OPTIONAL PURE / RECURRENCE RELATIONS

Behaviour and Limits of Sequences

Questions by topic 11 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

Question 1	*5 marks*	2
Question 2	*8 marks*	3
Question 3	*6 marks*	4
Question 4	*5 marks*	5
Question 5	*10 marks*	6
Question 6	*10 marks*	7
Question 7	*8 marks*	8
Question 8	*11 marks*	9
Question 9	*14 marks*	10
Question 10	*9 marks*	11
Question 11	*8 marks*	12

1. The sequence u_n is defined for $n \geq 1$ by

$$u_n = 3 + 8\left(-\frac{3}{4}\right)^n$$

(a) State the limit, α, of the sequence, and describe the behaviour of the sequence as n increases. **[2]**

(b) Using logarithms, find the smallest integer N such that $|u_n - \alpha| < 0.01$ for all $n \geq N$. **[3]**

2. Three sequences, a_n, b_n and c_n, are defined for $n \geq 1$ by the following recurrence relations.

$$a_{n+1}(1 - a_n) = 1 \qquad \text{with } a_1 = 2$$
$$b_{n+1} = \sqrt{b_n + 3} \qquad \text{with } b_1 = 1$$
$$c_{n+1} + nc_n = 1 \qquad \text{with } c_1 = 2$$

The output from a spreadsheet which presents the first 10 terms of a_n, b_n and c_n is shown below.

	A	B	C	D
1	n	a_n	b_n	c_n
2	1	2	1	2
3	2	−1	2	−1
4	3	0.5	2.23607	3
5	4	2	2.28825	−8
6	5	−1	2.29962	33
7	6	0.5	2.30209	−164
8	7	2	2.30263	985
9	8	−1	2.30274	−6894
10	9	0.5	2.30277	55 153
11	10	2	2.30277	−496 376

(a) **Without** attempting to solve any recurrence relations, describe the apparent behaviour, including as $n \to \infty$, of each of the three sequences. **[6]**

(b) Given that b_n converges to a limit L, use the recurrence relation for b_n to find the exact value of L. **[2]**

3. **(a)** The convergent sequence a_n is defined by $a_0 = 6$ and

$$a_{n+1} = \frac{3a_n}{4} + \frac{5}{a_n}$$

for $n \geq 0$.
Find the exact value of the limit of the sequence. **[2]**

(b) The convergent sequence b_n is defined by $b_0 = 5$ and

$$b_{n+1} = \frac{b_n}{2} + \frac{k}{b_n}$$

for $n \geq 0$, where k is a positive constant.
Given that the limit of the sequence is $k - 4$, determine the value of k. **[4]**

4. For all positive integers n, the term u_n of a sequence is the integer for which

$$u_n \equiv 2^n + 3n \pmod 9, \qquad 0 \leq u_n \leq 8$$

(a) Show that $u_{n+6} = u_n$ for all positive integers n. **[3]**

(b) Hence describe the behaviour of the sequence, justifying your answer. **[2]**

5. The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 5$ and, for $n \geq 1$,

$$u_{n+1} = \frac{12u_n + 15}{2u_n + 5}$$

(a) By considering $5 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 5$ for all positive integers n. **[5]**

(b) Show that $u_{n+1} > u_n$ for $n \geq 1$. **[3]**

(c) Given that u_n tends to a limit L as $n \to \infty$, form a quadratic equation satisfied by L and hence find the value of L, giving a reason for rejecting the other root of your equation. **[2]**

6. The sequence $a_1, a_2, a_3, \ldots$ is such that $2 < a_1 < 4$ and

$$a_{n+1} = \frac{a_n^2 + 8}{6}$$

for every positive integer n.

(a) Prove by mathematical induction that $2 < a_n < 4$ for every positive integer n. **[5]**

(b) Hence prove that $a_{n+1} < a_n$ for every positive integer n. **[3]**

(c) Given that the sequence converges to a limit L, find the value of L, giving a reason for your answer. **[2]**

7. A sequence u_n is defined by

$$u_1 = a \quad \text{and} \quad u_{n+1} = \frac{u_n - 3}{u_n + 1} \quad \text{for } n \geq 1$$

where a is a real constant.

(a) Determine, with justification, the behaviour of the sequence for all possible values of a. **[6]**

(b) Given that $a = 5$, find the sum of the first 100 terms of the sequence. **[2]**

8. The sequence of real numbers u_n is given by $u_1 = 3$ and

$$u_{n+1} = k + \frac{6}{u_n}$$

where k is a real constant.
For each of the following cases, either determine a value of k or prove that one does not exist.

(a) u_n is constant. **[2]**

(b) u_n is periodic, with period 2. **[4]**

(c) u_n is periodic, with period 3. **[5]**

9. The sequence u_n satisfies the recurrence relation

$$u_n = u_{n-1} + 6u_{n-2}, \quad n \geq 2 \qquad (*)$$

with $u_0 = 3$ and $u_1 = 4$.

(a) Find u_n in terms of n. **[5]**

A new sequence v_n is defined, for $n \geq 1$, by

$$v_n = \frac{u_n}{u_{n-1}}$$

(b) (i) Use $(*)$ to show that, for $n \geq 2$,

$$v_n = 1 + \frac{6}{v_{n-1}}$$ **[2]**

(ii) Deduce that, if v_n tends to a limit, then that limit is either 3 or -2. **[2]**

(c) Use your answer to part (a) to show that $\lim_{n\to\infty} v_n = 3$. **[3]**

(d) The sequence w_n satisfies the same recurrence relation, $w_n = w_{n-1} + 6w_{n-2}$ for $n \geq 2$, with $w_0 = 5$ and $w_1 = k$, where k is a constant.
Given that $\frac{w_n}{w_{n-1}}$ tends to a limit that is not 3, find the value of k. **[2]**

10. The sequence of positive real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \frac{a_n^2}{2 + a_n}$$

(a) Prove by mathematical induction that $\ln a_n \leq \left(1 - 2^{n-1}\right) \ln 2$ for all positive integers n. You may use the fact that $\ln(2 + x) > \ln 2$ for $x > 0$. **[6]**

(b) Show that $\ln a_n - \ln a_{n+1} > 2^{n-1} \ln 2$ for all positive integers n, and hence find the limit of $\frac{a_{n+1}}{a_n}$ as $n \to \infty$, giving a reason for your answer. **[3]**

11. The sequence A_n is given for all integers $n \geq 0$ by

$$A_n = \frac{(I_{n+1})^2}{I_n \, I_{n+2}}$$

where

$$I_n = \int_0^4 x^n \sqrt{4-x} \, \mathrm{d}x$$

- Show that A_n increases monotonically, that is, that $A_{n+1} > A_n$ for all $n \geq 0$.
- Show that A_n converges to a limit, A, whose exact value should be stated.

[8]