OPTIONAL PURE / RECURRENCE RELATIONS

1st Order Recurrence Relations

Worked solutions 12 questions · 123 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

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1. The sequence u_n satisfies a recurrence relation of the form $u_{n+1} = au_n + b$, for $n \geq 1$, where a and b are constants.
The first four terms of the sequence are $u_1 = 4$, $u_2 = -5$, $u_3 = 13$ and $u_4 = -23$.

(a) Find the values of a and b. **[2]**

(b) By solving the recurrence relation, find an expression for u_n in terms of n. **[5]**

(c) Hence find the value of n for which $u_n = 769$. **[2]**

Solution

(a) Using $u_2 = au_1 + b$ and $u_3 = au_2 + b$:

$$4a + b = -5$$
$$-5a + b = 13$$

Subtracting the second equation from the first gives $9a = -18$, so

$$a = -2, \qquad b = 3$$

(b) The recurrence relation is

$$u_{n+1} = -2u_n + 3$$

The complementary function solves $u_{n+1} = -2u_n$, so it is

$$A(-2)^n$$

where A is a constant.

For a particular solution, try $u_n = k$, a constant:

$$k = -2k + 3 \implies 3k = 3 \implies k = 1$$

The general solution is therefore

$$u_n = A(-2)^n + 1$$

Using $u_1 = 4$:

$$-2A + 1 = 4 \implies A = -\frac{3}{2}$$

Hence

$$u_n = 1 - \frac{3}{2}(-2)^n = 1 + 3(-2)^{n-1}$$

(c) Using part (b), set $u_n = 769$:

$$1 + 3(-2)^{n-1} = 769$$
$$(-2)^{n-1} = 256$$

Since $256 = 2^8$ is positive, $n - 1$ must be even, and $(-2)^8 = 256$, so $n - 1 = 8$, giving

$$n = 9$$

2. A student takes out a loan of £6000 from a bank.
At the end of each month

- the student repays £P;
- the bank then adds interest of 1% of the amount still owed after that repayment.

Let U_n be the amount of money, in pounds, owed n months after the loan was taken out, so that $U_0 = 6000$.

(a) Explain, in the context of the problem, why

$$U_n = 1.01U_{n-1} - 1.01P, \qquad n \geq 1$$ **[2]**

(b) Solve this recurrence relation, giving U_n in terms of n and P. **[5]**

(c) The loan is completely repaid at the end of exactly 24 months.
Hence find the value of P, giving your answer to the nearest penny. **[3]**

Solution

(a) At the end of month n, the amount owed at the start of that month is U_{n-1}. The student first repays £P, so the amount still owed is $U_{n-1} - P$. The bank then adds 1% of this amount, which multiplies it by 1.01, so the amount owed after n months is

$$U_n = 1.01(U_{n-1} - P) = 1.01U_{n-1} - 1.01P$$

This holds for every month $n \geq 1$, with the starting value $U_0 = 6000$ being the loan itself, as required.

(b) The complementary function solves $U_n = 1.01U_{n-1}$, in which each term is 1.01 times the previous one, so it is

$$A(1.01)^n$$

For a particular solution, try the constant $U_n = k$ for all n:

$$k = 1.01k - 1.01P$$

$$0.01k = 1.01P$$

$$k = 101P$$

The general solution is the complementary function plus this particular solution:

$$U_n = A(1.01)^n + 101P$$

Using $U_0 = 6000$:

$$6000 = A + 101P \quad \Longrightarrow \quad A = 6000 - 101P$$

Hence

$$U_n = (6000 - 101P)(1.01)^n + 101P$$

(c) The loan is repaid after 24 months, so $U_{24} = 0$. Using part (b):

$$(6000 - 101P)(1.01)^{24} + 101P = 0$$

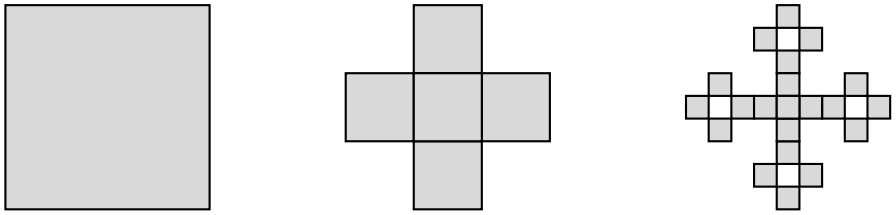
$$6000 \times (1.01)^{24} - 101P(1.01)^{24} + 101P = 0$$

$$101P\big((1.01)^{24} - 1\big) = 6000 \times (1.01)^{24}$$

Therefore

$$P = \frac{6000 \times (1.01)^{24}}{101((1.01)^{24} - 1)} \approx 279.64 \qquad (2 \text{ d.p.})$$

The monthly repayment is £279.64, to the nearest penny.



