OPTIONAL PURE / RECURRENCE RELATIONS

2nd Order Recurrence Relations

Worked solutions 12 questions · 124 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

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1. The sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_{n+2} = u_{n+1} + 12u_n, \qquad n \geq 1$$

Given that $u_1 = 11$ and $u_2 = 23$, find an expression for u_n in terms of n. **[5]**

Solution

Rearranging, the recurrence relation is $u_{n+2} - u_{n+1} - 12u_n = 0$. Trying $u_n = \lambda^n$ gives

$$\lambda^{n+2} - \lambda^{n+1} - 12\lambda^n = 0$$

so the auxiliary equation is

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\lambda = 4 \text{ or } \lambda = -3$$

The roots are real and distinct, so the general solution is

$$u_n = A \times 4^n + B(-3)^n$$

Now use the initial conditions. Substituting $n = 1$, $u_1 = 11$ and $n = 2$, $u_2 = 23$:

$$4A - 3B = 11$$

$$16A + 9B = 23$$

Adding three times the first equation to the second gives

$$28A = 56 \quad \Longrightarrow \quad A = 2$$

Then $4 \times 2 - 3B = 11$, so $B = -1$. Hence

$$u_n = 2 \times 4^n - (-3)^n$$

2. A sequence satisfies the recurrence relation

$$4H_{n+2} = 4H_{n+1} - H_n$$

for $n \geq 0$, with $H_0 = 5$ and $H_1 = 4$.

Solve the recurrence relation to find an expression for H_n in terms of n. **[5]**

Solution

Rearranging, the recurrence relation is

$$4H_{n+2} - 4H_{n+1} + H_n = 0$$

The auxiliary equation is

$$4\lambda^2 - 4\lambda + 1 = 0$$

$$(2\lambda - 1)^2 = 0$$

$$\lambda = \frac{1}{2} \quad \text{(repeated)}$$

The auxiliary equation has a repeated root, so the general solution is

$$H_n = (A + Bn)\left(\frac{1}{2}\right)^n$$

Now use the initial conditions. Substituting $n = 0$ and $H_0 = 5$:

$$A = 5$$

Substituting $n = 1$ and $H_1 = 4$:

$$\frac{1}{2}(A + B) = 4$$

$$5 + B = 8$$

$$B = 3$$

Therefore

$$H_n = (5 + 3n)\left(\frac{1}{2}\right)^n$$

3. A sequence u_n, where $n \geq 0$, satisfies the recurrence relation

$$3u_{n+2} = 5u_{n+1} + 2u_n - 15\left(\frac{1}{2}\right)^n$$

(a) Find the general solution of this recurrence relation. **[5]**

A particular solution of this recurrence relation has $u_0 = 7$ and $u_1 = k$, where k is a constant, and it is given that this sequence converges.

(b) Determine the value of k, and state the limit of the sequence. **[4]**

Solution

(a) First solve the homogeneous relation $3u_{n+2} - 5u_{n+1} - 2u_n = 0$. The auxiliary equation is

$$3\lambda^2 - 5\lambda - 2 = 0$$

$$(3\lambda + 1)(\lambda - 2) = 0$$

$$\lambda = 2 \text{ or } \lambda = -\frac{1}{3}$$

so the complementary function is

$$A \times 2^n + B\left(-\frac{1}{3}\right)^n$$

where A and B are arbitrary constants.

Since $\frac{1}{2}$ is not a root of the auxiliary equation, for a particular solution try

$$u_n = a\left(\frac{1}{2}\right)^n$$

Substituting into the recurrence relation:

$$3a\left(\frac{1}{2}\right)^{n+2} = 5a\left(\frac{1}{2}\right)^{n+1} + 2a\left(\frac{1}{2}\right)^n - 15\left(\frac{1}{2}\right)^n$$

Dividing through by $\left(\frac{1}{2}\right)^n$ gives

$$\frac{3a}{4} = \frac{5a}{2} + 2a - 15$$

$$3a = 10a + 8a - 60$$

$$15a = 60$$

$$a = 4$$

The general solution is the sum of the complementary function and this particular solution:

$$u_n = A \times 2^n + B\left(-\frac{1}{3}\right)^n + 4\left(\frac{1}{2}\right)^n$$

(b) As $n \to \infty$, $\left(-\frac{1}{3}\right)^n \to 0$ and $\left(\frac{1}{2}\right)^n \to 0$, but 2^n increases without limit. The sequence converges, so the term in 2^n must be absent:

$$A = 0$$

Using $u_0 = 7$ in the general solution from part (a):

$$B + 4 = 7 \quad \Longrightarrow \quad B = 3$$

so

$$u_n = 3\left(-\frac{1}{3}\right)^n + 4\left(\frac{1}{2}\right)^n$$

Substituting $n = 1$:

$$\begin{aligned} k &= u_1 \\ &= 3 \times \left(-\frac{1}{3}\right) + 4 \times \frac{1}{2} \\ &= -1 + 2 \\ &= 1 \end{aligned}$$

Both terms of u_n tend to 0 as $n \to \infty$, so $k = 1$ and the limit of the sequence is 0.

