OPTIONAL PURE / RECURRENCE RELATIONS

Proof by Induction - Recurrence Relations

Mark scheme

12 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · CIE 9231

Questions

Question	Marks by part	Marks	Page
Question 1		6	4
Question 2		6	5
Question 3		6	6
Question 4	(a) 2 (b) 6 (c) 1	9	7
Question 5	(a) 3 (b) 4	7	8
Question 6	(a) 6 (b) 3	9	9
Question 7		6	10
Question 8	(a) 6 (b) 4	10	11
Question 9		6	12
Question 10	(a) 6 (b) 4	10	13
Question 11	(a) 6 (b) 3 (c) 2	11	14
Question 12	(a) 6 (b) 2	8	15
	Total	**94**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$n = 1$: $(1-2) \times 2^1 + 4 = 2 = u_1$	**B1**	Evaluates the formula at $n = 1$ and compares with $u_1 = 2$.
	Assume $u_k = (k-2)2^k + 4$ $u_{k+1} = u_k + k2^k$	**M1**	Assumes the result for $n = k$ and uses the recurrence with $n = k$, soi by the substituted expression.
	$u_{k+1} = (k-2)2^k + 4 + k2^k$ $= (2k-2)2^k + 4$	**M1 A1**	**M1** Substitutes the hypothesis into the recurrence and collects the terms in 2^k. Condone slips in the collection. **A1** Correct collected form, oe.
	$u_{k+1} = (k-1)2^{k+1} + 4$ $= \big((k+1)-2\big)2^{k+1} + 4$	**A1**	cso. The formula with $n = k+1$, or $(k-1)2^{k+1} + 4$ with $k - 1 = (k+1) - 2$ shown.
	True for $n = 1$; true for $n = k$ implies true for $n = k + 1$ So true for all integers $n \geq 1$	**E1**	States true for $n = 1$, that truth for $n = k$ implies truth for $n = k + 1$, and so true for all integers $n \geq 1$.
	Total for Question 1	**6**	

Question	Scheme	Marks	Notes
2	$n = 1$: $1 \times \left(3^1 + 1\right) = 4 = u_1$	**B1**	Formula evaluated at $n = 1$ and compared with $u_1 = 4$.
	Assume $u_k = k\left(3^k + 1\right)$ $u_{k+1} = 3u_k + 3^{k+1} - 2k + 1$	**M1**	States the hypothesis for $n = k$ and writes u_{k+1} from the recurrence with $n = k$. Condone sign slips.
	$= 3k\left(3^k + 1\right) + 3^{k+1} - 2k + 1$ $= k \times 3^{k+1} + 3k + 3^{k+1} - 2k + 1$	**M1**	Substitutes their hypothesis for u_k into the recurrence and expands. Condone sign slips.
	$= k \times 3^{k+1} + 3^{k+1} + k + 1$	**A1**	Correct expression, oe.
	$= (k + 1)\left(3^{k+1} + 1\right)$	**A1**	cso. Factorised form, or the formula with $n = k + 1$ shown equal to their expression.
	True for $n = 1$, and true for $n = k$ $\Rightarrow$ true for $n = k + 1$, so true for all positive integers n	**E1**	States true for $n = 1$, true for $n = k$ implies true for $n = k + 1$, and so true for all positive integers n.
	Total for Question 2	**6**	

Question	Scheme	Marks	Notes
3	$3^1 - 2^1 + 1 = 2 = u_1$ $3^2 - 2^2 + 1 = 6 = u_2$	**B1**	Needs both $n = 1$ and $n = 2$, each with the formula's value shown equal to the given term.
	Assumes true for $n = k$ and $n = k + 1$: $u_{k+2} = 5u_{k+1} - 6u_k + 2$	**M1**	Assumes the result for two consecutive values, soi, and uses the recurrence for the next term. Accept $n = k - 1$ and $n = k$ leading to $n = k + 1$.
	$u_{k+2} = 5\left(3^{k+1} - 2^{k+1} + 1\right)$ $\quad - 6\left(3^k - 2^k + 1\right) + 2$ $\quad = 3^k(15 - 6) - 2^k(10 - 6) + 1$	**M1**	Substitutes both assumed forms into the recurrence and collects terms in like powers of 3 and of 2. Condone sign slips.
	$= 9 \times 3^k - 4 \times 2^k + 1$	**A1**	Correct expression, oe such as $3 \times 3^{k+1} - 2 \times 2^{k+1} + 1$.
	$= 3^{k+2} - 2^{k+2} + 1$	**A1**	cso. The form $3^{k+2} - 2^{k+2} + 1$ must be seen.
	True for $n = 1$ and $n = 2$ True for $n = k$ and $n = k+1 \Rightarrow$ true for $n = k+2$ So true for all $n \in \mathbb{Z}^+$	**E1**	States true for $n = 1$ and $n = 2$, true for $n = k$ and $n = k + 1$ implies true for $n = k + 2$, and hence true for all positive integers n.
	Total for Question 3	**6**	

