OPTIONAL PURE / RECURRENCE RELATIONS

Modelling with Recurrence Relations

Questions by topic 10 questions · 120 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

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1. A vet begins a course of treatment for an animal when a blood sample taken from it is found to contain 600 000 bacteria of a particular type. The aim of the treatment is to reduce the number of bacteria in such a sample to below 10 000, the level at which the animal's immune system can control the infection without help. If, at any time after the treatment has started, a sample contains 10 000 or more bacteria, then the animal needs further treatment.
The progress of the infection is modelled by the recurrence relation

$$P_{n+1} = \frac{3P_n}{n+2} + \frac{4n}{P_n}$$

for $n \geq 0$, with $P_0 = 600$, where P_n is the number of bacteria, **in thousands**, in a sample taken n days after the treatment started.
The table below shows the values of P_n given by the model for some values of n, correct to 2 decimal places where appropriate.

n	0	1	2	3	4	5	6	7	8	9
P_n	600	900	900.00	675.01	405.03			32.87	11.81	6.25
n	10	15	20	40	60	80	100	200	300	400
P_n	7.46	8.37	9.51	13.04	15.81	18.17	20.25	28.46	34.79	40.13

(a) Find the values of P_5 and P_6, giving each correct to 2 decimal places. **[3]**

(b) Assuming that the model is valid, and using the values of P_0 to P_{10},

(i) describe how the number of bacteria changes during the first 10 days of the treatment, **[1]**

(ii) state the number of days after the treatment started when the animal's immune system is first expected to be able to control the infection for itself. **[1]**

(c) (i) For large values of n, the values in the table suggest that P_n is approximately $f(n)$, where $f(n) = an^k$ for constants a and k. Use the values of P_{100} and P_{400} to suggest suitable values of a and k, where k is a simple fraction. **[2]**

(ii) Use your function f to estimate the number of days after the treatment started when the animal may once again need treatment. **[1]**

(iii) Using the value of P_{20} in the table and the recurrence relation, verify or correct your estimate in part **(c)(ii)**. **[2]**

(d) Give one reason why the predictions of the model for large values of n may be unreliable. **[1]**

2. In a conservation project in a nature reserve, scientists are modelling the population of one species of animal.
The initial population of the species, P_0, is 6000. After n years, the population is P_n. The scientists believe that the year-on-year change in the population can be modelled by a recurrence relation of the form

$$P_{n+1} = 2.5P_n(1 - kP_n), \qquad n \geq 0$$

where k is a constant.

(a) The initial aim of the project is to keep the population constant at 6000. Find the value of k for which this happens, according to this model. **[2]**

(b) A few years later, with the population still at 6000, the scientists propose adding 250 animals to the reserve at the end of each year. Using the value of k found in part **(a)**, the population in this scenario is modelled by the recurrence relation

$$P_0 = 6000, \qquad P_{n+1} = 2.5P_n(1 - 0.0001P_n) + 250, \qquad n \geq 0$$

Use your calculator to find the long-term behaviour of P_n predicted by this recurrence relation, describing the way in which P_n approaches its long-term value. **[2]**

(c) The scientists decide **not** to add any animals at the end of each year. Instead, further research predicts that certain factors will remove 1440 animals from the population each year.

(i) Write down a modified form of the recurrence relation given in part **(b)** that models the population when 1440 animals are removed each year and no animals are added. **[1]**

(ii) Starting from $P_0 = 6000$, use your calculator to describe the behaviour of P_n predicted by this modified recurrence relation over the next ten years. **[1]**

(iii) Find algebraically the two values of P for which $P_n = P$ gives $P_{n+1} = P$ in this modified recurrence relation. Explain which of these values is the long-term population predicted when $P_0 = 6000$. **[4]**

(iv) Determine the value that should replace 0.0001 in this modified recurrence relation so that P_n remains constant at 6000. **[2]**

3. An investment company offers customers a three-year investment scheme, in which a fixed bonus of £200 is added at the end of each year.
The scheme is modelled by the recurrence relation

$$I_0 = \alpha, \qquad I_{n+1} = 0.0008I_n^2 - 0.6I_n + 200, \qquad n \geq 0$$

where I_n is the value in pounds of the scheme n years after its start, and £α, where $\alpha > 0$, is the **integer** value of the initial sum invested.

