OPTIONAL PURE / RECURRENCE RELATIONS

1st Order Recurrence Relations

Questions by topic 12 questions · 123 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

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1. The sequence u_n satisfies a recurrence relation of the form $u_{n+1} = au_n + b$, for $n \geq 1$, where a and b are constants.
The first four terms of the sequence are $u_1 = 4$, $u_2 = -5$, $u_3 = 13$ and $u_4 = -23$.

(a) Find the values of a and b. **[2]**

(b) By solving the recurrence relation, find an expression for u_n in terms of n. **[5]**

(c) Hence find the value of n for which $u_n = 769$. **[2]**

2. A student takes out a loan of £6000 from a bank.
At the end of each month

- the student repays £P;
- the bank then adds interest of 1% of the amount still owed after that repayment.

Let U_n be the amount of money, in pounds, owed n months after the loan was taken out, so that $U_0 = 6000$.

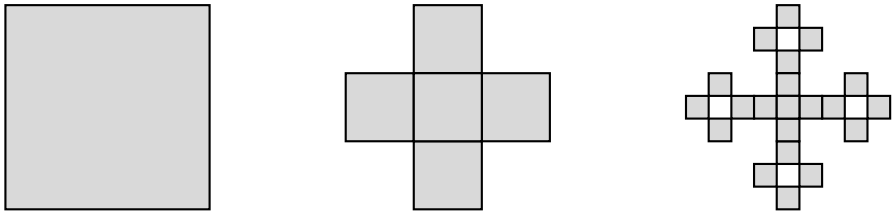
(a) Explain, in the context of the problem, why

$$U_n = 1.01U_{n-1} - 1.01P, \qquad n \geq 1$$

[2]

(b) Solve this recurrence relation, giving U_n in terms of n and P. **[5]**

(c) The loan is completely repaid at the end of exactly 24 months.
Hence find the value of P, giving your answer to the nearest penny. **[3]**



3. The diagram above shows the first three stages of a pattern that is created by a recursive process. The process starts with a single square and proceeds as follows:

- each square is divided into a 3×3 grid of 9 equal smaller squares, and is replaced by the 4 of these that lie at the middle of its sides;
- once every square has been replaced, one further square, of the same size as the new squares, is added to fill the gap at the centre of the pattern.

Let u_n be the number of squares in the pattern in stage n, where stage 1 is the original square.

(a) Explain why u_n satisfies the recurrence relation

$$u_1 = 1, \qquad u_{n+1} = 4u_n + 1, \qquad n \geq 1$$ **[2]**

(b) Solve this recurrence relation. **[5]**

(c) The initial square has side length 9.
Find, in terms of n, the total area of the squares in stage n, giving your answer in the form

$$p\left(\left(\frac{4}{9}\right)^n - \left(\frac{1}{9}\right)^n\right)$$

where p is an integer. **[3]**

4. A patient is treated with a course of injections of a medicine.
Before the course there was none of the medicine in the patient's body.
The injections are given at the same time each day. The first injection contains 60 mg of the medicine and every later injection contains 20 mg.
Between injections, the amount of the medicine in the patient's body decreases naturally by 50%.
Let v_n mg be the amount of the medicine in the patient's body immediately **before** an injection is given, n days after the first injection.

(a) Explain why v_n satisfies the recurrence relation

$$v_1 = 30, \qquad v_{n+1} = 0.5v_n + 10, \qquad n \geq 1$$ **[2]**

(b) Solve this recurrence relation to find v_n in terms of n. **[5]**

(c) The course is safe and effective only if, at **all times** after the first injection, the amount of the medicine in the patient's body is at most 65 mg and at least 22 mg.
Determine whether the course is safe and effective for this patient, giving a reason for your answer. **[4]**

5. A sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_{n+1} = 3u_n + 2^{n+2}, \qquad n \geq 1$$

(a) Find the general solution of this recurrence relation. **[4]**

(b) Given that $u_1 = 1$, find u_n in terms of n, and hence show that u_n is odd for every positive integer n. **[3]**

6. A lake is stocked with fish as part of a conservation project. The number of fish in the lake n years after the start of the project is F_n. At the start of the project, there are 2000 fish in the lake.
The first model for the number of fish is that, each year, a fixed proportion of the fish in the lake is lost, and then 300 fish are added to the lake at the end of the year. This model predicts that, once the fish have been added at the end of the first year, there will be 1900 fish in the lake.

(a) Use this model to express the number of fish as a first-order recurrence relation. **[1]**

In a second model, the proportion of fish lost each year is the same as in the first model, but the number of fish added increases each year: 300 fish are added at the end of the first year, and at the end of each later year 50 more fish are added than at the end of the year before.