3. The diagram above shows the first three stages of a pattern that is created by a recursive process. The process starts with a single square and proceeds as follows:

- each square is divided into a 3×3 grid of 9 equal smaller squares, and is replaced by the 4 of these that lie at the middle of its sides;
- once every square has been replaced, one further square, of the same size as the new squares, is added to fill the gap at the centre of the pattern.

Let u_n be the number of squares in the pattern in stage n, where stage 1 is the original square.

(a) Explain why u_n satisfies the recurrence relation

$$u_1 = 1, \qquad u_{n+1} = 4u_n + 1, \qquad n \geq 1$$ **[2]**

(b) Solve this recurrence relation. **[5]**

(c) The initial square has side length 9.
Find, in terms of n, the total area of the squares in stage n, giving your answer in the form

$$p\left(\left(\frac{4}{9}\right)^n - \left(\frac{1}{9}\right)^n\right)$$

where p is an integer. **[3]**

Solution

(a) Stage 1 is the single original square, so $u_1 = 1$.

In going from stage n to stage $n+1$, each of the u_n squares is replaced by 4 smaller squares, giving $4u_n$ squares. One further square is then added at the centre, so

$$u_{n+1} = 4u_n + 1$$

(b) The complementary function solves $u_{n+1} = 4u_n$, so it is

$$A \times 4^n$$

For a particular solution, try a constant $u_n = k$:

$$k = 4k + 1$$

$$-3k = 1$$

$$k = -\frac{1}{3}$$

The general solution is therefore

$$u_n = A \times 4^n - \frac{1}{3}$$

Using $u_1 = 1$:

$$4A - \frac{1}{3} = 1$$
$$4A = \frac{4}{3}$$
$$A = \frac{1}{3}$$

Hence

$$u_n = \frac{1}{3} \times 4^n - \frac{1}{3} = \frac{4^n - 1}{3}$$

(c) Each stage divides the side length of the squares by 3. Stage 1 has side length 9, so in stage n every square has side length

$$\frac{9}{3^{n-1}} = \frac{27}{3^n}$$

so each square has area

$$\left(\frac{27}{3^n}\right)^2 = \frac{729}{9^n}$$

All u_n squares in stage n are the same size, so using part (b) the total area is

$$\begin{aligned} u_n \times \frac{729}{9^n} &= \frac{4^n - 1}{3} \times \frac{729}{9^n} \\ &= \frac{243(4^n - 1)}{9^n} \\ &= 243\left(\left(\frac{4}{9}\right)^n - \left(\frac{1}{9}\right)^n\right) \end{aligned}$$

This is in the required form with $p = 243$.

4. A patient is treated with a course of injections of a medicine.
Before the course there was none of the medicine in the patient's body.
The injections are given at the same time each day. The first injection contains 60 mg of the medicine and every later injection contains 20 mg.
Between injections, the amount of the medicine in the patient's body decreases naturally by 50%.
Let v_n mg be the amount of the medicine in the patient's body immediately **before** an injection is given, n days after the first injection.

(a) Explain why v_n satisfies the recurrence relation

$$v_1 = 30, \qquad v_{n+1} = 0.5v_n + 10, \qquad n \geq 1$$ **[2]**

(b) Solve this recurrence relation to find v_n in terms of n. **[5]**

(c) The course is safe and effective only if, at **all times** after the first injection, the amount of the medicine in the patient's body is at most 65 mg and at least 22 mg.
Determine whether the course is safe and effective for this patient, giving a reason for your answer. **[4]**