(a) Determine the minimum initial sum invested that guarantees that the value of the scheme increases every year. Justify your answer. **[4]**

(b) The investment company decides to make the following change to the scheme. At the end of each year, the value of I_n is always rounded **down** to the nearest £10.

(i) Write down a modified recurrence relation that takes this change into account. **[1]**

(ii) A customer invests £1900 in the three-year investment scheme. Work out how much less the customer's investment would be worth after three years under the new scheme compared with the original scheme, giving your answer to the nearest penny. **[6]**

4. A streaming service models the number of subscribers that it will have at the end of each month after its launch. The service assumes that

- a constant proportion, r (where $0 < r < 1$), of the previous month's subscribers will cancel during the month
- a constant number, k, of new subscribers will join each month.

Let u_n (where $n \geq 0$) represent the number of subscribers that the service will have at the end of n months, so that u_0 is the number of subscribers at launch.
The service has 60 000 subscribers at launch.

(a) By setting up a first-order recurrence relation for u_{n+1} in terms of u_n, determine an expression for u_n in terms of n, r and k. **[6]**

The service now assumes that 4% of the previous month's subscribers will cancel each month and that 1200 new subscribers will join each month.
According to the model, the service will first have fewer than 32 000 subscribers at the end of the mth month.

(b) Using logarithms, determine the value of m, and explain why, according to the model, the number of subscribers will never fall below 30 000. **[4]**

5. A free online coding course is launched, and 40 000 learners enrol on its first run, which lasts 8 weeks. Data from similar courses suggest that, of every 50 000 learners who enrol, only 5006 are still taking part at the end of the 8 weeks, when they complete the course.
It is assumed that the steady decline in the number of learners taking part, from all causes, can be explained by a weekly drop-out factor, r, which is constant across each week of the course.
Let T_k denote the number of learners still taking part at the end of k weeks after the start of the course.

(a) (i) Write down a recurrence relation for T_k, stating the values of k for which it is valid, and hence show that $T_k = 40\,000(1-r)^k$ for $0 \leq k \leq 8$. **[3]**

(ii) Show that $r = 0.250$, correct to 3 significant figures. **[2]**

Learners who complete the course join an online community, and a proportion p of them, where $0 < p < 1$, act as volunteer tutors for future runs of the course. In a simple model of the community, the following assumptions are made.

- From the end of the course, the weekly drop-out factor for members of the community is $\frac{1}{5}r$, and it is the same for tutors and for other members.
- The course can continue to run provided there are at least **200** volunteer tutors still active 26 weeks after the start of the first run.

(b) (i) Find the smallest value of p for which the course can continue to run according to the model. **[4]**

(ii) Write down one assumption that has been made in order to obtain this result. **[1]**

Each volunteer tutor still active 26 weeks after the start of the first run then recruits 150 new learners, who enrol on the second run of the course.

(c) In the case where exactly 200 volunteer tutors are active 26 weeks after the start of the first run, find the total number of learners who enrol on the second run. By comparing this with the first run, give one criticism of the assumption that the course can continue to run provided there are at least **200** volunteer tutors active at that time, and find the least number of volunteer tutors needed for the second run to have at least as many learners as the first. **[3]**

6. Priya borrows £12 000 from a bank to buy a car.
At the start of each year, Priya pays the bank a fixed amount, £F, or the whole of the amount owed if this is less than £F. At the end of each year, the bank adds 4% interest to the amount still owed.
Let £A_n be the amount owed at the start of year n, before Priya's payment for that year, so that $A_1 = 12\,000$.

(a) Write down an expression for A_{n+1} in terms of A_n and F, valid when $A_n \geq F$. **[1]**

(b) Prove by induction that, for each $n \geq 1$ such that all of Priya's payments before year n are the full £F,

$$A_n = (12\,000 - 26F)\,1.04^{n-1} + 26F$$ **[6]**

(c) Given that $F = 1000$, find the number of payments that Priya makes and the amount of her final payment, giving this amount to the nearest penny. **[5]**

7. Aisha opens a bank account to help her save for a car. The account is empty when she opens it.
In the first month she pays £k into the account, and in each month after that she pays in £20 more than she paid in the month before. In each month after the first, she also allows herself to spend one fifth of what was in the account at the end of the previous month.
Let u_n, where $n \geq 1$, be the amount, in pounds, in the account at the end of month n, so that $u_1 = k$.

