OPTIONAL PURE / RECURRENCE RELATIONS

Behaviour and Limits of Sequences

Mark scheme 11 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

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	Total	**94**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$(\alpha =)\ 3$	**B1**	
	Oscillates, with terms alternately below and above 3, and converges to 3	**E1**	States that the terms alternate either side of 3, oe "oscillates"; convergence may be taken from their stated limit. Not for "converges" alone.
		[2]	
1(b)	$8\left(\frac{3}{4}\right)^n < 0.01$	**M1**	Forms the inequality, or the matching equation, from $\lvert u_n - 3\rvert$. Condone any inequality sign, and $\left(-\frac{3}{4}\right)^n$ kept inside a modulus.
	$n\ln\left(\frac{3}{4}\right) < \ln(0.00125)$ $\Rightarrow n > \ldots\ [= 23.2]$	**dM1**	Uses logarithms, to any base, on a form $p\left(\frac{3}{4}\right)^n = q$ with $p, q > 0$, reaching a value for n. Condone any inequality sign. Not for trial of values alone.
	$N = 24$	**A1**	24 given as the answer, oe $n = 24$; an equation solved in place of the inequality is accepted. Not for $n > 23.2$ left.
		[3]	
Total for Question 1		**5**	

Question	Scheme	Marks	Notes
2(a)	a_n is periodic, with period 3	**E1**	States that a_n is periodic, or repeats in a cycle, and gives the period as 3. Accept "repeats every three terms". Not for "oscillates" alone.
	a_n does not converge to a limit	**E1**	States that a_n has no limit as $n \to \infty$, oe such as "does not converge" or "never settles to one value". Condone "divergent".
	b_n is increasing	**E1**	States that the terms of b_n increase, oe such as "each term is larger than the one before".
	b_n converges, with $b_n \to 2.303$	**E1**	States that b_n converges and gives its limit, awrt 2.3 or the exact value from (b). Not for "converges" alone.
	c_n oscillates, its terms alternating in sign	**E1**	States that c_n oscillates, or that its terms alternate in sign. Either is enough.
	c_n is divergent and unbounded	**E1**	States that c_n diverges, or that its terms grow in size without bound. Not for $c_n \to \infty$.
		[6]	
2(b)	$L = \sqrt{L+3} \Rightarrow L^2 - L - 3 = 0 \Rightarrow L = \ldots$	**M1**	Replaces both b_{n+1} and b_n by L, forms a three-term quadratic and solves it by any method. Condone sign slips in rearranging.
	$L = \frac{1+\sqrt{13}}{2}$	**A1**	oe, given as the only value of L. No reason for rejecting $\frac{1-\sqrt{13}}{2}$ is needed.
		[2]	
	Total for Question 2	**8**	

Question	Scheme	Marks	Notes
3(a)	$L = \frac{3L}{4} + \frac{5}{L} \Rightarrow L^2 = \ldots$ $[= 20]$	**M1**	Replaces a_n and a_{n+1} by one letter and rearranges to $L^2 = \ldots$ or $L = \ldots$ Condone slips in rearranging.
	$L = 2\sqrt{5}$	**A1**	$2\sqrt{5}$ or $\sqrt{20}$ only, with $-2\sqrt{5}$ rejected or never given. No reason is needed.
		[2]	
3(b)	$k - 4 = \frac{k-4}{2} + \frac{k}{k-4}$	**M1**	Replaces b_n and b_{n+1} by $k - 4$ to form an equation in k only. Condone one copying slip.
	$(k-4)^2 = 2k \Rightarrow k^2 - 10k + 16 = 0$	**A1**	Any correct quadratic with no fractions, such as $(k-4)^2 = 2k$. Need not be collected.
	$(k-2)(k-8) = 0 \Rightarrow k = \ldots$	**dM1**	Solves their three-term quadratic in k by any method, calculator included, reaching at least one value of k.
	$k = 2 \Rightarrow k - 4 = -2 < 0$, rejected $k = 8$	**A1**	$k = 8$ only, with $k = 2$ rejected because its limit $k - 4 = -2$ is negative, oe. Need not show that the terms are positive.
		[4]	
	Total for Question 3	**6**	

