive, so the limit cannot be negative. Hence

$$L = 2\sqrt{5}$$

(b) As $n \to \infty$, both $b_n \to k - 4$ and $b_{n+1} \to k - 4$, so

$$k - 4 = \frac{k-4}{2} + \frac{k}{k-4}$$

Rearranging:

$$\frac{k-4}{2} = \frac{k}{k-4}$$

$$(k-4)^2 = 2k$$

$$k^2 - 8k + 16 = 2k$$

$$k^2 - 10k + 16 = 0$$

$$(k-2)(k-8) = 0$$

so $k = 2$ or $k = 8$.

Since $b_0 = 5 > 0$ and $k > 0$, $b_{n+1} > 0$ whenever $b_n > 0$, so every term of the sequence is positive and the limit cannot be negative. If $k = 2$, the limit would be $k - 4 = -2$, which is not possible. If $k = 8$, the limit is $k - 4 = 4$, which is positive. Hence

$$k = 8$$

4. For all positive integers n, the term u_n of a sequence is the integer for which

$$u_n \equiv 2^n + 3n \pmod 9, \qquad 0 \leq u_n \leq 8$$

(a) Show that $u_{n+6} = u_n$ for all positive integers n. **[3]**

(b) Hence describe the behaviour of the sequence, justifying your answer. **[2]**

Solution

(a) Now $2^6 = 64 = 7 \times 9 + 1$, so $2^6 \equiv 1 \pmod 9$. Multiplying by 2^n gives

$$2^{n+6} = 2^6 \times 2^n \equiv 2^n \pmod 9$$

Also

$$3(n+6) = 3n + 2 \times 9 \equiv 3n \pmod 9$$

Adding these two congruences:

$$2^{n+6} + 3(n+6) \equiv 2^n + 3n \pmod 9$$

so $u_{n+6} \equiv u_n \pmod 9$, which means that $u_{n+6} - u_n$ is a multiple of 9.

Both u_{n+6} and u_n lie between 0 and 8 inclusive, so

$$-8 \leq u_{n+6} - u_n \leq 8$$

The only multiple of 9 in this range is 0, so $u_{n+6} = u_n$ for all positive integers n, as required.

(b) Using part (a), the terms repeat every 6 terms, so the sequence is periodic.

For $n = 1, 2, \ldots, 6$, the values of $2^n + 3n$ are 5, 10, 17, 28, 47, 82. Taking the remainder of each on division by 9 gives

$$u_1 = 5, \quad u_2 = 1, \quad u_3 = 8, \quad u_4 = 1, \quad u_5 = 2, \quad u_6 = 1$$

The value 5 appears only once among these six terms, so the terms do not repeat any sooner than every 6 terms.

The sequence is periodic with period 6, repeating the values 5, 1, 8, 1, 2, 1 in that order, so it does not converge to a limit.

5. The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 5$ and, for $n \geq 1$,

$$u_{n+1} = \frac{12u_n + 15}{2u_n + 5}$$

(a) By considering $5 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 5$ for all positive integers n. **[5]**

(b) Show that $u_{n+1} > u_n$ for $n \geq 1$. **[3]**

(c) Given that u_n tends to a limit L as $n \to \infty$, form a quadratic equation satisfied by L and hence find the value of L, giving a reason for rejecting the other root of your equation. **[2]**

Solution

(a) The statement to be proved is $u_n < 5$.

Base case: When $n = 1$, the question gives $u_1 < 5$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so

$$u_k < 5$$

Inductive step: Writing $5 - u_{k+1}$ as a single fraction:

$$\begin{aligned} 5 - u_{k+1} &= 5 - \frac{12u_k + 15}{2u_k + 5} \\ &= \frac{5(2u_k + 5) - (12u_k + 15)}{2u_k + 5} \\ &= \frac{10 - 2u_k}{2u_k + 5} \\ &= \frac{2(5 - u_k)}{2u_k + 5} \end{aligned}$$

By the assumption $u_k < 5$, the numerator satisfies $2(5 - u_k) > 0$. The terms of the sequence are positive, so $u_k > 0$ and the denominator satisfies $2u_k + 5 > 0$. Therefore

$$5 - u_{k+1} > 0$$

so $u_{k+1} < 5$, which is the statement for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