4. **(a)** Determine the general solution of the recurrence relation

$$u_{n+2} - 3u_{n+1} + 2u_n = 0 \qquad \textbf{[2]}$$

(b) Consider the recurrence relation

$$u_{n+2} - 3u_{n+1} + 2u_n = 2 - 6n \qquad (*)$$

(i) Show that $(*)$ has no particular solution of the form $u_n = pn + q$, where p and q are constants. **[2]**

(ii) Using your answer to part (a), determine the general solution of $(*)$. **[3]**

In the rest of this question the sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation $(*)$.
You are given that $u_0 = 3$ and $u_1 = 5$.

(c) Determine the particular solution for u_n. **[3]**

You are given that, as n increases, once the values of u_n start to decrease, the sequence is a decreasing sequence from that point onwards.

(d) Use your answer to part (c) to determine, by direct calculation, the greatest value taken by terms of the sequence. You should show any values that you rely on in your argument. **[2]**

Solution

(a) The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 1)(\lambda - 2) = 0$$
$$\lambda = 1 \text{ or } \lambda = 2$$

The general solution is $u_n = A \times 1^n + B \times 2^n$, where A and B are arbitrary constants, so

$$u_n = A + B \times 2^n$$

(b) (i) Substituting $u_n = pn + q$, so that $u_{n+1} = p(n+1) + q$ and $u_{n+2} = p(n+2) + q$, into the left-hand side of $(*)$:

$$\begin{aligned} u_{n+2} - 3u_{n+1} + 2u_n &= p(n+2) + q - 3\big(p(n+1) + q\big) + 2(pn + q) \\ &= (p - 3p + 2p)n + (2p - 3p) + (q - 3q + 2q) \\ &= -p \end{aligned}$$

The left-hand side is the constant $-p$ for every n, with no term in n, whereas the right-hand side $2 - 6n$ has coefficient of n equal to -6. The two sides cannot be equal for all n, whatever the values of p and q, so $(*)$ has no particular solution of the form $u_n = pn + q$.

(ii) By part (a), the complementary function is $A + B \times 2^n$. It already contains a constant term, which is why the linear trial in part (b)(i) fails, so multiply the linear trial by n and try a particular solution of the form

$$u_n = an^2 + bn$$

Substituting into the left-hand side of $(*)$:

$$\begin{aligned} u_{n+2} - 3u_{n+1} + 2u_n &= a(n+2)^2 + b(n+2) - 3a(n+1)^2 - 3b(n+1) + 2an^2 + 2bn \\ &= a\big(n^2 + 4n + 4 - 3n^2 - 6n - 3 + 2n^2\big) + b(n + 2 - 3n - 3 + 2n) \\ &= a(1 - 2n) - b \\ &= -2an + (a - b) \end{aligned}$$

This must equal $2 - 6n$ for all n. Equating coefficients of n:

$$-2a = -6 \quad \Longrightarrow \quad a = 3$$

Equating constant terms:

$$a - b = 2 \quad \Longrightarrow \quad b = 1$$

A particular solution is $u_n = 3n^2 + n$. Adding the complementary function, the general solution of $(*)$ is

$$u_n = A + B \times 2^n + 3n^2 + n$$

(c) Using $u_0 = 3$ in the general solution from part (b)(ii):

$$A + B = 3$$

Using $u_1 = 5$:

$$A + 2B + 3 + 1 = 5 \quad \Longrightarrow \quad A + 2B = 1$$

Subtracting the first equation from the second gives $B = -2$, and then $A = 5$. Since $-2 \times 2^n = -2^{n+1}$, the particular solution is

$$u_n = 5 - 2^{n+1} + 3n^2 + n$$

(d) Using the formula from part (c), together with the given values $u_0 = 3$ and $u_1 = 5$:

$$u_2 = 5 - 2^3 + 3 \times 2^2 + 2 = 11$$

$$u_3 = 5 - 2^4 + 3 \times 3^2 + 3 = 19$$

$$u_4 = 5 - 2^5 + 3 \times 4^2 + 4 = 25$$

$$u_5 = 5 - 2^6 + 3 \times 5^2 + 5 = 21$$

The terms increase from $u_0 = 3$ up to $u_4 = 25$, and then $u_5 < u_4$, so the values start to decrease after u_4. From the given information, the sequence is decreasing from that point onwards, so no later term exceeds u_4. The greatest value taken by terms of the sequence is 25, which is the term u_4.

5. The sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n + 15 \times 3^n, \qquad n \geq 0$$

(a) Determine the general solution of this recurrence relation. **[5]**

(b) Hence find u_n in terms of n, given that $u_0 = 1$ and $u_2 = 2u_1$. **[3]**

Solution

(a) Write the recurrence relation as

$$u_{n+2} - u_{n+1} - 6u_n = 15 \times 3^n$$

The auxiliary equation is

$$\lambda^2 - \lambda - 6 = 0$$
$$(\lambda - 3)(\lambda + 2) = 0$$
$$\lambda = 3 \text{ or } \lambda = -2$$

so the complementary function is

$$A \times 3^n + B(-2)^n$$

where A and B are arbitrary constants.