Question	Scheme	Marks	Notes
4(a)	$2^6 = 64 = 7 \times 9 + 1 \equiv 1 \pmod 9$ $\Rightarrow 2^{n+6} = 2^6 \times 2^n \equiv 2^n \pmod 9$	**M1**	Uses $2^{n+6} = 64 \times 2^n$ with 64 reduced modulo 9, or $2^{n+6} - 2^n = 63 \times 2^n$. Not for particular values of n alone.
	$3(n+6) = 3n + 2 \times 9 \equiv 3n \pmod 9$ $\Rightarrow u_{n+6} \equiv u_n \pmod 9$	**A1**	Needs $3(n+6) \equiv 3n$ shown from 18 being a multiple of 9, and the congruence of u_{n+6} and u_n, or of the two expressions, stated.
	$0 \leq u_n \leq 8$ and $0 \leq u_{n+6} \leq 8$ $\Rightarrow -8 \leq u_{n+6} - u_n \leq 8$ $\Rightarrow u_{n+6} = u_n$	**A1 AG**	Correct congruence $u_{n+6} \equiv u_n \pmod 9$, then both terms lying from 0 to 8 used to reach $u_{n+6} = u_n$. Accept "remainders on division by 9 are unique" oe.
		[3]	
4(b)	$2^n + 3n = 5,\ 10,\ 17,\ 28,\ 47,\ 82$ $\Rightarrow u_1, \ldots, u_6 = \ldots$	**M1**	Finds terms as remainders of $2^n + 3n$ on division by 9, at least u_1 to u_4. Condone arithmetic slips.
	$u_1, \ldots, u_6 = 5,\ 1,\ 8,\ 1,\ 2,\ 1$ Periodic with period 6 [so does not converge]	**A1**	Periodic with period 6, oe such as "repeats every 6 terms", with correct terms that rule out periods 1, 2 and 3; u_1 to u_6 listed is enough.
		[2]	
	Total for Question 4	**5**	

Question	Scheme	Marks	Notes
5(a)	$u_1 < 5$ is given, so true for $n = 1$	**B1**	Refers to the given $u_1 < 5$. Not for "true for $n = 1$" alone.
	Assumes $u_k < 5$ and considers $5 - u_{k+1}$: $5 - u_{k+1} = \frac{5(2u_k+5)-(12u_k+15)}{2u_k+5}$	**M1**	Forms $5 - u_{k+1}$ as a single fraction in u_k, with the assumption $u_k < 5$ stated or used later; oe on another valid route. Condone sign slips.
	$5 - u_{k+1} = \frac{2(5-u_k)}{2u_k+5}$	**A1**	oe, such as the numerator $10 - 2u_k$.
	$5 - u_k > 0$ and $2u_k + 5 > 0$ $\Rightarrow 5 - u_{k+1} > 0 \Rightarrow u_{k+1} < 5$	**A1**	Needs the numerator positive from the assumption, the denominator positive from the terms being positive, and $u_{k+1} < 5$ stated.
	True for $n = 1$ True for $n = k \Rightarrow$ true for $n = k + 1$ $\Rightarrow u_n < 5$ for all positive integers n	**E1**	States all three points: true for $n = 1$; if true for $n = k$ then true for $n = k + 1$; so true for all positive integers n.
		[5]	
5(b)	$u_{n+1} - u_n = \frac{12u_n+15-u_n(2u_n+5)}{2u_n+5}$	**M1**	Forms $u_{n+1} - u_n$ as a single fraction in u_n, or an equivalent inequality with the denominator cleared. Condone sign slips.
	$u_{n+1} - u_n = \frac{(5-u_n)(2u_n+3)}{2u_n+5}$	**A1**	Numerator factorised, oe, or its roots 5 and $-\frac{3}{2}$ found.
	$5 - u_n > 0$ by (a) $2u_n + 3 > 0$ and $2u_n + 5 > 0$ $\Rightarrow u_{n+1} - u_n > 0 \Rightarrow u_{n+1} > u_n$	**A1 AG**	Correct fraction, with $5 - u_n > 0$ from part (a) and the other factors positive as $u_n > 0$, leading without error to $u_{n+1} > u_n$.
		[3]	
5(c)	$L = \frac{12L+15}{2L+5} \Rightarrow 2L^2 - 7L - 15 = 0$ $\Rightarrow L = \ldots$	**M1**	Forms a quadratic in L by replacing both u_{n+1} and u_n by L, and solves it by any method. Condone sign slips.
	$L = 5$ or $L = -\frac{3}{2}$ Terms positive $\Rightarrow L \neq -\frac{3}{2}$, so $L = 5$	**A1**	$L = 5$ only, with $-\frac{3}{2}$ rejected because the terms are positive, oe. Not for "L is positive" with no reference to the terms.
		[2]	
	Total for Question 5	**10**	