(b) Writing $u_{n+1} - u_n$ as a single fraction:

$$\begin{aligned} u_{n+1} - u_n &= \frac{12u_n + 15}{2u_n + 5} - u_n \\ &= \frac{12u_n + 15 - u_n(2u_n + 5)}{2u_n + 5} \\ &= \frac{-2u_n^2 + 7u_n + 15}{2u_n + 5} \\ &= \frac{(5 - u_n)(2u_n + 3)}{2u_n + 5} \end{aligned}$$

By part (a), $u_n < 5$, so $5 - u_n > 0$. The terms are positive, so $2u_n + 3 > 0$ and $2u_n + 5 > 0$. Every factor is positive, so

$$u_{n+1} - u_n > 0$$

which gives $u_{n+1} > u_n$ for $n \geq 1$, as required.

(c) As $n \to \infty$, both $u_n \to L$ and $u_{n+1} \to L$, so the recurrence relation gives

$$L = \frac{12L + 15}{2L + 5}$$

$$2L^2 + 5L = 12L + 15$$

$$2L^2 - 7L - 15 = 0$$

$$(2L + 3)(L - 5) = 0$$

so $L = -\frac{3}{2}$ or $L = 5$.

Every term of the sequence is positive, so the limit cannot be negative, and $L = -\frac{3}{2}$ is rejected. Hence

$$L = 5$$

6. The sequence $a_1, a_2, a_3, \ldots$ is such that $2 < a_1 < 4$ and

$$a_{n+1} = \frac{a_n^2 + 8}{6}$$

for every positive integer n.

(a) Prove by mathematical induction that $2 < a_n < 4$ for every positive integer n. **[5]**

(b) Hence prove that $a_{n+1} < a_n$ for every positive integer n. **[3]**

(c) Given that the sequence converges to a limit L, find the value of L, giving a reason for your answer. **[2]**

Solution

(a) The statement to be proved is $2 < a_n < 4$.

Base case: When $n = 1$, the question gives $2 < a_1 < 4$, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$2 < a_k < 4$$

Inductive step: By the assumption, a_k is positive, so squaring keeps the order of the inequalities:

$$4 < a_k^2 < 16$$

Adding 8 throughout and then dividing by 6:

$$12 < a_k^2 + 8 < 24$$

$$2 < \frac{a_k^2 + 8}{6} < 4$$

Since $a_{k+1} = \frac{a_k^2+8}{6}$, this is

$$2 < a_{k+1} < 4$$

so the statement is true for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for every positive integer n.

(b) Consider the difference between consecutive terms:

$$\begin{aligned} a_{n+1} - a_n &= \frac{a_n^2 + 8}{6} - a_n \\ &= \frac{a_n^2 - 6a_n + 8}{6} \\ &= \frac{(a_n - 2)(a_n - 4)}{6} \end{aligned}$$

By part (a), $2 < a_n < 4$, so $a_n - 2 > 0$ and $a_n - 4 < 0$. The product $(a_n - 2)(a_n - 4)$ is therefore negative, so

$$a_{n+1} - a_n < 0 \quad \Longrightarrow \quad a_{n+1} < a_n$$

for every positive integer n, as required.

(c) As $n \to \infty$, both $a_n \to L$ and $a_{n+1} \to L$, so the recurrence relation gives

$$L = \frac{L^2 + 8}{6}$$

$$L^2 - 6L + 8 = 0$$

$$(L - 2)(L - 4) = 0$$

so $L = 2$ or $L = 4$.

By part (b), the sequence is decreasing, so $a_n \leq a_1$ for every positive integer n, and the limit satisfies $L \leq a_1$. Since $a_1 < 4$, the limit is less than 4, so $L = 4$ is rejected and

$$L = 2$$

7. A sequence u_n is defined by

$$u_1 = a \quad \text{and} \quad u_{n+1} = \frac{u_n - 3}{u_n + 1} \quad \text{for } n \geq 1$$

where a is a real constant.