Solution

(a) Immediately after the first injection there is 60 mg in the body. This halves before the next injection one day later, so

$$v_1 = 0.5 \times 60 = 30$$

Immediately before the injection n days after the first, there is v_n mg in the body. The injection adds 20 mg, giving $(v_n + 20)$ mg, and this halves before the next injection, so

$$v_{n+1} = 0.5(v_n + 20) = 0.5v_n + 10, \qquad n \geq 1$$

(b) The complementary function solves $v_{n+1} = 0.5v_n$, so it is

$$A(0.5)^n$$

For a particular solution, try a constant $v_n = k$:

$$k = 0.5k + 10$$

$$0.5k = 10$$

$$k = 20$$

The general solution is

$$v_n = A(0.5)^n + 20$$

Using $v_1 = 30$:

$$0.5A + 20 = 30$$

$$A = 20$$

Hence

$$v_n = 20 + 20(0.5)^n = 20 + \frac{20}{2^n}$$

(c) Between injections the amount only decreases, so the greatest amounts occur immediately after injections and the least amounts immediately before injections.

Safe: immediately after the first injection the amount is 60 mg. Immediately after each later injection the amount is

$$v_n + 20 = 40 + 20(0.5)^n \leq 40 + 20 \times 0.5 = 50$$

so the greatest amount at any time is 60 mg $\leq$ 65 mg, and the course is safe.

Effective: the amount immediately before an injection is $v_n = 20 + 20(0.5)^n$, which decreases as n increases. It is at least 22 mg only when

$$20(0.5)^n \geq 2$$

$$2^n \leq 10$$

which holds only for $n \leq 3$. Immediately before the injection on day 4 the amount is given by

$$v_4 = 20 + \frac{20}{16} = 21.25 < 22$$

so the amount falls below 22 mg. The course is safe but not effective, so it is not safe and effective for this patient.

5. A sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_{n+1} = 3u_n + 2^{n+2}, \qquad n \geq 1$$

(a) Find the general solution of this recurrence relation. **[4]**

(b) Given that $u_1 = 1$, find u_n in terms of n, and hence show that u_n is odd for every positive integer n. **[3]**

Solution

(a) The recurrence relation can be written as

$$u_{n+1} - 3u_n = 4 \times 2^n$$

The complementary function solves $u_{n+1} = 3u_n$, so it is

$$A \times 3^n$$

For a particular solution, try $u_n = k \times 2^n$, so that $u_{n+1} = k \times 2^{n+1} = 2k \times 2^n$. Substituting into the recurrence relation:

$$2k \times 2^n = 3k \times 2^n + 4 \times 2^n$$

$$2k = 3k + 4$$

$$k = -4$$

So a particular solution is $u_n = -4 \times 2^n = -2^{n+2}$. The general solution is the complementary function plus this particular solution:

$$u_n = A \times 3^n - 2^{n+2}$$

(b) Using $u_1 = 1$ in the general solution from part (a):

$$3A - 2^3 = 1$$

$$3A = 9$$

$$A = 3$$

So

$$u_n = 3 \times 3^n - 2^{n+2} = 3^{n+1} - 2^{n+2}$$

For every positive integer n, 3^{n+1} is a product of odd numbers, so it is odd, and 2^{n+2} has a factor of 2, so it is even. An odd number minus an even number is odd, so u_n is odd for every positive integer n, as required.

6. A lake is stocked with fish as part of a conservation project. The number of fish in the lake n years after the start of the project is F_n. At the start of the project, there are 2000 fish in the lake.
The first model for the number of fish is that, each year, a fixed proportion of the fish in the lake is lost, and then 300 fish are added to the lake at the end of the year. This model predicts that, once the fish have been added at the end of the first year, there will be 1900 fish in the lake.

(a) Use this model to express the number of fish as a first-order recurrence relation. **[1]**

In a second model, the proportion of fish lost each year is the same as in the first model, but the number of fish added increases each year: 300 fish are added at the end of the first year, and at the end of each later year 50 more fish are added than at the end of the year before.