Since 3^n is already part of the complementary function, for a particular solution try $u_n = an \times 3^n$. Substituting into the recurrence relation:

$$a(n+2) \times 3^{n+2} - a(n+1) \times 3^{n+1} - 6an \times 3^n = 15 \times 3^n$$
$$a \times 3^n[9(n+2) - 3(n+1) - 6n] = 15 \times 3^n$$
$$a \times 3^n[9n + 18 - 3n - 3 - 6n] = 15 \times 3^n$$
$$15a \times 3^n = 15 \times 3^n$$
$$a = 1$$

A particular solution is $u_n = n \times 3^n$, so the general solution is

$$u_n = A \times 3^n + B(-2)^n + n \times 3^n$$

(b) Using the general solution from part (a) with $n = 0$, $n = 1$ and $n = 2$:

$$u_0 = A + B$$
$$u_1 = 3A - 2B + 3$$
$$u_2 = 9A + 4B + 18$$

Using $u_0 = 1$:

$$A + B = 1$$

Using $u_2 = 2u_1$:

$$\begin{aligned} 9A + 4B + 18 &= 2(3A - 2B + 3) \\ 9A + 4B + 18 &= 6A - 4B + 6 \\ 3A + 8B &= -12 \end{aligned}$$

Substituting $A = 1 - B$:

$$\begin{aligned} 3(1 - B) + 8B &= -12 \\ 5B &= -15 \\ B &= -3 \end{aligned}$$

so $A = 4$. Hence

$$\begin{aligned} u_n &= 4 \times 3^n - 3(-2)^n + n \times 3^n \\ &= (n + 4) \times 3^n - 3(-2)^n \end{aligned}$$

6. The general solution of the second-order recurrence relation

$$u_{n+2} + k_1 u_{n+1} + k_2 u_n = 0, \qquad n \geq 0$$

is given by

$$u_n = \left(2\sqrt{3}\right)^n \left(A\cos\left(\frac{5n\pi}{6}\right) + B\sin\left(\frac{5n\pi}{6}\right)\right)$$

where k_1 and k_2 are real constants and A and B are arbitrary constants.

(a) Find the value of k_1 and the value of k_2. **[3]**

(b) Given that $u_1 = 3$ and $u_2 = 6$, find the exact values of A and B. **[3]**

Solution

(a) The auxiliary equation of the recurrence relation is

$$\lambda^2 + k_1\lambda + k_2 = 0$$

When the auxiliary equation has complex roots $\lambda = r(\cos\theta \pm i\sin\theta)$, the general solution is

$$u_n = r^n(A\cos n\theta + B\sin n\theta)$$

Comparing with the given general solution, $r = 2\sqrt{3}$ and $\theta = \frac{5\pi}{6}$, so the roots are

$$\begin{aligned}\lambda &= 2\sqrt{3}\left(\cos\frac{5\pi}{6} \pm i\sin\frac{5\pi}{6}\right)\\ &= 2\sqrt{3}\left(-\frac{\sqrt{3}}{2} \pm \frac{1}{2}i\right)\\ &= -3 \pm \sqrt{3}\,i\end{aligned}$$

The auxiliary equation with these roots is

$$\begin{aligned}\left(\lambda + 3 - \sqrt{3}\,i\right)\left(\lambda + 3 + \sqrt{3}\,i\right) &= 0\\ (\lambda+3)^2 + 3 &= 0\\ \lambda^2 + 6\lambda + 12 &= 0\end{aligned}$$

Equating coefficients with $\lambda^2 + k_1\lambda + k_2 = 0$ gives

$$k_1 = 6, \qquad k_2 = 12$$

(b) Substituting $n = 1$ and $u_1 = 3$ into the general solution:

$$\begin{aligned}2\sqrt{3}\left(A\cos\frac{5\pi}{6} + B\sin\frac{5\pi}{6}\right) &= 3\\ 2\sqrt{3}\left(-\frac{\sqrt{3}}{2}A + \frac{1}{2}B\right) &= 3\\ -3A + \sqrt{3}\,B &= 3\end{aligned}$$

Substituting $n = 2$ and $u_2 = 6$, with $\left(2\sqrt{3}\right)^2 = 12$:

$$12\left(A\cos\frac{5\pi}{3} + B\sin\frac{5\pi}{3}\right) = 6$$

$$12\left(\frac{1}{2}A - \frac{\sqrt{3}}{2}B\right) = 6$$

$$A - \sqrt{3}\,B = 1$$

Adding the two equations gives

$$-2A = 4 \quad \Longrightarrow \quad A = -2$$

Substituting $A = -2$ into $A - \sqrt{3}\,B = 1$:

$$-2 - \sqrt{3}\,B = 1 \quad \Longrightarrow \quad B = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$

Therefore

$$A = -2, \qquad B = -\sqrt{3}$$

7. A sequence is defined by the recurrence relation

$$u_{n+2} = 2u_{n+1} - 4u_n$$

for $n \geq 0$, with $u_0 = 1$ and $u_1 = 4$.