(a) By setting up a first-order recurrence relation for u_{n+1} in terms of u_n and n, determine an expression for u_n in terms of n and k. **[8]**

At the end of the 12th month, Aisha needs to have at least £2500 in the account to pay for the car.

(b) Determine, to the nearest penny, the minimum amount that Aisha should pay into the account in the first month. **[2]**

8. A colony of seabirds is established on a small island. The number of birds in the colony t years after it is established is denoted by N_t, and the initial number of birds is N_0.
When the colony is small, few of its chicks survive, so the following discrete population model is proposed for N_t.

$$\text{Model I: } N_{t+1} = \frac{200N_t^2}{6400 + N_t^2}$$

(a) (i) Find the three steady state values of the number of birds for Model I. **[4]**

(ii) Show that $0 < N_{t+1} - N_t < 160 - N_t$ when $40 < N_t < 160$. **[4]**

(iii) Hence describe the long-term behaviour of the number of birds predicted by Model I when $40 < N_0 < 160$. **[2]**

An alternative discrete population model is proposed for N_t, where $\text{INT}(x)$ denotes the integer part of x.

$$\text{Model II: } N_{t+1} = \text{INT}\left(\frac{200N_t^2}{6400 + N_t^2}\right)$$

(b) (i) Given that $N_0 = 80$, find the value of N_3 for each of the two models, giving the value for Model I correct to 1 decimal place. **[2]**

(ii) Show that 159 is a steady state value of the number of birds for Model II. Given also that, for Model II, $N_{t+1} > N_t$ whenever N_t is an integer with $80 \leq N_t \leq 158$, state how the long-term behaviour predicted by Model II when $N_0 = 80$ differs from that predicted by Model I. **[2]**

9. A young tree is planted in a garden, and its height is measured on the first day of each year.
Let H_n metres be the height of the tree on the first day of year n. The measurements give $H_1 = 1.2$, $H_2 = 1.8$ and $H_3 = 2.2$.
A gardener models the growth of the tree by assuming that the increase in its height during each year is a fixed fraction p of the increase in its height during the previous year, where $0 < p < 1$.

(a) Explain why the model gives

$$H_{n+2} = (1+p)H_{n+1} - pH_n, \qquad n \geq 1$$

and use the measurements to show that $p = \frac{2}{3}$. **[3]**

(b) Prove by induction that

$$H_n = 3 - 1.8\left(\frac{2}{3}\right)^{n-1}, \qquad n \geq 1$$ **[6]**

(c) Explain why, according to the model, the height of the tree is always less than 3 metres, and find the height of the tree in the long term. **[2]**

(d) A second tree of the same species is planted in richer soil. Its heights on the first days of years 1 and 2 are also 1.2 metres and 1.8 metres, and its growth follows the same model with a different value of p. The gardener predicts that its height in the long term is 4 metres.
Find, as a percentage to 3 significant figures, the value of p for the second tree. **[2]**

10. A sequence u_n, where $n \geq 0$, satisfies the first-order recurrence relation

$$u_{n+1} = \alpha u_n - \beta\alpha^n \qquad (*)$$

where α and β are non-zero constants.

(a) Find, in terms of α, β and n, the general solution of $(*)$. **[4]**

(b) Given that $u_0 = L$ and $u_T = 0$, where T is a positive integer, show that

$$u_n = L\alpha^n \left(1 - \frac{n}{T}\right) \quad \textbf{[4]}$$

Jordan borrows a sum of money, L, to buy a flat.
On each anniversary of the loan being taken out, 6% interest is charged on the outstanding balance and Jordan then makes that year's repayment.
Each annual repayment is 6% larger than the one before, and the whole debt is cleared in exactly 25 years.
The recurrence relation $(*)$ can be used to model this situation, where u_n is the outstanding debt n years after the loan is taken out, immediately after that year's repayment.

(c) (i) State the value of α. **[1]**

(ii) Explain what β represents in the context of the loan. **[1]**

(iii) Find, as a multiple of L, the amount of Jordan's first annual repayment. **[1]**

(d) Find, according to the model, the number of complete years after which the outstanding debt first falls below the original loan L. **[3]**