Question	Scheme	Marks	Notes
4(a)	$p_2 = \frac{2}{5} \times 0 + \frac{3}{5} \times \frac{2}{5}$ $= \frac{6}{25}$	**M1** **A1 AG**	**M1** Multiplies probabilities along the routes from B to A via an outer flower, adding any taken separately, as $\frac{3}{5} \times \frac{2}{5}$ or $2 \times \frac{3}{10} \times \frac{2}{5}$. **A1 AG** Correct products leading without error to $\frac{6}{25}$, with no nonzero probability given to the route via A.
		[2]	
4(b)	$n = 1$: $\frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right) = \frac{2}{7} + \frac{4}{35}$ $= \frac{2}{5} = p_1$	**B1**	Substitutes $n = 1$ into the formula and obtains $\frac{2}{5}$. B0 for $\frac{2}{5}$ stated with no substitution seen.
	Assumes $p_k = \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^k$ $p_{k+1} = \frac{2}{5}(1 - p_k)$	**M1**	Assumes the result for $n = k$ and uses the recurrence with $n = k$ for p_{k+1}, soi by the substituted expression.
	$p_{k+1} = \frac{2}{5}\left(1 - \frac{2}{7} + \frac{2}{7}\left(-\frac{2}{5}\right)^k\right)$ $= \frac{2}{5}\left(\frac{5}{7} + \frac{2}{7}\left(-\frac{2}{5}\right)^k\right)$	**M1**	Multiplies out their substituted expression, beyond the substitution itself, towards a constant plus a multiple of $\left(-\frac{2}{5}\right)^k$. Condone one sign slip.
	$= \frac{2}{7} + \frac{4}{35}\left(-\frac{2}{5}\right)^k$	**A1**	Correct expression, oe.
	$= \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)^k$ $= \frac{2}{7} - \frac{2}{7}\left(-\frac{2}{5}\right)^{k+1}$	**A1**	cso, with $\frac{4}{35}$ shown as $-\frac{2}{7}\left(-\frac{2}{5}\right)$ or the power combined explicitly.
	True for $n = 1$; true for $n = k$ implies true for $n = k + 1$ so true for all $n \geq 1$	**E1**	States all three points, true for $n = 1$, the implication from $n = k$ to $n = k + 1$, and true for all $n \geq 1$.
		[6]	
4(c)	$\left\|-\frac{2}{5}\right\| < 1 \Rightarrow \left(-\frac{2}{5}\right)^n \to 0$ $\Rightarrow p_n \to \frac{2}{7}$	**B1**	$\frac{2}{7}$ with the reason $\left(-\frac{2}{5}\right)^n \to 0$, or $\left\|-\frac{2}{5}\right\| < 1$. B0 for $\frac{2}{7}$ alone.
		[1]	
	Total for Question 4	**9**	

Question	Scheme	Marks	Notes
5(a)	$4k,\ 4k+1,\ 4k+2 \geq 3$, so the recurrence applies $L_{4k} = L_{4k-1} + L_{4k-2}$ $L_{4k+1} = L_{4k} + L_{4k-1} = \ldots$	**M1**	Applies the recurrence at least twice, working forwards or reducing subscripts, towards terms in L_{4k-1} and L_{4k-2}. Condone slips in collecting terms.
	$L_{4k+1} = 2L_{4k-1} + L_{4k-2}$	**A1**	Correct L_{4k} and L_{4k+1}, or an equivalent correct intermediate stage on another route.
	$L_{4k+2} = L_{4k+1} + L_{4k}$ $= (2L_{4k-1} + L_{4k-2}) + (L_{4k-1} + L_{4k-2})$ $= 3L_{4k-1} + 2L_{4k-2}$	**A1 AG**	Correct substitution into $L_{4k+2} = L_{4k+1} + L_{4k}$ oe, leading without error to the printed result. Condone the subscripts' range not stated.
		[3]	
5(b)	$n = 1$: $L_2 = 3$, divisible by 3	**B1**	
	Assume $L_{4k-2} = 3m$ $L_{4k+2} = 3L_{4k-1} + 2(3m)$	**M1**	Assumes the result for $n = k$, in any form, and substitutes it into an expression for L_{4k+2}, such as part (a).
	$= 3(L_{4k-1} + 2m)$ $L_{4k-1} + 2m$ is an integer, since every L_n is an integer	**A1**	Shows L_{4k+2} a multiple of 3, with the bracket stated to be an integer.
	True for $n = 1$, and true for $n = k$ $\Rightarrow$ true for $n = k + 1$ So true for all positive integers n	**E1**	States all three points, the base case, truth for $n = k$ implying truth for $n = k + 1$, and truth for all positive integers n.
		[4]	
Total for Question 5		**7**	