(a) Find an exact real expression for u_n in terms of n. **[7]**

A second sequence is defined by $v_n = a^n u_n$ for $n \geq 0$, where a is a positive constant.

(b) In each of the following cases, describe the behaviour of v_n as $n \to \infty$, stating the period of the sequence in any case where it is periodic.

- $a = \frac{1}{3}$
- $a = \frac{1}{2}$
- $a = \frac{2}{3}$

[6]

Solution

(a) Write the recurrence relation as $u_{n+2} - 2u_{n+1} + 4u_n = 0$. The auxiliary equation is

$$\lambda^2 - 2\lambda + 4 = 0$$

Using the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{4 - 16}}{2} = 1 \pm \sqrt{3}\, i$$

The root $1 + \sqrt{3}\, i$ has modulus $\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$. It lies in the first quadrant, so its argument is $\arctan \sqrt{3} = \frac{\pi}{3}$. Hence

$$\lambda = 2\left(\cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3}\right)$$

When the auxiliary equation has complex roots $\lambda = r(\cos\theta \pm i \sin\theta)$, the general solution is

$$u_n = r^n (A \cos n\theta + B \sin n\theta)$$

Here $r = 2$ and $\theta = \frac{\pi}{3}$, so

$$u_n = 2^n \left(A \cos\left(\frac{n\pi}{3}\right) + B \sin\left(\frac{n\pi}{3}\right)\right)$$

Now use the initial conditions. Using $u_0 = 1$:

$$A = 1$$

Using $u_1 = 4$:

$$2\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) = 4$$

$$1 + \sqrt{3}\, B = 4$$

$$B = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Therefore

$$u_n = 2^n\left(\cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\sin\left(\frac{n\pi}{3}\right)\right)$$

(b) Using part (a):

$$v_n = a^n u_n = (2a)^n w_n, \qquad \text{where } w_n = \cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\sin\left(\frac{n\pi}{3}\right)$$

Increasing n by 6 increases the angle $\frac{n\pi}{3}$ by 2π, so $w_{n+6} = w_n$. The first six values are

$$w_0 = 1, \quad w_1 = 2, \quad w_2 = 1, \quad w_3 = -1, \quad w_4 = -2, \quad w_5 = -1$$

These six values then repeat, and no shorter block of values repeats, so w_n is periodic with period 6. Also $1 \le |w_n| \le 2$ for all n.

When $a = \frac{1}{3}$:

$$v_n = \left(\frac{2}{3}\right)^n w_n \quad \Longrightarrow \quad |v_n| \le 2\left(\frac{2}{3}\right)^n$$

Since $\left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$, the sequence converges: $v_n \to 0$. The terms change sign every three terms, so v_n oscillates about 0 with decreasing amplitude. It is not periodic.

When $a = \frac{1}{2}$:

$$v_n = 1^n \times w_n = w_n$$

The sequence is periodic with period 6, repeating the values $1,\ 2,\ 1,\ -1,\ -2,\ -1$. It stays between -2 and 2 but does not converge to a limit.

When $a = \frac{2}{3}$:

$$v_n = \left(\frac{4}{3}\right)^n w_n \quad \Longrightarrow \quad |v_n| \ge \left(\frac{4}{3}\right)^n$$

Since $\left(\frac{4}{3}\right)^n \to \infty$ as $n \to \infty$, the size of the terms increases without limit, while the terms still change sign every three terms. The sequence diverges, oscillating with increasing amplitude, so it tends neither to a finite limit nor to $\pm\infty$. It is not periodic.

8. The sequence $T_0, T_1, T_2, \ldots$ satisfies the second-order recurrence relation

$$T_{n+2} + 2T_{n+1} + 4T_n = 7n + 25$$

for $n \geq 0$, where $T_0 = 5$ and $T_1 = 8$.

(a) Solve the recurrence relation to find an expression for T_n in terms of n. **[9]**

(b) Determine the value of T_{16}. **[2]**

Solution

(a) The auxiliary equation is

$$\lambda^2 + 2\lambda + 4 = 0$$

so

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 4}}{2} = -1 \pm \sqrt{3}\, i$$

The root $\lambda = -1 + \sqrt{3}\, i$ has modulus

$$\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$$

It lies in the second quadrant, so its argument is

$$\pi - \arctan\sqrt{3} = \frac{2\pi}{3}$$

Hence the roots are $\lambda = 2\left(\cos\frac{2\pi}{3} \pm i \sin\frac{2\pi}{3}\right)$. When the auxiliary equation has complex roots $\lambda = r(\cos\theta \pm i\sin\theta)$, the complementary function is

$$r^n(A\cos n\theta + B\sin n\theta)$$

Here $r = 2$ and $\theta = \frac{2\pi}{3}$, so it is

$$2^n\left(A\cos\left(\frac{2n\pi}{3}\right) + B\sin\left(\frac{2n\pi}{3}\right)\right)$$

The right-hand side is linear in n, so for a particular solution try $T_n = an + b$. Then $T_{n+1} = an + a + b$ and $T_{n+2} = an + 2a + b$. Substituting into the recurrence relation:

$$(an + 2a + b) + 2(an + a + b) + 4(an + b) = 7n + 25$$

$$7an + 4a + 7b = 7n + 25$$

Equating coefficients of n gives $7a = 7$, so $a = 1$. Equating constant terms gives $4 + 7b = 25$, so $b = 3$.

