OPTIONAL PURE / RECURRENCE RELATIONS

Proof by Induction - Recurrence Relations

Questions by topic 12 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · CIE 9231

Questions

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1. The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2, \qquad u_{n+1} = u_n + n2^n$$

Prove by induction that, for all integers $n \geq 1$,

$$u_n = (n-2)2^n + 4$$

[6]

2. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 4, \qquad u_{n+1} = 3u_n + 3^{n+1} - 2n + 1, \qquad n \geq 1$$

Prove by induction that $u_n = n\,(3^n + 1)$ for all positive integers n. **[6]**

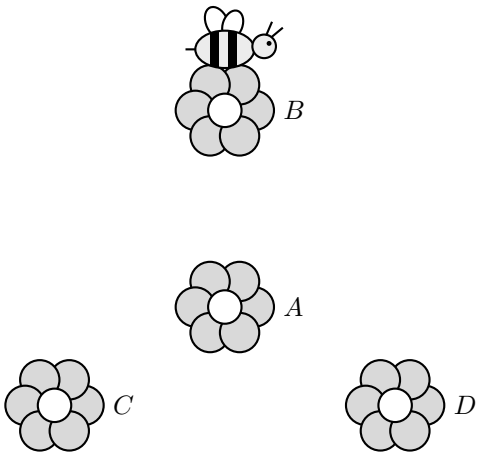
3. A sequence of numbers is defined by

$$u_1 = 2, \qquad u_2 = 6$$

$$u_{n+2} = 5u_{n+1} - 6u_n + 2, \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 3^n - 2^n + 1$$ **[6]**



4. Four flowers, A, B, C and D, grow in a flower bed, with A in the middle and B, C and D around it, as shown in the diagram above. A bee flies repeatedly from one flower to another.
The bee starts on flower B, and in a model of its flights

- when the bee is on A, it flies to each of B, C and D with probability $\frac{1}{3}$
- when the bee is on one of B, C or D, it flies to A with probability $\frac{2}{5}$ and to each of the other two outer flowers with probability $\frac{3}{10}$.

Let p_n be the probability that the bee is on flower A after n flights.

(a) Show that $p_2 = \frac{6}{25}$. **[2]**

The probabilities $p_1, p_2, p_3, \ldots$ satisfy the recurrence relation

$$p_1 = \frac{2}{5}, \qquad p_{n+1} = \frac{2}{5}\left(1 - p_n\right), \qquad n \geq 1$$

(b) Prove by induction that, for $n \geq 1$,

$$p_n = \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^n$$ **[6]**

(c) Use the result in part (b) to find, with a reason, the long-term probability that the bee is on flower A. **[1]**

5. For positive integers n, the sequence of Lucas numbers, L_n, starts with the terms $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, ... and is given by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

(a) Show that $L_{4k+2} = 3L_{4k-1} + 2L_{4k-2}$ for every positive integer k. **[3]**

(b) Prove by induction that L_{4n-2} is divisible by 3 for all positive integers n. **[4]**

6. The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$ and, for $n \geq 1$,

$$u_{n+1} = \frac{5u_n + 6}{u_n + 4}$$

(a) By considering $u_{n+1} - 3$, or otherwise, prove by mathematical induction that $0 < u_n < 3$ for all positive integers n. **[6]**

(b) Show that $u_{n+1} > u_n$ for all positive integers n. **[3]**

7. A sequence of integers is defined by $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$.
Prove by induction that, for each positive integer n,

$$\left(1 + \sqrt{2}\right)^n = u_n + u_{n-1} + u_n\sqrt{2}$$

[6]

8. The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 16$ and

$$u_{n+1} = 5u_n + 16(n+1), \qquad n \geq 1$$

(a) Prove by induction that $u_n = 5^{n+1} - 4n - 5$ for all positive integers n. **[6]**

(b) Using the recurrence relation, deduce that $5^{n+1} - 4n - 5$ is divisible by 16 for all positive integers n. **[4]**

9. A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$, $u_2 = 10$ and

$$u_{n+2} = 3(n+2)u_{n+1} - 2(n+1)(n+2)u_n, \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = n!\,(2^n + 1)$$

[6]

10. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 197, \qquad u_2 = 409, \qquad u_{n+2} = 6u_n - u_{n+1}, \qquad n \geq 1$$

(a) Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = 100(2^n) + (-3)^n$$ **[6]**

(b) Hence find the value of the first negative term of the sequence. **[4]**

11. The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$ and

$$u_{n+1} = \frac{u_n^2 + 5}{2u_n}, \qquad n \geq 1$$

(a) Prove by induction that $u_n > \sqrt{5}$ for all positive integers n. **[6]**

(b) By considering $u_{n+1} - u_n$, use the result of part (a) to show that $u_{n+1} < u_n$ for all positive integers n. **[3]**

(c) Show that $u_{n+1} - \sqrt{5} < \frac{1}{2}\left(u_n - \sqrt{5}\right)$ for all positive integers n. **[2]**

12. The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and, for all integers $n \geq 1$,

$$a_{n+1} = 2a_n^2 + 1$$

(a) Prove by mathematical induction that $\ln a_n \leq \left(2^{n-1} - 1\right) \ln 3$ for all integers $n \geq 1$. **[6]**

(b) Hence show that $\ln a_{n+1} - \ln a_n \leq 2^{n-1} \ln 3$ for all integers $n \geq 1$. **[2]**