(b) (i) Show that the recurrence relation for the second model is

$$F_0 = 2000, \qquad F_{n+1} = 0.8F_n + 50n + 300$$ **[1]**

(ii) Solve this recurrence relation. **[6]**

(iii) Give one criticism of the second model for predicting the number of fish in the lake in the long term. **[1]**

Solution

(a) Let k be the proportion of fish that remain each year. Then $F_{n+1} = kF_n + 300$, and $F_1 = 1900$ gives

$$2000k + 300 = 1900 \implies k = 0.8$$

So the recurrence relation is

$$F_0 = 2000, \qquad F_{n+1} = 0.8F_n + 300$$

(b) (i) The proportion lost is the same as in the first model, so each year $0.8F_n$ fish remain before the new fish are added. The numbers added at the ends of years $1, 2, 3, \ldots$ are $300, 350, 400, \ldots$, an arithmetic sequence with first term 300 and common difference 50. The number added at the end of year $n+1$ is therefore $300 + 50n$, so

$$F_0 = 2000, \qquad F_{n+1} = 0.8F_n + 50n + 300$$

as required.

(ii) The complementary function solves $F_{n+1} = 0.8F_n$, so it is

$$A(0.8)^n$$

For a particular solution, try $F_n = an + b$. Substituting into the recurrence relation:

$$a(n+1) + b = 0.8(an + b) + 50n + 300$$

$$an + a + b = 0.8an + 0.8b + 50n + 300$$

Equating coefficients of n:

$$a = 0.8a + 50 \implies 0.2a = 50 \implies a = 250$$

Equating constant terms:

$$a + b = 0.8b + 300 \implies 250 + 0.2b = 300 \implies b = 250$$

So the general solution is

$$F_n = A(0.8)^n + 250n + 250$$

Using $F_0 = 2000$:

$$A + 250 = 2000 \quad \Longrightarrow \quad A = 1750$$

Hence

$$F_n = 1750(0.8)^n + 250n + 250$$

(iii) As $n \to \infty$, $(0.8)^n \to 0$ but $250n \to \infty$, so the model predicts that the number of fish grows without limit. This is unrealistic, since the lake has limited space and food and can only support a limited number of fish.

7. A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = 4$ and

$$u_{n+1} = \frac{3u_n}{n+2}, \qquad n \geq 0$$

The solution of this recurrence relation can be written in the form

$$u_n = \frac{a\,k^n}{(n+b)!}$$

where a, k and b are constants. Find the values of a, k and b, and verify that the resulting expression satisfies both the recurrence relation and the initial condition. **[6]**

Solution

For $n \geq 1$, replacing n by $n-1$ in the recurrence relation gives

$$u_n = \frac{3}{n+1}\,u_{n-1}$$

Applying this repeatedly, each step brings in a factor of 3 and a denominator one less than the step before:

$$\begin{aligned} u_n &= \frac{3}{n+1} \times u_{n-1} \\ &= \frac{3}{n+1} \times \frac{3}{n} \times u_{n-2} \\ &= \frac{3}{n+1} \times \frac{3}{n} \times \frac{3}{n-1} \times \cdots \times \frac{3}{2} \times u_0 \\ &= \frac{3^n}{(n+1) \times n \times (n-1) \times \cdots \times 2} \times 4 \end{aligned}$$

There are n factors of 3, one for each step from u_0 to u_n. Since $(n+1) \times n \times \cdots \times 2 \times 1 = (n+1)!$, the denominator is $(n+1)!$, so

$$u_n = \frac{4 \times 3^n}{(n+1)!}$$

This also holds for $n = 0$, since it gives $u_0 = 4$. The solution is therefore in the given form, with

$$a = 4, \qquad k = 3, \qquad b = 1$$

Recurrence relation: replacing n by $n+1$ in the formula gives

$$\begin{aligned} u_{n+1} &= \frac{4 \times 3^{n+1}}{(n+2)!} \\ &= \frac{3 \times 4 \times 3^n}{(n+2) \times (n+1)!} \\ &= \frac{3}{n+2} \times \frac{4 \times 3^n}{(n+1)!} \\ &= \frac{3u_n}{n+2} \end{aligned}$$

so the formula satisfies the recurrence relation.

Initial condition: substituting $n = 0$ into the formula gives

$$u_0 = \frac{4 \times 3^0}{(0+1)!} = \frac{4 \times 1}{1} = 4$$

as required.