Question	Scheme	Marks	Notes
6(a)	$n = 1$: $u_1 = 1$ and $0 < 1 < 3$	**B1**	
	Assumes $0 < u_k < 3$ $u_{k+1} - 3 = \frac{5u_k+6-3(u_k+4)}{u_k+4}$	**M1**	Assumes the statement for $n = k$ and combines $u_{k+1} - 3$, or $3 - u_{k+1}$, over a common denominator, or writes $u_{k+1} = 5 - \frac{14}{u_k+4}$. Condone sign slips.
	$= \frac{2(u_k-3)}{u_k+4}$	**A1**	oe, such as $\frac{2u_k-6}{u_k+4}$.
	$u_k < 3 \Rightarrow 2(u_k - 3) < 0$ $u_k > 0 \Rightarrow u_k + 4 > 0$	**M1**	Uses the hypothesis to give the sign of the numerator and of the denominator of their single fraction, or bounds $u_k + 4$ between 4 and 7.
	$u_{k+1} - 3 < 0 \Rightarrow u_{k+1} < 3$ $u_k > 0 \Rightarrow 5u_k + 6 > 0 \Rightarrow u_{k+1} > 0$	**A1**	Both $u_{k+1} < 3$ and $u_{k+1} > 0$ deduced, each with its reason from the hypothesis.
	True for $n = 1$, and true for $n = k$ $\Rightarrow$ true for $n = k + 1$, so true for all positive integers n	**E1**	States true for $n = 1$, that true for $n = k$ implies true for $n = k + 1$, and so true for all positive integers n.
		[6]	
6(b)	$u_{n+1} - u_n = \frac{5u_n+6-u_n(u_n+4)}{u_n+4}$	**M1**	Combines $u_{n+1} - u_n$, or $u_n - u_{n+1}$, over a common denominator. Condone sign slips.
	$= \frac{(3-u_n)(2+u_n)}{u_n+4}$	**A1**	oe, such as $-\frac{(u_n-3)(u_n+2)}{u_n+4}$.
	$0 < u_n < 3 \Rightarrow 3 - u_n > 0$, $2 + u_n > 0$, $u_n + 4 > 0 \Rightarrow u_{n+1} - u_n > 0$ $\Rightarrow u_{n+1} > u_n$	**A1 AG**	Correct factorised fraction, with each factor's sign justified by part (a), leading without error to $u_{n+1} > u_n$.
		[3]	
	Total for Question 6	**9**	

Question	Scheme	Marks	Notes
7	$n = 1$: LHS $= 1 + \sqrt{2}$ RHS $= u_1 + u_0 + u_1\sqrt{2} = 1 + 0 + \sqrt{2} = 1 + \sqrt{2}$	**B1**	Both sides shown equal to $1 + \sqrt{2}$, using $u_0 = 0$ and $u_1 = 1$. Not for $1 + \sqrt{2}$ stated once.
	Assume $\left(1 + \sqrt{2}\right)^k = u_k + u_{k-1} + u_k\sqrt{2}$ $\left(1 + \sqrt{2}\right)^{k+1} = \left(1 + \sqrt{2}\right)^k \left(1 + \sqrt{2}\right)$ $= \left(u_k + u_{k-1} + u_k\sqrt{2}\right)\left(1 + \sqrt{2}\right)$	**M1**	Assumes the result for $n = k$ and multiplies its right-hand side by $1 + \sqrt{2}$. Not for assuming the result for $n = k + 1$.
	$= u_k + u_{k-1} + (u_k + u_{k-1})\sqrt{2}$ $+ u_k\sqrt{2} + 2u_k$	**M1**	Expands their product to at least four terms, using $\sqrt{2} \times \sqrt{2} = 2$, and collects rational and $\sqrt{2}$ parts. Condone sign or coefficient slips.
	$= (3u_k + u_{k-1}) + (2u_k + u_{k-1})\sqrt{2}$	**A1**	Correct collected form, oe.
	$u_{k+1} = 2u_k + u_{k-1}$ as $k \geq 1$ $\Rightarrow 3u_k + u_{k-1} = u_{k+1} + u_k$ $\left(1 + \sqrt{2}\right)^{k+1} = u_{k+1} + u_k + u_{k+1}\sqrt{2}$	**A1**	cso. Uses the recurrence to write both parts in u_{k+1} and u_k, reaching the $n = k+1$ form. Condone $k \geq 1$ not stated.
	True for $n = 1$; if true for $n = k$, true for $n = k + 1$ So true for every positive integer n	**E1**	States true for $n = 1$, true for $n = k + 1$ if true for $n = k$, and so true for every positive integer n.
	Total for Question 7	**6**	