(a) Determine, with justification, the behaviour of the sequence for all possible values of a. **[6]**

(b) Given that $a = 5$, find the sum of the first 100 terms of the sequence. **[2]**

Solution

(a) The first term is $u_1 = a$. Provided $a \neq -1$, the second term is

$$u_2 = \frac{a-3}{a+1}$$

Substituting this into the recurrence relation, and multiplying the numerator and denominator by $a + 1$:

$$\begin{aligned} u_3 = \frac{u_2 - 3}{u_2 + 1} &= \frac{\dfrac{a-3}{a+1} - 3}{\dfrac{a-3}{a+1} + 1} \\ &= \frac{(a-3) - 3(a+1)}{(a-3) + (a+1)} \\ &= \frac{-2a - 6}{2a - 2} \\ &= \frac{a+3}{1-a} \end{aligned}$$

This is defined provided $a \neq 1$. In the same way, multiplying the numerator and denominator by $1 - a$:

$$\begin{aligned} u_4 = \frac{u_3 - 3}{u_3 + 1} &= \frac{(a+3) - 3(1-a)}{(a+3) + (1-a)} \\ &= \frac{4a}{4} \\ &= a \end{aligned}$$

The denominator here is 4, which is never zero, so u_4 is defined whenever u_3 is. Since $u_4 = u_1$ and each term depends only on the term before it, the terms repeat in a cycle of three, so $u_{n+3} = u_n$ for all $n \geq 1$.

Two consecutive terms are equal only if some term u_n satisfies

$$u_n = \frac{u_n - 3}{u_n + 1} \quad \Longrightarrow \quad u_n^2 + u_n = u_n - 3 \quad \Longrightarrow \quad u_n^2 = -3$$

This has no real solutions, so no term is equal to the term before it. Hence $u_2 \neq u_1$, $u_3 \neq u_2$ and $u_1 = u_4 \neq u_3$, so the three terms in the cycle are all different.

The two excluded values are considered separately. When $a = -1$, the denominator of u_2 is zero. When $a = 1$, the second term is

$$u_2 = \frac{1-3}{1+1} = -1$$

so the denominator of u_3 is zero.

The behaviour is therefore as follows. If $a = -1$, then u_2 is undefined and the sequence stops after one term. If $a = 1$, then u_3 is undefined and the sequence stops after the two terms $1, -1$. For every other real value of a, the sequence is periodic with period 3, taking the values a, $\frac{a-3}{a+1}$, $\frac{a+3}{1-a}$ in turn, and so it does not converge.

(b) Using part (a) with $a = 5$:

$$u_1 = 5, \qquad u_2 = \frac{5-3}{5+1} = \frac{1}{3}, \qquad u_3 = \frac{5+3}{1-5} = -2$$

The sequence has period 3, and the sum of the terms in one cycle is

$$5 + \frac{1}{3} - 2 = \frac{10}{3}$$

Since $100 = 3 \times 33 + 1$, the first 100 terms consist of 33 complete cycles followed by $u_{100} = u_1 = 5$. Hence

$$\sum_{n=1}^{100} u_n = 33 \times \frac{10}{3} + 5 = 115$$

8. The sequence of real numbers u_n is given by $u_1 = 3$ and

$$u_{n+1} = k + \frac{6}{u_n}$$

where k is a real constant.
For each of the following cases, either determine a value of k or prove that one does not exist.

(a) u_n is constant. **[2]**

(b) u_n is periodic, with period 2. **[4]**

(c) u_n is periodic, with period 3. **[5]**

Solution

(a) The sequence is constant exactly when every term equals $u_1 = 3$, that is, when $u_n = 3$ gives $u_{n+1} = 3$. Substituting $u_n = 3$ and $u_{n+1} = 3$ into the recurrence relation:

$$3 = k + \frac{6}{3} \quad \Longrightarrow \quad k = 1$$

(b) A sequence with period 2 has $u_{n+2} = u_n$ for every n, with $u_2 \neq u_1$. In particular, $u_3 = u_1 = 3$.

The second and third terms are

$$u_2 = k + \frac{6}{3} = k + 2$$

$$u_3 = k + \frac{6}{k+2}$$

where $k \neq -2$, since otherwise $u_2 = 0$ and u_3 is undefined. Setting $u_3 = 3$ and multiplying through by $k + 2$:

$$k + \frac{6}{k+2} = 3$$

$$k(k+2) + 6 = 3(k+2)$$

$$k^2 - k = 0$$

$$k(k-1) = 0$$

so $k = 0$ or $k = 1$. By part (a), $k = 1$ gives the constant sequence, for which $u_2 = u_1$, so it does not have period 2.

With $k = 0$, the recurrence relation is $u_{n+1} = \frac{6}{u_n}$, so

$$u_{n+2} = \frac{6}{u_{n+1}} = 6 \div \frac{6}{u_n} = u_n$$

for every n. The sequence is $3, 2, 3, 2, \ldots$, which is periodic with period 2, so $k = 0$.