8. The sequence u_n, where k is a constant with $k \neq 0$ and $k \neq 1$, satisfies the first-order recurrence relation

$$u_{n+1} = ku_n + k + 1, \qquad n \geq 0$$

(a) Find, in terms of k, the general solution of the recurrence relation. **[3]**

(b) Hence find, in terms of k, the value of u_0 for which every term of the sequence is equal, and find the value of k for which this common value is 3. **[2]**

(c) Given instead that $u_0 = 1$ and $u_2 = 13$, find the two possible values of k. **[4]**

Solution

(a) The complementary function solves $u_{n+1} = ku_n$, so it is

$$Ak^n$$

For a particular solution, try a constant $u_n = p$ for all n. Substituting into the recurrence relation:

$$p = kp + k + 1$$

$$p(1-k) = 1 + k$$

$$p = \frac{1+k}{1-k}$$

This is valid since $k \neq 1$. Adding the complementary function and this particular solution, the general solution is

$$u_n = Ak^n + \frac{1+k}{1-k}$$

where A is an arbitrary constant.

(b) Since $k \neq 0$ and $k \neq 1$, the term Ak^n takes the same value for every n only when $A = 0$. So every term is equal exactly when $A = 0$. Putting $n = 0$ in the general solution from part (a) gives $u_0 = A + \frac{1+k}{1-k}$, so with $A = 0$:

$$u_0 = \frac{1+k}{1-k}$$

For this common value to be 3:

$$\frac{1+k}{1-k} = 3$$

$$1 + k = 3 - 3k$$

$$4k = 2$$

$$k = \frac{1}{2}$$

(c) Using the recurrence relation with $u_0 = 1$:

$$u_1 = k \times 1 + k + 1 = 2k + 1$$

$$u_2 = k(2k+1) + k + 1 = 2k^2 + 2k + 1$$

Setting $u_2 = 13$:

$$2k^2 + 2k + 1 = 13$$
$$2k^2 + 2k - 12 = 0$$
$$k^2 + k - 6 = 0$$
$$(k + 3)(k - 2) = 0$$

Both values satisfy $k \neq 0$ and $k \neq 1$, so

$$k = 2 \quad \text{or} \quad k = -3$$

9. The sequence u_n is defined by $u_1 = 4$ and $u_{n+1} = 3u_n - 4n^2$ for $n \geq 1$.

(a) Find u_n in terms of n. **[7]**

(b) A second sequence v_n satisfies the same recurrence relation, $v_{n+1} = 3v_n - 4n^2$ for $n \geq 1$, but has first term $v_1 = k$, where k is a constant.
Find the value of k for which v_n is a quadratic expression in n. **[2]**

Solution

(a) The recurrence relation can be written as

$$u_{n+1} - 3u_n = -4n^2$$

The complementary function solves $u_{n+1} = 3u_n$, so it is

$$A \times 3^n$$

where A is a constant. The right-hand side is a quadratic in n, so try a particular solution of the form

$$u_n = an^2 + bn + c$$

Then

$$\begin{aligned} u_{n+1} &= a(n+1)^2 + b(n+1) + c \\ &= an^2 + (2a+b)n + (a+b+c) \end{aligned}$$

Substituting into $u_{n+1} = 3u_n - 4n^2$ gives

$$an^2 + (2a+b)n + (a+b+c) = (3a-4)n^2 + 3bn + 3c$$

Equating coefficients of n^2, n and the constant terms:

$$\begin{aligned} a = 3a - 4 \quad &\implies \quad a = 2 \\ 2a + b = 3b \quad &\implies \quad b = a = 2 \\ a + b + c = 3c \quad &\implies \quad 2c = 4 \quad \implies \quad c = 2 \end{aligned}$$

So the general solution is

$$u_n = A \times 3^n + 2n^2 + 2n + 2$$

Using $u_1 = 4$:

$$3A + 2 + 2 + 2 = 4 \quad \implies \quad A = -\frac{2}{3}$$

Since $-\frac{2}{3} \times 3^n = -2 \times 3^{n-1}$, the solution is

$$u_n = 2n^2 + 2n + 2 - 2 \times 3^{n-1}$$

(b) The sequence v_n satisfies the same recurrence relation, so by part (a) its general term is

$$v_n = B \times 3^n + 2n^2 + 2n + 2$$

for some constant B. This is a quadratic expression in n only when the exponential term vanishes, that is when $B = 0$. Then $v_n = 2n^2 + 2n + 2$, so