The general solution is

$$T_n = 2^n\left(A\cos\left(\frac{2n\pi}{3}\right) + B\sin\left(\frac{2n\pi}{3}\right)\right) + n + 3$$

Now use the initial conditions. Using $T_0 = 5$:

$$A + 3 = 5 \quad \Longrightarrow \quad A = 2$$

Using $T_1 = 8$, with $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$:

$$2\left(-\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) + 4 = 8$$

$$-A + \sqrt{3}\,B = 4$$

$$\sqrt{3}\,B = 6$$

$$B = 2\sqrt{3}$$

Therefore

$$T_n = 2^n\left(2\cos\left(\frac{2n\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{2n\pi}{3}\right)\right) + n + 3$$

(b) Using part (a) with $n = 16$, the angle is

$$\frac{2 \times 16\pi}{3} = \frac{32\pi}{3} = 10\pi + \frac{2\pi}{3}$$

so $\cos\frac{32\pi}{3} = \cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{32\pi}{3} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. Hence

$$T_{16} = 2^{16}\left(2 \times \left(-\frac{1}{2}\right) + 2\sqrt{3} \times \frac{\sqrt{3}}{2}\right) + 16 + 3$$

$$= 2^{16} \times 2 + 19$$

$$= 131\,072 + 19$$

$$= 131\,091$$

9. **(a)** Determine the general solution of the recurrence relation

$$16u_{n+2} - 24u_{n+1} + 9u_n = c, \qquad n \geq 0$$

where c is a constant. **[4]**

The manager of a company uses the recurrence relation in part (a) to model the number of containers held by the company, using the sequence $u_0, u_1, u_2, \ldots$ which satisfies this recurrence relation.

The quantity u_0 represents the number of containers held by the company at the **start** of January 2020. The quantity u_n, for $n \geq 1$, represents the number of containers held by the company at the **end** of month n, where the first month is January 2020. If necessary, the quantity u_n is rounded to the nearest integer.

The company holds 30 containers at the start of January 2020 and 60 containers at the end of January 2020.

(b) In an initial model the manager uses $c = -6$.

(i) Determine the particular solution for u_n when $c = -6$. **[3]**

(ii) The model predicts a single peak value for u_n.
Find, by direct calculation, the largest number of containers held by the company according to the model. **[2]**

(iii) Determine the month in which the model in part (b)(i) breaks down, giving a reason for your answer. **[4]**

(c) In a second model the manager uses $c = 24$ instead of $c = -6$.
Show that the second model predicts that eventually the number of containers held by the company will be a constant k, where k is to be determined. **[2]**

Solution

(a) The auxiliary equation is

$$16\lambda^2 - 24\lambda + 9 = 0$$

Factorising:

$$(4\lambda - 3)^2 = 0 \quad \Longrightarrow \quad \lambda = \frac{3}{4} \quad \text{(repeated)}$$

The root is repeated, so the complementary function is

$$(A + Bn)\left(\frac{3}{4}\right)^n$$

where A and B are arbitrary constants.

The right-hand side is a constant, so for a particular solution try the constant $u_n = p$. Then $u_{n+2} = u_{n+1} = u_n = p$, and substituting gives

$$16p - 24p + 9p = c \quad \Longrightarrow \quad p = c$$

The general solution is

$$u_n = (A + Bn)\left(\frac{3}{4}\right)^n + c$$

(b) (i) With $c = -6$, the general solution in part (a) is

$$u_n = (A + Bn)\left(\frac{3}{4}\right)^n - 6$$

Using $u_0 = 30$:

$$A - 6 = 30 \quad \Longrightarrow \quad A = 36$$

Using $u_1 = 60$:

$$\frac{3}{4}(36 + B) - 6 = 60 \quad \Longrightarrow \quad 36 + B = 88 \quad \Longrightarrow \quad B = 52$$

The particular solution is

$$u_n = (36 + 52n)\left(\frac{3}{4}\right)^n - 6$$

(ii) Calculating the next terms from part (b)(i):

$$u_2 = 140 \times \left(\frac{3}{4}\right)^2 - 6 = 72.75$$

$$u_3 = 192 \times \left(\frac{3}{4}\right)^3 - 6 = 75$$

$$u_4 = 244 \times \left(\frac{3}{4}\right)^4 - 6 \approx 71.2$$

The terms increase up to u_3 and then decrease, and the model has a single peak, so the largest number of containers held by the company is 75, at the end of March 2020.