Question	Scheme	Marks	Notes
8(a)	$n = 1$: $5^2 - 4(1) - 5 = 16 = u_1$	**B1**	Formula evaluated at $n = 1$ and compared with $u_1 = 16$.
	Assume $u_k = 5^{k+1} - 4k - 5$ $u_{k+1} = 5u_k + 16(k+1)$	**M1**	States the hypothesis for $n = k$ and writes u_{k+1} by the recurrence with $n = k$.
	$u_{k+1} = 5\left(5^{k+1} - 4k - 5\right) + 16(k+1)$ $= 5^{k+2} - 20k - 25 + 16k + 16$	**M1**	Substitutes their hypothesis into the recurrence and expands. Condone sign slips.
	$= 5^{k+2} - 4k - 9$	**A1**	oe, such as $25 \times 5^k - 4k - 9$.
	$= 5^{(k+1)+1} - 4(k+1) - 5$	**A1**	cso. Expression in terms of $k+1$; $5^{k+2} - 4(k+1) - 5$ is enough.
	True for $n = 1$ True for $n = k \Rightarrow$ true for $n = k + 1$ So true for all positive integers n	**E1**	States true for $n = 1$, that truth for $n = k$ implies truth for $n = k + 1$, and so true for all positive integers n. Not if any is missing.
		[6]	
8(b)	$u_1 = 16 = 16 \times 1$	**B1**	
	Assume $u_k = 16m$, m an integer $u_{k+1} = 5(16m) + 16(k+1)$	**M1**	Uses u_k, or $5^{k+1} - 4k - 5$, as a multiple of 16 in the recurrence with $n = k$.
	$= 16(5m + k + 1)$, $5m + k + 1$ an integer	**A1**	Needs $5m + k + 1$, oe, stated to be an integer.
	True for $n = 1$ True for $n = k \Rightarrow$ true for $n = k + 1$ $u_n = 5^{n+1} - 4n - 5$ by (a), so $16 \mid 5^{n+1} - 4n - 5$ for all n	**E1**	States the three points of the induction and links u_n to $5^{n+1} - 4n - 5$ by part (a). Not if any is missing.
		[4]	
	Total for Question 8	**10**	

Question	Scheme	Marks	Notes
9	$n = 1:\ 1!\left(2^1+1\right) = 3 = u_1$ $n = 2:\ 2!\left(2^2+1\right) = 10 = u_2$	**B1**	Both $n = 1$ and $n = 2$ checked against u_1 and u_2. B0 for one case alone.
	Assumes $u_k = k!\left(2^k+1\right)$ and $u_{k+1} = (k+1)!\left(2^{k+1}+1\right)$: $u_{k+2} = 3(k+2)(k+1)!\left(2^{k+1}+1\right)$ $\quad - 2(k+1)(k+2)k!\left(2^k+1\right)$	**M1**	Assumes the result for two consecutive values and substitutes both into the recurrence for the next term. Condone a slip in one term. Not for assuming one value only.
	$(k+2)(k+1)! = (k+2)! = (k+1)(k+2)k!$ $u_{k+2} = (k+2)!\left[3\left(2^{k+1}+1\right) - 2\left(2^k+1\right)\right]$	**M1 A1**	**M1** Uses $(k+2)(k+1)! = (k+2)!$ and $(k+1)(k+2)k! = (k+2)!$, soi, to take out the common factor $(k+2)!$. Condone slips inside the bracket. **A1** Correct, oe.
	$u_{k+2} = (k+2)!\left(2\times 2^{k+1}+1\right)$ $\quad = (k+2)!\left(2^{k+2}+1\right)$	**A1**	cso. Must reach $(k+2)!\left(2^{k+2}+1\right)$ explicitly.
	True for $n = 1$, 2; true for $n = k$, $k+1$ $\Rightarrow$ true for $n = k+2$, so true for all $n \in \mathbb{N}$	**E1**	States true for $n = 1$ and $n = 2$, that truth for $n = k$ and $n = k+1$ implies truth for $n = k+2$, and so true for all $n \in \mathbb{N}$.
	Total for Question 9	**6**	

