OPTIONAL PURE / RECURRENCE RELATIONS

2nd Order Recurrence Relations

Questions by topic 12 questions · 124 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

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1. The sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_{n+2} = u_{n+1} + 12u_n, \qquad n \geq 1$$

Given that $u_1 = 11$ and $u_2 = 23$, find an expression for u_n in terms of n. **[5]**

2. A sequence satisfies the recurrence relation

$$4H_{n+2} = 4H_{n+1} - H_n$$

for $n \geq 0$, with $H_0 = 5$ and $H_1 = 4$.

Solve the recurrence relation to find an expression for H_n in terms of n. **[5]**

3. A sequence u_n, where $n \geq 0$, satisfies the recurrence relation

$$3u_{n+2} = 5u_{n+1} + 2u_n - 15\left(\frac{1}{2}\right)^n$$

(a) Find the general solution of this recurrence relation. **[5]**

A particular solution of this recurrence relation has $u_0 = 7$ and $u_1 = k$, where k is a constant, and it is given that this sequence converges.

(b) Determine the value of k, and state the limit of the sequence. **[4]**

4. **(a)** Determine the general solution of the recurrence relation

$$u_{n+2} - 3u_{n+1} + 2u_n = 0$$ **[2]**

(b) Consider the recurrence relation

$$u_{n+2} - 3u_{n+1} + 2u_n = 2 - 6n \qquad (*)$$

(i) Show that $(*)$ has no particular solution of the form $u_n = pn + q$, where p and q are constants. **[2]**

(ii) Using your answer to part (a), determine the general solution of $(*)$. **[3]**

In the rest of this question the sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation $(*)$.
You are given that $u_0 = 3$ and $u_1 = 5$.

(c) Determine the particular solution for u_n. **[3]**

You are given that, as n increases, once the values of u_n start to decrease, the sequence is a decreasing sequence from that point onwards.

(d) Use your answer to part (c) to determine, by direct calculation, the greatest value taken by terms of the sequence. You should show any values that you rely on in your argument. **[2]**

5. The sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n + 15 \times 3^n, \qquad n \geq 0$$

(a) Determine the general solution of this recurrence relation. **[5]**

(b) Hence find u_n in terms of n, given that $u_0 = 1$ and $u_2 = 2u_1$. **[3]**

6. The general solution of the second-order recurrence relation

$$u_{n+2} + k_1 u_{n+1} + k_2 u_n = 0, \qquad n \geq 0$$

is given by

$$u_n = \left(2\sqrt{3}\right)^n \left(A\cos\left(\frac{5n\pi}{6}\right) + B\sin\left(\frac{5n\pi}{6}\right)\right)$$

where k_1 and k_2 are real constants and A and B are arbitrary constants.

(a) Find the value of k_1 and the value of k_2. **[3]**

(b) Given that $u_1 = 3$ and $u_2 = 6$, find the exact values of A and B. **[3]**

7. A sequence is defined by the recurrence relation

$$u_{n+2} = 2u_{n+1} - 4u_n$$

for $n \geq 0$, with $u_0 = 1$ and $u_1 = 4$.

(a) Find an exact real expression for u_n in terms of n. **[7]**

A second sequence is defined by $v_n = a^n u_n$ for $n \geq 0$, where a is a positive constant.

(b) In each of the following cases, describe the behaviour of v_n as $n \to \infty$, stating the period of the sequence in any case where it is periodic.

- $a = \frac{1}{3}$
- $a = \frac{1}{2}$
- $a = \frac{2}{3}$

[6]

8. The sequence $T_0, T_1, T_2, \ldots$ satisfies the second-order recurrence relation

$$T_{n+2} + 2T_{n+1} + 4T_n = 7n + 25$$

for $n \geq 0$, where $T_0 = 5$ and $T_1 = 8$.

(a) Solve the recurrence relation to find an expression for T_n in terms of n. **[9]**

(b) Determine the value of T_{16}. **[2]**

9. **(a)** Determine the general solution of the recurrence relation

$$16u_{n+2} - 24u_{n+1} + 9u_n = c, \qquad n \geq 0$$

where c is a constant. **[4]**

The manager of a company uses the recurrence relation in part (a) to model the number of containers held by the company, using the sequence $u_0, u_1, u_2, \ldots$ which satisfies this recurrence relation.

The quantity u_0 represents the number of containers held by the company at the **start** of January 2020. The quantity u_n, for $n \geq 1$, represents the number of containers held by the company at the **end** of month n, where the first month is January 2020. If necessary, the quantity u_n is rounded to the nearest integer.

The company holds 30 containers at the start of January 2020 and 60 containers at the end of January 2020.

(b) In an initial model the manager uses $c = -6$.

(i) Determine the particular solution for u_n when $c = -6$. **[3]**

(ii) The model predicts a single peak value for u_n.
Find, by direct calculation, the largest number of containers held by the company according to the model. **[2]**

(iii) Determine the month in which the model in part (b)(i) breaks down, giving a reason for your answer. **[4]**

(c) In a second model the manager uses $c = 24$ instead of $c = -6$.
Show that the second model predicts that eventually the number of containers held by the company will be a constant k, where k is to be determined. **[2]**

10. The equation

$$x^2 - 4x + 1 = 0$$

has roots α and β, where $\alpha > \beta$.

The sequence S_n is defined by $S_n = \alpha^n + \beta^n$ for $n \geq 0$.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

(b) (i) Determine the values of S_2 and S_4. **[3]**

(ii) Show that $S_{n+2} = 4S_{n+1} - S_n$ for $n \geq 0$. **[2]**

(iii) Deduce that S_n is an even integer for all $n \geq 0$. **[2]**

(c) A student approximates the terms of the sequence S_n using the formula $T_n = \alpha^n$.

(i) Explain why $T_n < S_n$ for every $n \geq 1$. **[1]**

(ii) State, with a reason, how the value of T_n should be rounded so that the result is equal to S_n for every $n \geq 1$. **[2]**

11. The sequences x_n and y_n are defined, for $n \geq 1$, by

$$x_{n+1} = x_n + 2y_n + 4 \quad \text{and} \quad y_{n+1} = 3x_n + 2y_n - 1$$

with $x_1 = 3$ and $y_1 = a$, where a is a constant.

(a) Show that $x_{n+2} - 3x_{n+1} - 4x_n = -6$. **[2]**

(b) Solve the second-order recurrence relation given in part (a) to obtain an expression for x_n in terms of a and n. **[8]**

Given that $x_5 = 1023$,

(c) find the value of a. **[2]**

(d) find an expression for y_n in terms of n. **[2]**

12. The sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation

$$2u_{n+2} - 7u_{n+1} + 3u_n = 4n^2 + 6n - 3$$

(a) Determine the general solution of the recurrence relation. **[6]**

(b) Hence determine the particular solution of the recurrence relation for which $u_0 = 5$ and $u_1 = 4$. **[3]**

(c) Show, by direct calculation, that your solution in part (b) gives the correct value for u_2. **[1]**

The sequence $v_0, v_1, v_2, \ldots$ is defined by $v_n = u_{n+1} - 3u_n$, where u_n denotes the particular solution found in part (b).
You are given that, for some constant p, the sequence with nth term $\frac{v_n}{n^p}$, for $n \geq 1$, converges to a finite non-zero limit, q, as $n \to \infty$.

(d) Determine p and q. **[5]**