(iii) After the peak at $n = 3$ the terms decrease, so the model breaks down when u_n first becomes negative. Calculating terms from part (b)(i):

$$u_{17} = 920 \times \left(\frac{3}{4}\right)^{17} - 6 \approx 0.916$$

$$u_{18} = 972 \times \left(\frac{3}{4}\right)^{18} - 6 \approx -0.520$$

So u_{17} rounds to 1, but u_{18} is negative and rounds to -1. The company cannot hold a negative number of containers, so the model breaks down at $n = 18$.

Month 12 is December 2020, so month 18 is June 2021. The model breaks down in June 2021, because it predicts a negative number of containers at the end of that month.

(c) With $c = 24$, the general solution in part (a) is

$$u_n = (A + Bn)\left(\frac{3}{4}\right)^n + 24$$

As $n \to \infty$, $\left(\frac{3}{4}\right)^n \to 0$ and $n\left(\frac{3}{4}\right)^n \to 0$. Whatever the values of A and B, this gives

$$(A + Bn)\left(\frac{3}{4}\right)^n \to 0 \quad \Longrightarrow \quad u_n \to 24$$

Since u_n is rounded to the nearest integer, the number of containers predicted is eventually constant at 24, so $k = 24$.

10. The equation

$$x^2 - 4x + 1 = 0$$

has roots α and β, where $\alpha > \beta$.

The sequence S_n is defined by $S_n = \alpha^n + \beta^n$ for $n \geq 0$.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

(b) (i) Determine the values of S_2 and S_4. **[3]**

(ii) Show that $S_{n+2} = 4S_{n+1} - S_n$ for $n \geq 0$. **[2]**

(iii) Deduce that S_n is an even integer for all $n \geq 0$. **[2]**

(c) A student approximates the terms of the sequence S_n using the formula $T_n = \alpha^n$.

(i) Explain why $T_n < S_n$ for every $n \geq 1$. **[1]**

(ii) State, with a reason, how the value of T_n should be rounded so that the result is equal to S_n for every $n \geq 1$. **[2]**

Solution

(a) For the quadratic $ax^2 + bx + c = 0$ with roots α and β, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Here $a = 1$, $b = -4$ and $c = 1$, so

$$\alpha + \beta = 4, \qquad \alpha\beta = 1$$

(b) (i) Using the values from part (a):

$$\begin{aligned} S_2 &= \alpha^2 + \beta^2 \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= 4^2 - 2 \times 1 \\ &= 14 \end{aligned}$$

Squaring $\alpha^2 + \beta^2$ gives $\alpha^4 + 2\alpha^2\beta^2 + \beta^4$, so

$$\begin{aligned} S_4 &= \alpha^4 + \beta^4 \\ &= \left(\alpha^2 + \beta^2\right)^2 - 2(\alpha\beta)^2 \\ &= 14^2 - 2 \times 1^2 \\ &= 194 \end{aligned}$$

(ii) Each root satisfies the equation $x^2 - 4x + 1 = 0$, so

$$\alpha^2 = 4\alpha - 1, \qquad \beta^2 = 4\beta - 1$$

Multiplying the first equation by α^n and the second by β^n:

$$\alpha^{n+2} = 4\alpha^{n+1} - \alpha^n, \qquad \beta^{n+2} = 4\beta^{n+1} - \beta^n$$

Adding these two equations gives

$$\alpha^{n+2} + \beta^{n+2} = 4\big(\alpha^{n+1} + \beta^{n+1}\big) - (\alpha^n + \beta^n)$$
$$S_{n+2} = 4S_{n+1} - S_n$$

as required.

(iii) The statement is that S_n is an even integer.

Base case: $S_0 = \alpha^0 + \beta^0 = 2$ and $S_1 = \alpha + \beta = 4$, which are both even integers, so the statement is true for $n = 0$ and $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$ and $n = k + 1$, so that

$$S_k = 2p, \qquad S_{k+1} = 2q$$

for some integers p and q.

Inductive step: Using the result of part (b)(ii) with $n = k$, and then the assumption:

$$\begin{aligned} S_{k+2} &= 4S_{k+1} - S_k \\ &= 4(2q) - 2p \\ &= 2(4q - p) \end{aligned}$$

Since $4q - p$ is an integer, S_{k+2} is an even integer, so the statement is true for $n = k + 2$.

Conclusion: If the statement is true for $n = k$ and $n = k + 1$, then it is true for $n = k + 2$. Since it is true for $n = 0$ and $n = 1$, by the principle of mathematical induction the statement is true for all $n \geq 0$.

(c) (i) Solving $x^2 - 4x + 1 = 0$ gives $x = 2 \pm \sqrt{3}$, and $\alpha > \beta$, so

$$\alpha = 2 + \sqrt{3}, \qquad \beta = 2 - \sqrt{3} \approx 0.268$$

Since $\beta > 0$, $\beta^n > 0$ for every $n \geq 1$. Then

$$T_n = \alpha^n = S_n - \beta^n$$

which is S_n with a positive number subtracted, so $T_n < S_n$ for every $n \geq 1$.