Question	Scheme	Marks	Notes
10(a)	$n = 1$: $100(2) + (-3) = 197 = u_1$ $n = 2$: $100(4) + 9 = 409 = u_2$	**B1**	Both $n = 1$ and $n = 2$ checked against the given values. B0 for one case only.
	Assume $u_k = 100(2^k) + (-3)^k$ and $u_{k+1} = 100(2^{k+1}) + (-3)^{k+1}$ $u_{k+2} = 6u_k - u_{k+1}$	**M1**	Assumes the result for two consecutive values, such as $n = k$ and $n = k + 1$, and writes the next term by the recurrence.
	$u_{k+2} = 6\left[100(2^k) + (-3)^k\right]$ $\quad - \left[100(2^{k+1}) + (-3)^{k+1}\right]$	**M1**	Substitutes both assumed forms into the recurrence and collects like powers of 2 and of -3. Condone sign slips.
	$= 400(2^k) + 9(-3)^k$	**A1**	Or any correct collected equivalent.
	$= 100(2^{k+2}) + (-3)^{k+2}$	**A1**	cso.
	True for $n = 1$ and $n = 2$; true for $n = k$, $k + 1$ $\Rightarrow$ true for $n = k + 2$; so true for all $n \in \mathbb{N}$	**E1**	States the base cases, the implication from $n = k$ and $n = k + 1$ to $n = k + 2$, and the result for all $n \in \mathbb{N}$.
		[6]	
10(b)	n even $\Rightarrow (-3)^n = 3^n > 0 \Rightarrow u_n > 0$ n odd: $u_n < 0 \Rightarrow 3^n > 100(2^n)$	**M1**	Uses $(-3)^n > 0$ for even n and sets $3^n > 100(2^n)$ oe for odd n. Condone $=$ for $>$.
	$n \ln 1.5 > \ln 100 \Rightarrow n > \ldots$ $[\approx 11.36]$	**dM1**	Takes logarithms to find the critical n, or compares 1.5^n with 100 at consecutive n, or consecutive odd n, either side. Condone $=$ for $>$.
	$n = 13$	**A1**	
	$u_{13} = 100(2^{13}) - 3^{13}$ $\quad = -775\,123$	**A1**	cao.
		[4]	
	Total for Question 10	**10**	