(c) A sequence with period 3 has $u_{n+3} = u_n$ for every n. In particular, $u_4 = u_1 = 3$.

From part (b), for $k \neq -2$:

$$u_3 = k + \frac{6}{k+2} = \frac{k^2 + 2k + 6}{k+2}$$

so

$$u_4 = k + \frac{6}{u_3} = k + \frac{6(k+2)}{k^2+2k+6}$$

Here $k^2+2k+6 = (k+1)^2+5 > 0$, so this denominator is never zero. Setting $u_4 = 3$ and multiplying through by k^2+2k+6:

$$k\big(k^2+2k+6\big) + 6(k+2) = 3\big(k^2+2k+6\big)$$

$$k^3+2k^2+12k+12 = 3k^2+6k+18$$

$$k^3-k^2+6k-6 = 0$$

$$k^2(k-1)+6(k-1) = 0$$

$$(k-1)\big(k^2+6\big) = 0$$

Since k is real, $k^2+6 \geq 6$, so $k^2+6 \neq 0$ and the only solution is $k = 1$.

By part (a), $k = 1$ gives the constant sequence $3, 3, 3, \ldots$, which does not have period 3. Every other real value of k gives $u_4 \neq u_1$, or a sequence that is undefined from u_3 onwards when $k = -2$. Hence no value of k exists for which the sequence is periodic with period 3.

9. The sequence u_n satisfies the recurrence relation

$$u_n = u_{n-1} + 6u_{n-2}, \quad n \geq 2 \qquad (*)$$

with $u_0 = 3$ and $u_1 = 4$.

(a) Find u_n in terms of n. **[5]**

A new sequence v_n is defined, for $n \geq 1$, by

$$v_n = \frac{u_n}{u_{n-1}}$$

(b) (i) Use $(*)$ to show that, for $n \geq 2$,

$$v_n = 1 + \frac{6}{v_{n-1}}$$ **[2]**

(ii) Deduce that, if v_n tends to a limit, then that limit is either 3 or -2. **[2]**

(c) Use your answer to part (a) to show that $\lim_{n\to\infty} v_n = 3$. **[3]**

(d) The sequence w_n satisfies the same recurrence relation, $w_n = w_{n-1} + 6w_{n-2}$ for $n \geq 2$, with $w_0 = 5$ and $w_1 = k$, where k is a constant.
Given that $\frac{w_n}{w_{n-1}}$ tends to a limit that is not 3, find the value of k. **[2]**

Solution

(a) The auxiliary equation is

$$\lambda^2 - \lambda - 6 = 0 \implies (\lambda - 3)(\lambda + 2) = 0 \implies \lambda = 3 \text{ or } \lambda = -2$$

so the general solution is

$$u_n = A \times 3^n + B(-2)^n$$

Using $u_0 = 3$ and $u_1 = 4$:

$$A + B = 3$$
$$3A - 2B = 4$$

Adding twice the first equation to the second gives $5A = 10$, so $A = 2$ and $B = 1$. Hence

$$u_n = 2 \times 3^n + (-2)^n$$

(b) (i) Since u_0 and u_1 are positive, $(*)$ shows that every term of the sequence is positive, so no term is zero. Dividing $(*)$ by u_{n-1}:

$$\frac{u_n}{u_{n-1}} = 1 + \frac{6u_{n-2}}{u_{n-1}}$$

For $n \geq 2$, $v_{n-1} = \frac{u_{n-1}}{u_{n-2}}$, so $\frac{u_{n-2}}{u_{n-1}} = \frac{1}{v_{n-1}}$. The left-hand side is v_n, so

$$v_n = 1 + \frac{6}{v_{n-1}}$$

as required.

(ii) Suppose that $v_n \to L$ as $n \to \infty$. Then $v_{n-1} \to L$ as well. Multiplying the result of part (b)(i) by v_{n-1} gives

$$v_n v_{n-1} = v_{n-1} + 6$$

Letting $n \to \infty$ on both sides:

$$L^2 = L + 6$$

$$L^2 - L - 6 = 0$$

$$(L-3)(L+2) = 0$$

so the limit is either 3 or -2, as required.