(ii) From part (c)(i), $0 < \beta < 1$, so $0 < \beta^n < 1$ for every $n \geq 1$. Since $T_n = S_n - \beta^n$, this gives

$$S_n - 1 < T_n < S_n$$

By part (b)(iii), S_n is an integer, so T_n lies strictly between the consecutive integers $S_n - 1$ and S_n. Hence T_n should be rounded up to the next integer, which is S_n.

11. The sequences x_n and y_n are defined, for $n \geq 1$, by

$$x_{n+1} = x_n + 2y_n + 4 \quad \text{and} \quad y_{n+1} = 3x_n + 2y_n - 1$$

with $x_1 = 3$ and $y_1 = a$, where a is a constant.

(a) Show that $x_{n+2} - 3x_{n+1} - 4x_n = -6$. **[2]**

(b) Solve the second-order recurrence relation given in part (a) to obtain an expression for x_n in terms of a and n. **[8]**

Given that $x_5 = 1023$,

(c) find the value of a. **[2]**

(d) find an expression for y_n in terms of n. **[2]**

Solution

(a) Replacing n by $n+1$ in the first relation, then substituting the second relation for y_{n+1}:

$$\begin{aligned} x_{n+2} &= x_{n+1} + 2y_{n+1} + 4 \\ &= x_{n+1} + 2(3x_n + 2y_n - 1) + 4 \\ &= x_{n+1} + 6x_n + 4y_n + 2 \end{aligned}$$

Rearranging the first relation gives

$$2y_n = x_{n+1} - x_n - 4 \quad \Longrightarrow \quad 4y_n = 2x_{n+1} - 2x_n - 8$$

Substituting this for $4y_n$:

$$\begin{aligned} x_{n+2} &= x_{n+1} + 6x_n + (2x_{n+1} - 2x_n - 8) + 2 \\ &= 3x_{n+1} + 4x_n - 6 \end{aligned}$$

so

$$x_{n+2} - 3x_{n+1} - 4x_n = -6$$

as required.

(b) The auxiliary equation is

$$\lambda^2 - 3\lambda - 4 = 0 \quad \Longrightarrow \quad (\lambda - 4)(\lambda + 1) = 0 \quad \Longrightarrow \quad \lambda = 4 \text{ or } \lambda = -1$$

so the complementary function is

$$A \times 4^n + B(-1)^n$$

For a particular solution, try the constant $x_n = k$:

$$k - 3k - 4k = -6 \quad \Longrightarrow \quad -6k = -6 \quad \Longrightarrow \quad k = 1$$

The general solution is

$$x_n = A \times 4^n + B(-1)^n + 1$$

Two starting values are needed. The first is $x_1 = 3$, and the first relation with $n = 1$ gives

$$x_2 = x_1 + 2y_1 + 4 = 3 + 2a + 4 = 7 + 2a$$

Substituting $n = 1$ and $n = 2$ into the general solution:

$$4A - B + 1 = 3 \quad \Longrightarrow \quad 4A - B = 2$$

$$16A + B + 1 = 7 + 2a \quad \Longrightarrow \quad 16A + B = 6 + 2a$$

Adding these two equations gives

$$20A = 8 + 2a \quad \Longrightarrow \quad A = \frac{a+4}{10}$$

Then

$$B = 4A - 2 = \frac{2a+8}{5} - 2 = \frac{2a-2}{5}$$

Hence

$$x_n = \frac{a+4}{10} \times 4^n + \frac{2a-2}{5}(-1)^n + 1$$

(c) Substituting $n = 5$ into the result of part (b), with $4^5 = 1024$ and $(-1)^5 = -1$:

$$\frac{1024(a+4)}{10} - \frac{2a-2}{5} + 1 = 1023$$

Multiplying through by 10:

$$1024(a+4) - 2(2a-2) + 10 = 10\,230$$

$$1020a + 4110 = 10\,230$$

$$1020a = 6120$$

$$a = 6$$

(d) With $a = 6$, the constants in part (b) are $A = 1$ and $B = 2$, so

$$x_n = 4^n + 2(-1)^n + 1$$

Replacing n by $n + 1$, with $4^{n+1} = 4 \times 4^n$ and $(-1)^{n+1} = -(-1)^n$:

$$x_{n+1} = 4 \times 4^n - 2(-1)^n + 1$$

Rearranging the first relation gives $y_n = \frac{1}{2}(x_{n+1} - x_n - 4)$, so

$$y_n = \frac{1}{2}(4 \times 4^n - 2(-1)^n + 1 - 4^n - 2(-1)^n - 1 - 4)$$

$$= \frac{1}{2}(3 \times 4^n - 4(-1)^n - 4)$$

$$= \frac{3}{2} \times 4^n - 2(-1)^n - 2$$

12. The sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation

$$2u_{n+2} - 7u_{n+1} + 3u_n = 4n^2 + 6n - 3$$

(a) Determine the general solution of the recurrence relation. **[6]**

(b) Hence determine the particular solution of the recurrence relation for which $u_0 = 5$ and $u_1 = 4$. **[3]**

(c) Show, by direct calculation, that your solution in part (b) gives the correct value for u_2. **[1]**

The sequence $v_0, v_1, v_2, \ldots$ is defined by $v_n = u_{n+1} - 3u_n$, where u_n denotes the particular solution found in part (b).
You are given that, for some constant p, the sequence with nth term $\frac{v_n}{n^p}$, for $n \geq 1$, converges to a finite non-zero limit, q, as $n \to \infty$.