Question	Scheme	Marks	Notes
11(a)	$u_1 = 3 > \sqrt{5}$ [as $3^2 = 9 > 5$]	**B1**	
	Assume $u_k > \sqrt{5}$ $u_{k+1} - \sqrt{5} = \frac{u_k^2+5}{2u_k} - \sqrt{5}$ $= \frac{u_k^2-2\sqrt{5}\,u_k+5}{2u_k}$	**M1**	Assumes true for $n = k$ and writes $u_{k+1} - \sqrt{5}$ as a single fraction in u_k, or compares $u_k^2 + 5$ with $2\sqrt{5}\,u_k$. Condone slips.
	$u_{k+1} - \sqrt{5} = \frac{\left(u_k-\sqrt{5}\right)^2}{2u_k}$	**A1**	Correct square numerator, oe.
	$\left(u_k - \sqrt{5}\right)^2 > 0$ as $u_k \neq \sqrt{5}$ $2u_k > 0$ as $u_k > \sqrt{5} > 0$	**M1**	Uses the hypothesis to give the signs of both numerator and denominator, for a numerator written as a square. Condone ≥ 0 for the numerator.
	$u_{k+1} - \sqrt{5} > 0 \Rightarrow u_{k+1} > \sqrt{5}$	**A1**	Needs the numerator strictly positive, from $u_k \neq \sqrt{5}$ oe.
	True for $n = 1$, and true for $n = k \Rightarrow$ true for $n = k + 1$, so true for all positive integers n	**E1**	States the three points, the base case, the step from $n = k$ to $n = k + 1$ and truth for all positive integers n.
		[6]	
11(b)	$u_{n+1} - u_n = \frac{u_n^2+5}{2u_n} - u_n = \ldots$	**M1**	Substitutes the recurrence and combines into a single fraction in u_n. Condone sign slips.
	$u_{n+1} - u_n = \frac{5-u_n^2}{2u_n}$	**A1**	
	$u_n > \sqrt{5} > 0 \Rightarrow u_n^2 > 5,\ 2u_n > 0$ $\Rightarrow u_{n+1} - u_n < 0 \Rightarrow u_{n+1} < u_n$	**A1 AG**	Correct fraction with the signs of numerator and denominator both justified from part (a), then $u_{n+1} < u_n$ concluded.
		[3]	
11(c)	$u_{n+1} - \sqrt{5} = \frac{\left(u_n-\sqrt{5}\right)^2}{2u_n}$ $\frac{1}{2}\left(u_n - \sqrt{5}\right) - \left(u_{n+1} - \sqrt{5}\right)$ $= \frac{u_n\left(u_n-\sqrt{5}\right)-\left(u_n-\sqrt{5}\right)^2}{2u_n}$	**M1**	Forms the difference, or the ratio, of $u_{n+1} - \sqrt{5}$ and $\frac{1}{2}\left(u_n - \sqrt{5}\right)$ as a single fraction in u_n, from the recurrence or part (a). Condone slips.
	$= \frac{\sqrt{5}\left(u_n-\sqrt{5}\right)}{2u_n} > 0$ as $u_n > \sqrt{5} > 0$ $\Rightarrow u_{n+1} - \sqrt{5} < \frac{1}{2}\left(u_n - \sqrt{5}\right)$	**A1 AG**	Correct simplified difference, or ratio below $\frac{1}{2}$, shown by signs from part (a), with the printed inequality concluded.
		[2]	
	Total for Question 11	**11**	

Question	Scheme	Marks	Notes
12(a)	$n = 1$: $\ln a_1 = \ln 1 = 0$ $\left(2^0 - 1\right)\ln 3 = 0$	**B1**	Both sides evaluated as 0, or $\ln a_1 \leq \left(2^0 - 1\right)\ln 3$ with both sides shown.
	Assume $\ln a_k \leq \left(2^{k-1} - 1\right)\ln 3$ $a_k \geq 1$ (as $a_1 = 1$ and $a_{n+1} \geq 1$) $\Rightarrow a_{k+1} = 2a_k^2 + 1 \leq 3a_k^2$	**M1**	Assumes the result for $n = k$ and bounds a_{k+1} above, by $3a_k^2$ using $a_k^2 \geq 1$, or using the hypothesis as $a_k \leq 3^{2^{k-1}-1}$. Condone $a_k \geq 1$ stated without reason.
	$\ln a_{k+1} \leq \ln 3 + 2\ln a_k \leq \ldots$	**M1**	Takes logs of their bound, uses log laws and substitutes the hypothesis for $\ln a_k$.
	$\ln a_{k+1} \leq \ln 3 + 2\left(2^{k-1} - 1\right)\ln 3$	**A1**	
	$\ln a_{k+1} \leq \left(2^k - 1\right)\ln 3$ $= \left(2^{(k+1)-1} - 1\right)\ln 3$	**A1**	cso, with the inequality correct throughout. Either form accepted.
	True for $n = 1$; true for $n = k$ $\Rightarrow$ true for $n = k + 1$ So true for all integers $n \geq 1$	**E1**	States the result true for $n = 1$, that truth for $n = k$ implies truth for $n = k + 1$, and so true for all integers $n \geq 1$.
		[6]	
12(b)	$\ln a_{n+1} - \ln a_n \leq \ln 3 + \ln a_n$ $\leq \ln 3 + \left(2^{n-1} - 1\right)\ln 3$	**M1**	Uses $\ln a_{n+1} \leq \ln 3 + 2\ln a_n$, subtracts $\ln a_n$ and applies the result of (a) to the remaining $\ln a_n$.
	$\ln a_{n+1} - \ln a_n \leq 2^{n-1}\ln 3$	**A1 AG**	Correct substituted inequality leading without error to the printed result; combined steps accepted.
		[2]	
	Total for Question 12	**8**	