(c) Using part (a):

$$v_n = \frac{u_n}{u_{n-1}} = \frac{2 \times 3^n + (-2)^n}{2 \times 3^{n-1} + (-2)^{n-1}}$$

Dividing the numerator and the denominator by 3^n, and using $\frac{(-2)^{n-1}}{3^n} = -\frac{1}{2}\left(-\frac{2}{3}\right)^n$:

$$v_n = \frac{2 + \left(-\frac{2}{3}\right)^n}{\frac{2}{3} - \frac{1}{2}\left(-\frac{2}{3}\right)^n}$$

Since $\left|-\frac{2}{3}\right| < 1$, $\left(-\frac{2}{3}\right)^n \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} v_n = \frac{2+0}{\frac{2}{3} - 0} = 3$$

as required.

(d) The recurrence relation is the same, so the general solution has the same form:

$$w_n = C \times 3^n + D(-2)^n$$

Using $w_0 = 5$ and $w_1 = k$:

$$C + D = 5$$

$$3C - 2D = k$$

If $C \neq 0$, dividing the numerator and the denominator by 3^n as in part (c) gives

$$\frac{w_n}{w_{n-1}} = \frac{C + D\left(-\frac{2}{3}\right)^n}{\frac{1}{3}C - \frac{1}{2}D\left(-\frac{2}{3}\right)^n} \to \frac{C}{\frac{1}{3}C} = 3$$

which is not the case here, so $C = 0$. Then $w_n = D(-2)^n$, so $\frac{w_n}{w_{n-1}} = -2$ for every n, which tends to the limit -2. With $C = 0$ the two equations give $D = 5$ and

$$k = -2D = -10$$

10. The sequence of positive real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \frac{a_n^2}{2 + a_n}$$

(a) Prove by mathematical induction that $\ln a_n \leq \left(1 - 2^{n-1}\right) \ln 2$ for all positive integers n. You may use the fact that $\ln(2 + x) > \ln 2$ for $x > 0$. **[6]**

(b) Show that $\ln a_n - \ln a_{n+1} > 2^{n-1} \ln 2$ for all positive integers n, and hence find the limit of $\frac{a_{n+1}}{a_n}$ as $n \to \infty$, giving a reason for your answer. **[3]**

Solution

(a) The statement to be proved is $\ln a_n \leq \left(1 - 2^{n-1}\right) \ln 2$.

Base case: When $n = 1$, $\ln a_1 = \ln 1 = 0$ and $\left(1 - 2^0\right) \ln 2 = 0$. Since $0 \leq 0$, the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, so that

$$\ln a_k \leq \left(1 - 2^{k-1}\right) \ln 2$$

Inductive step: Taking logarithms of the recurrence relation with $n = k$, which is possible because every term is positive:

$$\begin{aligned} \ln a_{k+1} &= \ln\left(\frac{a_k^2}{2 + a_k}\right) \\ &= 2 \ln a_k - \ln(2 + a_k) \end{aligned}$$

By the inductive hypothesis:

$$2 \ln a_k \leq 2\left(1 - 2^{k-1}\right) \ln 2 = \left(2 - 2^k\right) \ln 2$$

Since $a_k > 0$, the given fact with $x = a_k$ gives $\ln(2 + a_k) > \ln 2$, so

$$-\ln(2 + a_k) < -\ln 2$$

Adding these two inequalities:

$$\begin{aligned} \ln a_{k+1} &< \left(2 - 2^k\right) \ln 2 - \ln 2 \\ &= \left(1 - 2^k\right) \ln 2 \\ &= \left(1 - 2^{(k+1)-1}\right) \ln 2 \end{aligned}$$

So $\ln a_{k+1} \leq \left(1 - 2^{(k+1)-1}\right) \ln 2$, which is the statement for $n = k + 1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

(b) Taking logarithms of the recurrence relation gives $\ln a_{n+1} = 2 \ln a_n - \ln(2 + a_n)$, so

$$\begin{aligned} \ln a_n - \ln a_{n+1} &= \ln a_n - 2 \ln a_n + \ln(2 + a_n) \\ &= \ln(2 + a_n) - \ln a_n \end{aligned}$$

Since $a_n > 0$, $\ln(2 + a_n) > \ln 2$. From part (a), $-\ln a_n \geq \left(2^{n-1} - 1\right) \ln 2$. Adding these:

$$\begin{aligned} \ln a_n - \ln a_{n+1} &> \ln 2 + \left(2^{n-1} - 1\right) \ln 2 \\ &= 2^{n-1} \ln 2 \end{aligned}$$

as required.