(d) Determine p and q. **[5]**

Solution

(a) The auxiliary equation is

$$2\lambda^2 - 7\lambda + 3 = 0$$
$$(2\lambda - 1)(\lambda - 3) = 0$$
$$\lambda = \frac{1}{2} \text{ or } \lambda = 3$$

so the complementary function is

$$A\left(\frac{1}{2}\right)^n + B \times 3^n$$

where A and B are arbitrary constants.

The right-hand side is a quadratic in n, so for a particular solution try $u_n = an^2 + bn + c$. Then

$$u_{n+1} = a(n+1)^2 + b(n+1) + c = an^2 + (2a+b)n + (a+b+c)$$
$$u_{n+2} = a(n+2)^2 + b(n+2) + c = an^2 + (4a+b)n + (4a+2b+c)$$

Substituting into the left-hand side of the recurrence relation:

$$\begin{aligned} 2u_{n+2} - 7u_{n+1} + 3u_n &= (2a - 7a + 3a)n^2 + (8a + 2b - 14a - 7b + 3b)n \\ &\quad + (8a + 4b + 2c - 7a - 7b - 7c + 3c) \\ &= -2an^2 + (-6a - 2b)n + (a - 3b - 2c) \end{aligned}$$

This must equal $4n^2 + 6n - 3$ for all n. Equating coefficients of n^2:

$$-2a = 4 \implies a = -2$$

Equating coefficients of n:

$$-6a - 2b = 6 \implies 12 - 2b = 6 \implies b = 3$$

Equating constant terms:

$$a - 3b - 2c = -3 \quad \Longrightarrow \quad -11 - 2c = -3 \quad \Longrightarrow \quad c = -4$$

A particular solution is $u_n = -2n^2 + 3n - 4$, so the general solution is

$$u_n = A\left(\frac{1}{2}\right)^n + B \times 3^n - 2n^2 + 3n - 4$$

(b) Using the general solution from part (a) with $u_0 = 5$:

$$A + B - 4 = 5 \quad \Longrightarrow \quad A + B = 9$$

Using $u_1 = 4$:

$$\frac{1}{2}A + 3B - 2 + 3 - 4 = 4 \quad \Longrightarrow \quad A + 6B = 14$$

Subtracting the first equation from the second gives $5B = 5$, so $B = 1$ and $A = 8$. The particular solution is

$$u_n = 8\left(\frac{1}{2}\right)^n + 3^n - 2n^2 + 3n - 4$$

(c) Putting $n = 0$ in the recurrence relation:

$$\begin{aligned} 2u_2 - 7u_1 + 3u_0 &= -3 \\ 2u_2 &= 7 \times 4 - 3 \times 5 - 3 = 10 \\ u_2 &= 5 \end{aligned}$$

Putting $n = 2$ in the solution from part (b):

$$u_2 = 8\left(\frac{1}{2}\right)^2 + 3^2 - 2 \times 2^2 + 3 \times 2 - 4 = 2 + 9 - 8 + 6 - 4 = 5$$

The two values agree, so the solution in part (b) gives the correct value for u_2.

(d) From part (b), replacing n by $n + 1$:

$$\begin{aligned} u_{n+1} &= 8\left(\frac{1}{2}\right)^{n+1} + 3^{n+1} - 2(n+1)^2 + 3(n+1) - 4 \\ &= 4\left(\frac{1}{2}\right)^n + 3 \times 3^n - 2n^2 - n - 3 \end{aligned}$$

Also

$$3u_n = 24\left(\frac{1}{2}\right)^n + 3 \times 3^n - 6n^2 + 9n - 12$$

Subtracting, the terms in 3^n cancel:

$$\begin{aligned} v_n &= u_{n+1} - 3u_n \\ &= 4n^2 - 10n + 9 - 20\left(\frac{1}{2}\right)^n \end{aligned}$$

Taking out a factor of n^2 and dividing by n^p:

$$\frac{v_n}{n^p} = n^{2-p}\left(4 - \frac{10}{n} + \frac{9}{n^2} - \frac{20}{n^2}\left(\frac{1}{2}\right)^n\right)$$

As $n \to \infty$, $\frac{10}{n} \to 0$, $\frac{9}{n^2} \to 0$ and $\frac{20}{n^2}\left(\frac{1}{2}\right)^n \to 0$, so the bracket tends to 4.

If $p < 2$, then $n^{2-p} \to \infty$, so the sequence does not converge. If $p > 2$, then $n^{2-p} \to 0$, so the limit is zero. Neither is allowed, so $2 - p = 0$, and then $n^{2-p} = 1$ for all $n \geq 1$ and the limit is 4. Hence

$$p = 2, \qquad q = 4$$