$$k = v_1 = 2 + 2 + 2 = 6$$

10. A sequence $t_1, t_2, t_3, \ldots$ is defined by $t_1 = 4$ and

$$t_{n+1} = 3t_n + 2^n - 4n, \qquad n \geq 1$$

(a) Solve the recurrence relation to find t_n in terms of n. **[7]**

A second sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 5$ and

$$(n+1)u_{n+1} - 2nu_n = 6n^2 + 6n - 2n^3 - 1, \qquad n \geq 1$$

(b) (i) Explain why the standard method of a complementary function plus a particular solution, used for non-homogeneous first-order recurrence relations, **cannot** be applied directly to the recurrence relation for u_n. **[1]**

(ii) Verify that the solution of this recurrence relation is given by $u_n = an^2 + \frac{b}{n}$, where a and b are constants whose values are to be determined. **[5]**

Solution

(a) The complementary function solves $t_{n+1} = 3t_n$, so it is

$$A \times 3^n$$

For a particular solution, try $t_n = p \times 2^n + qn + r$. Then $t_{n+1} = 2p \times 2^n + qn + q + r$, and substituting into the recurrence relation gives

$$2p \times 2^n + qn + q + r = 3p \times 2^n + 3qn + 3r + 2^n - 4n$$

Equating coefficients:

$$\begin{aligned} 2^n: &\quad 2p = 3p + 1 &\implies &\quad p = -1 \\ n: &\quad q = 3q - 4 &\implies &\quad q = 2 \\ \text{constant}: &\quad q + r = 3r &\implies &\quad r = 1 \end{aligned}$$

A particular solution is $-2^n + 2n + 1$, so the general solution is

$$t_n = A \times 3^n - 2^n + 2n + 1$$

Using $t_1 = 4$:

$$3A - 2 + 2 + 1 = 4 \quad \implies \quad A = 1$$

Therefore

$$t_n = 3^n - 2^n + 2n + 1$$

(b) (i) The coefficients of u_{n+1} and u_n, namely $n+1$ and $2n$, depend on n. Written as $u_{n+1} = \frac{2n}{n+1}u_n + \cdots$, the multiplier of u_n is not a constant, so the relation is not of the form $u_{n+1} = ku_n + f(n)$ with k constant, and there is no complementary function of the form $A \times k^n$.

(ii) With $u_n = an^2 + \frac{b}{n}$, we have $u_{n+1} = a(n+1)^2 + \frac{b}{n+1}$, so

$$\begin{aligned}(n+1)u_{n+1} &= a(n+1)^3 + b \\ &= an^3 + 3an^2 + 3an + a + b \\ 2nu_n &= 2an^3 + 2b\end{aligned}$$

Subtracting:

$$(n+1)u_{n+1} - 2nu_n = -an^3 + 3an^2 + 3an + (a - b)$$

This must equal $-2n^3 + 6n^2 + 6n - 1$ for all $n \geq 1$. Equating coefficients:

$$\begin{aligned}n^3\text{:} \quad & -a = -2 \quad \Longrightarrow \quad a = 2 \\ n^2\text{:} \quad & 3a = 6 \\ n\text{:} \quad & 3a = 6 \\ \text{constant:} \quad & a - b = -1 \quad \Longrightarrow \quad b = 3\end{aligned}$$

The n^2 and n coefficients are both satisfied by $a = 2$, so $u_n = 2n^2 + \frac{3}{n}$ satisfies the recurrence relation. It also satisfies the initial condition, since

$$u_1 = 2 \times 1^2 + \frac{3}{1} = 5$$

The recurrence relation and u_1 determine the sequence, so the solution is $u_n = an^2 + \frac{b}{n}$ with $a = 2$, $b = 3$.

11. A customer takes out a loan of £P from a bank. Interest is charged at a rate of 0.5% per month and is added to the outstanding amount of the loan at the end of each month. Immediately after the interest is added, the customer makes a monthly payment.
The payment made at the end of the first month is £M, and each monthly payment after that is £2 more than the payment before it.
Let L_n denote the outstanding amount of the loan at the end of month n **after** that month's payment has been made, with $L_0 = P$.