Multiplying this inequality by -1 reverses it:

$$\ln\left(\frac{a_{n+1}}{a_n}\right) = \ln a_{n+1} - \ln a_n < -2^{n-1} \ln 2 = \ln\left(2^{-2^{n-1}}\right)$$

The function ln is increasing, and both terms are positive, so

$$0 < \frac{a_{n+1}}{a_n} < \frac{1}{2^{2^{n-1}}}$$

As $n \to \infty$, $2^{n-1} \to \infty$, so $2^{2^{n-1}} \to \infty$ and $\frac{1}{2^{2^{n-1}}} \to 0$. The ratio $\frac{a_{n+1}}{a_n}$ lies between 0 and a quantity that tends to 0, so

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0$$

11. The sequence A_n is given for all integers $n \geq 0$ by

$$A_n = \frac{(I_{n+1})^2}{I_n\, I_{n+2}}$$

where

$$I_n = \int_0^4 x^n \sqrt{4-x}\,\mathrm{d}x$$

- Show that A_n increases monotonically, that is, that $A_{n+1} > A_n$ for all $n \geq 0$.
- Show that A_n converges to a limit, A, whose exact value should be stated.

[8]

Solution

First find a relation between I_n and I_{n-1} for $n \geq 1$. The formula for integration by parts is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Take $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = (4-x)^{\frac{1}{2}}$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$ and $v = -\frac{2}{3}(4-x)^{\frac{3}{2}}$. Then

$$I_n = \left[-\frac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_0^4 + \frac{2n}{3}\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,\mathrm{d}x$$

The bracket is zero at both limits, since $n \geq 1$. Writing $(4-x)^{\frac{3}{2}} = (4-x)\sqrt{4-x}$:

$$\begin{aligned} I_n &= \frac{2n}{3}\int_0^4 \left(4x^{n-1} - x^n\right)\sqrt{4-x}\,\mathrm{d}x \\ &= \frac{2n}{3}(4I_{n-1} - I_n) \\ 3I_n &= 8nI_{n-1} - 2nI_n \\ (2n+3)I_n &= 8nI_{n-1} \end{aligned}$$

For $0 < x < 4$ the integrand $x^n\sqrt{4-x}$ is positive, so $I_n > 0$ for every $n \geq 0$ and it is valid to divide by I_{n-1}. Hence, for $n \geq 1$, the ratio of consecutive integrals is

$$\frac{I_n}{I_{n-1}} = \frac{8n}{2n+3}$$

For $n \geq 0$, replacing n by $n+1$ and by $n+2$ gives

$$\frac{I_{n+1}}{I_n} = \frac{8(n+1)}{2n+5}, \qquad \frac{I_{n+2}}{I_{n+1}} = \frac{8(n+2)}{2n+7}$$

Therefore

$$\begin{aligned} A_n &= \frac{I_{n+1}}{I_n} \times \frac{I_{n+1}}{I_{n+2}} \\ &= \frac{8(n+1)}{2n+5} \times \frac{2n+7}{8(n+2)} \\ &= \frac{(n+1)(2n+7)}{(n+2)(2n+5)} \\ &= \frac{2n^2+9n+7}{2n^2+9n+10} \\ &= 1 - \frac{3}{2n^2+9n+10} \end{aligned}$$

To show that the sequence increases, consider the difference between consecutive terms. Replacing n by $n+1$ in the denominator gives $2(n+1)^2+9(n+1)+10 = 2n^2+13n+21$, so

$$\begin{aligned} A_{n+1} - A_n &= \frac{3}{2n^2+9n+10} - \frac{3}{2n^2+13n+21} \\ &= \frac{3\big(2n^2+13n+21\big) - 3\big(2n^2+9n+10\big)}{(2n^2+9n+10)(2n^2+13n+21)} \\ &= \frac{3(4n+11)}{(2n^2+9n+10)(2n^2+13n+21)} \end{aligned}$$

For $n \geq 0$, the numerator and both factors of the denominator are positive, so $A_{n+1} - A_n > 0$. Hence $A_{n+1} > A_n$ for all integers $n \geq 0$, and A_n increases monotonically.

For the limit, use

$$A_n = 1 - \frac{3}{2n^2+9n+10}$$

As $n \to \infty$, $2n^2+9n+10 \to \infty$, so $\frac{3}{2n^2+9n+10} \to 0$ and $A_n \to 1$. Hence A_n converges, with limit $A = 1$.