(b) (i) Show that the recurrence relation for the second model is

$$F_0 = 2000, \qquad F_{n+1} = 0.8F_n + 50n + 300$$ **[1]**

(ii) Solve this recurrence relation. **[6]**

(iii) Give one criticism of the second model for predicting the number of fish in the lake in the long term. **[1]**

7. A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = 4$ and

$$u_{n+1} = \frac{3u_n}{n+2}, \qquad n \geq 0$$

The solution of this recurrence relation can be written in the form

$$u_n = \frac{a\,k^n}{(n+b)!}$$

where a, k and b are constants. Find the values of a, k and b, and verify that the resulting expression satisfies both the recurrence relation and the initial condition. **[6]**

8. The sequence u_n, where k is a constant with $k \neq 0$ and $k \neq 1$, satisfies the first-order recurrence relation

$$u_{n+1} = ku_n + k + 1, \qquad n \geq 0$$

(a) Find, in terms of k, the general solution of the recurrence relation. **[3]**

(b) Hence find, in terms of k, the value of u_0 for which every term of the sequence is equal, and find the value of k for which this common value is 3. **[2]**

(c) Given instead that $u_0 = 1$ and $u_2 = 13$, find the two possible values of k. **[4]**

9. The sequence u_n is defined by $u_1 = 4$ and $u_{n+1} = 3u_n - 4n^2$ for $n \geq 1$.

(a) Find u_n in terms of n. **[7]**

(b) A second sequence v_n satisfies the same recurrence relation, $v_{n+1} = 3v_n - 4n^2$ for $n \geq 1$, but has first term $v_1 = k$, where k is a constant.
Find the value of k for which v_n is a quadratic expression in n. **[2]**

10. A sequence $t_1, t_2, t_3, \ldots$ is defined by $t_1 = 4$ and

$$t_{n+1} = 3t_n + 2^n - 4n, \qquad n \geq 1$$

(a) Solve the recurrence relation to find t_n in terms of n. **[7]**

A second sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 5$ and

$$(n+1)u_{n+1} - 2nu_n = 6n^2 + 6n - 2n^3 - 1, \qquad n \geq 1$$

(b) (i) Explain why the standard method of a complementary function plus a particular solution, used for non-homogeneous first-order recurrence relations, **cannot** be applied directly to the recurrence relation for u_n. **[1]**

(ii) Verify that the solution of this recurrence relation is given by $u_n = an^2 + \frac{b}{n}$, where a and b are constants whose values are to be determined. **[5]**

11. A customer takes out a loan of £P from a bank. Interest is charged at a rate of 0.5% per month and is added to the outstanding amount of the loan at the end of each month. Immediately after the interest is added, the customer makes a monthly payment.
The payment made at the end of the first month is £M, and each monthly payment after that is £2 more than the payment before it.
Let L_n denote the outstanding amount of the loan at the end of month n **after** that month's payment has been made, with $L_0 = P$.

(a) Explain why the outstanding amount of the loan is modelled by the recurrence relation

$$L_{n+1} = 1.005L_n - M - 2n \qquad (*)$$

for $n \geq 0$, with $L_0 = P$. **[3]**

(b) Solve the recurrence relation $(*)$, giving L_n in terms of n, M and P. **[7]**

(c) The loan amount is £12 000 and the loan is fully repaid by the payment made at the end of month 48. Find, to the nearest pound, the value of M, and hence find the amount of the final monthly payment. **[4]**

12. A building society models the repayment of a loan in the following way.

- A loan of £C is taken out at the start of the first year.
- At the start of each year, the borrower makes a repayment of £R, which is subtracted from the amount owed.
- At the end of each year, interest of $\alpha\%$ ($\alpha > 0$) of the amount owed at that time is added to the amount owed.
- The amount owed immediately before the $(n+1)$th repayment is £L_n, so that $L_0 = C$.
- When the amount owed immediately before a repayment is £R or less, the borrower instead pays the whole amount owed, and the loan is cleared.

(a) Show that, while the loan is being repaid, $L_{n+1} = aL_n + b$, giving a and b in terms of α and R. **[2]**

(b) Find the solution of the recurrence relation $L_{n+1} = aL_n + b$ with $L_0 = C$, giving L_n in terms of a, b, C and n. **[5]**

(c) Deduce from parts (a) and (b) that, for the loan to be cleared,

$$R > \frac{\alpha C}{100 + \alpha}$$ **[3]**

A borrower takes out a loan of £20 000 at 5% interest and agrees to repay £2000 at the start of each year.

(d) (i) Use an algebraic method to find the number of repayments the borrower makes, including the final repayment. **[3]**

(ii) Find the total amount that the borrower repays over the lifetime of the loan. **[3]**