(a) Explain why the outstanding amount of the loan is modelled by the recurrence relation

$$L_{n+1} = 1.005L_n - M - 2n \qquad (*)$$

for $n \geq 0$, with $L_0 = P$. **[3]**

(b) Solve the recurrence relation $(*)$, giving L_n in terms of n, M and P. **[7]**

(c) The loan amount is £12 000 and the loan is fully repaid by the payment made at the end of month 48. Find, to the nearest pound, the value of M, and hence find the amount of the final monthly payment. **[4]**

Solution

(a) At the end of month n, after the payment, the amount outstanding is £L_n. During month $n+1$, interest of 0.5% is added, so the amount outstanding becomes

$$L_n + 0.005L_n = 1.005L_n$$

The payments are £M in month 1, £$(M+2)$ in month 2, £$(M+4)$ in month 3, and so on, increasing by £2 each month. The payment in month k is therefore £$\big(M + 2(k-1)\big)$, so the payment in month $n+1$ is £$(M+2n)$. This payment is made immediately after the interest is added, so

$$L_{n+1} = 1.005L_n - (M + 2n) = 1.005L_n - M - 2n$$

The amount outstanding at the start, before any interest or payment, is the amount borrowed, so $L_0 = P$.

(b) The complementary function solves $L_{n+1} = 1.005L_n$, so it is

$$A(1.005)^n$$

where A is a constant. The non-homogeneous term $-M - 2n$ is linear in n, so try a particular solution $L_n = an + b$. Substituting into $(*)$ gives

$$a(n+1) + b = 1.005(an + b) - M - 2n$$

$$an + a + b = 1.005an + 1.005b - M - 2n$$

Equating coefficients of n:

$$a = 1.005a - 2 \quad \Longrightarrow \quad 0.005a = 2 \quad \Longrightarrow \quad a = 400$$

Equating constant terms:

$$a + b = 1.005b - M$$

$$400 + M = 0.005b$$

$$b = 80\,000 + 200M$$

The general solution is the sum of the complementary function and this particular solution:

$$L_n = A(1.005)^n + 400n + 80\,000 + 200M$$

Using $L_0 = P$:

$$P = A + 80\,000 + 200M \quad \Longrightarrow \quad A = P - 80\,000 - 200M$$

Therefore

$$L_n = (P - 80\,000 - 200M)(1.005)^n + 400n + 80\,000 + 200M$$

(c) With $P = 12\,000$, the solution from part (b) becomes

$$L_n = (-68\,000 - 200M)(1.005)^n + 400n + 80\,000 + 200M$$

The loan is fully repaid by the payment at the end of month 48, so $L_{48} = 0$:

$$(-68\,000 - 200M)(1.005)^{48} + 400 \times 48 + 80\,000 + 200M = 0$$

$$-68\,000(1.005)^{48} - 200M(1.005)^{48} + 99\,200 + 200M = 0$$

$$200M\left((1.005)^{48} - 1\right) = 99\,200 - 68\,000(1.005)^{48}$$

so

$$M = \frac{99\,200 - 68\,000(1.005)^{48}}{200((1.005)^{48} - 1)} \approx 236.73$$

To the nearest pound, $M = 237$.

The final payment is made in month 48, so in pounds it is $M + 2 \times 47 = M + 94$:

$$M + 94 \approx 236.73 + 94 = 330.73$$

The final monthly payment is £331 to the nearest pound.

12. A building society models the repayment of a loan in the following way.

- A loan of £C is taken out at the start of the first year.
- At the start of each year, the borrower makes a repayment of £R, which is subtracted from the amount owed.
- At the end of each year, interest of $\alpha\%$ ($\alpha > 0$) of the amount owed at that time is added to the amount owed.
- The amount owed immediately before the $(n+1)$th repayment is £L_n, so that $L_0 = C$.
- When the amount owed immediately before a repayment is £R or less, the borrower instead pays the whole amount owed, and the loan is cleared.

(a) Show that, while the loan is being repaid, $L_{n+1} = aL_n + b$, giving a and b in terms of α and R. **[2]**

(b) Find the solution of the recurrence relation $L_{n+1} = aL_n + b$ with $L_0 = C$, giving L_n in terms of a, b, C and n. **[5]**

(c) Deduce from parts (a) and (b) that, for the loan to be cleared,

$$R > \frac{\alpha C}{100 + \alpha}$$ **[3]**

A borrower takes out a loan of £20 000 at 5% interest and agrees to repay £2000 at the start of each year.

(d) (i) Use an algebraic method to find the number of repayments the borrower makes, including the final repayment. **[3]**

(ii) Find the total amount that the borrower repays over the lifetime of the loan. **[3]**

Solution

(a) Immediately before the $(n+1)$th repayment, the amount owed is £L_n. After the repayment it is £$(L_n - R)$, and adding interest of $\alpha\%$ at the end of the year multiplies this by $1 + \frac{\alpha}{100}$, so

$$L_{n+1} = \left(1 + \frac{\alpha}{100}\right)(L_n - R) = \left(1 + \frac{\alpha}{100}\right)L_n - \left(1 + \frac{\alpha}{100}\right)R$$

This is of the form $L_{n+1} = aL_n + b$ with

$$a = 1 + \frac{\alpha}{100} = \frac{100 + \alpha}{100}, \qquad b = -\left(1 + \frac{\alpha}{100}\right)R = -\frac{(100 + \alpha)R}{100}$$

(b) The complementary function solves $L_{n+1} = aL_n$, so it is

$$Da^n$$

where D is a constant.

For a particular solution, try $L_n = k$, a constant. Substituting into the recurrence relation:

$$k = ak + b$$

$$k(1 - a) = b$$

$$k = \frac{b}{1 - a}$$

This is valid since $a = 1 + \frac{\alpha}{100} \neq 1$. The general solution is therefore

$$L_n = Da^n + \frac{b}{1-a}$$

Using $L_0 = C$:

$$C = D + \frac{b}{1-a} \quad \Longrightarrow \quad D = C - \frac{b}{1-a}$$

so

$$L_n = \left(C - \frac{b}{1-a}\right)a^n + \frac{b}{1-a}$$

(c) From part (a), $1 - a = -\frac{\alpha}{100}$, so

$$\frac{b}{1-a} = \frac{-(100+\alpha)R/100}{-\alpha/100} = \frac{(100+\alpha)R}{\alpha}$$

Let

$$k = \frac{(100+\alpha)R}{\alpha}$$

By part (b), while the loan is being repaid, the amount owed is

$$L_n = (C-k)a^n + k$$

Since $100 + \alpha > \alpha$, $k > R$. Also $a > 1$, so $a^n > 0$ for every n.

Suppose $C \geq k$. Then $(C-k)a^n \geq 0$, so $L_n \geq k > R$ for every n: the amount owed before each repayment is always more than R, so the borrower never pays off the whole amount and the loan is never cleared.

Hence, for the loan to be cleared, $C < k$:

$$C < \frac{(100+\alpha)R}{\alpha} \quad \Longrightarrow \quad R > \frac{\alpha C}{100+\alpha}$$

as required.

(d) (i) Here $\alpha = 5$, $R = 2000$ and $C = 20\,000$, so parts (a) and (c) give

$$a = 1.05, \qquad b = -\frac{105 \times 2000}{100} = -2100, \qquad \frac{b}{1-a} = \frac{105 \times 2000}{5} = 42\,000$$

By part (b):

$$L_n = (20\,000 - 42\,000) \times (1.05)^n + 42\,000 = 42\,000 - 22\,000 \times (1.05)^n$$

The final repayment is made at the first n for which $L_n \leq 2000$:

$$42\,000 - 22\,000 \times (1.05)^n \leq 2000$$

$$22\,000 \times (1.05)^n \geq 40\,000$$

$$(1.05)^n \geq \frac{20}{11}$$

$$n \geq \frac{\ln\left(\frac{20}{11}\right)}{\ln 1.05} \approx 12.25$$

The last step takes logarithms and divides by $\ln 1.05$, which is positive. The smallest such integer is $n = 13$, and L_{13} is the amount owed immediately before the 14th repayment, which is the final one. The borrower makes 14 repayments.

(ii) The borrower makes 13 repayments of £2000, then pays the whole amount owed, £L_{13}, where

$$L_{13} = 42\,000 - 22\,000 \times (1.05)^{13} \approx 515.72$$

The total amount repaid is $13 \times 2000 + L_{13} \approx 26\,000 + 515.72 = 26\,515.72$, so the borrower repays £26 515.72 in total.