Question	Scheme	Marks	Notes
6(a)	$n = 1$: $2 < a_1 < 4$ is given, so true for $n = 1$	**B1**	Needs the given $2 < a_1 < 4$ cited or written, not only "true for $n = 1$".
	Assume $2 < a_k < 4$ $\Rightarrow 4 < a_k^2 < 16$	**M1**	Assumes the result for $n = k$ and uses it to bound a_k^2, or $a_{k+1} - 2$ and $4 - a_{k+1}$. Condone one bound wrong.
	$a_k^2 > 4 \Rightarrow a_{k+1} > \frac{4+8}{6} = 2$	**A1**	$a_{k+1} > 2$ from $a_k > 2$, from correct working, oe such as $a_{k+1} - 2 = \frac{a_k^2 - 4}{6} > 0$. Both bounds may come in one chain.
	$a_k^2 < 16 \Rightarrow a_{k+1} < \frac{16+8}{6} = 4$	**A1**	$a_{k+1} < 4$ from $a_k < 4$, from correct working. Need not say that $a_k > 0$ lets the squaring keep the order.
	True for $n = 1$, and true for $n = k$ $\Rightarrow$ true for $n = k + 1$, so true for every positive integer n	**E1**	States all three points, that it is true for $n = 1$, that true for $n = k$ implies true for $n = k + 1$, and so true for all positive integers n.
		[5]	
6(b)	$a_{n+1} - a_n = \frac{a_n^2+8}{6} - a_n$ $= \frac{a_n^2-6a_n+8}{6}$	**M1**	Forms $a_{n+1} - a_n$, its negative or the inequality $a_{n+1} < a_n$ in terms of a_n, and collects a three-term quadratic. Condone sign slips.
	$= \frac{(a_n-2)(a_n-4)}{6}$	**A1**	Correct factorised form, oe. Accept $\frac{(a_n-3)^2-1}{6}$.
	$2 < a_n < 4 \Rightarrow a_n - 2 > 0,\ a_n - 4 < 0$ $\Rightarrow a_{n+1} - a_n < 0 \Rightarrow a_{n+1} < a_n$	**A1 AG**	Signs of both factors, oe, from $2 < a_n < 4$, giving a negative difference and $a_{n+1} < a_n$ without error. A0 if part (a) is not used.
		[3]	
6(c)	$L = \frac{L^2+8}{6} \Rightarrow L^2 - 6L + 8 = 0$ $\Rightarrow L = \ldots$	**M1**	Replaces both a_{n+1} and a_n by L and solves their three-term quadratic by any means, a calculator included.
	$L = 2$ or $L = 4$ Decreasing and $a_1 < 4 \Rightarrow L < 4$, so $L = 2$	**A1**	$L = 2$ only, with 4 rejected because the sequence decreases from $a_1 < 4$, oe such as $a_n \leq a_1 < 4$. Not for "$a_n < 4$" alone.
		[2]	
	Total for Question 6	**10**	

Question	Scheme	Marks	Notes
7(a)	$u_2 = \frac{a-3}{a+1}$ $u_3 = \frac{u_2-3}{u_2+1} = \frac{(a-3)-3(a+1)}{(a-3)+(a+1)}$	**M1**	Substitutes u_2 in terms of a into the recurrence to form u_3, or u_{n+1} in terms of u_n to form u_{n+2}. Need not be simplified.
	$u_3 = \frac{-2a-6}{2a-2} = \frac{a+3}{1-a}$	**A1**	Any equivalent single fraction, such as $\frac{-2a-6}{2a-2}$.
	$u_4 = \frac{u_3-3}{u_3+1} = \frac{(a+3)-3(1-a)}{(a+3)+(1-a)}$	**dM1**	Substitutes their u_3, a single fraction in a, into the recurrence to form u_4, or their u_{n+2} to form u_{n+3}. Need not be simplified.
	$u_4 = \frac{4a}{4} = a \Rightarrow u_{n+3} = u_n$	**A1**	$u_4 = a$ from correct working, with repetition every three terms stated here or implied by the conclusion.
	$a = -1$: u_2 undefined $a = 1$: $u_2 = -1$, u_3 undefined	**B1**	Both values excluded, with an undefined term or a zero denominator as the reason. Need not say which term.
	$u_n = \frac{u_n-3}{u_n+1} \Rightarrow u_n^2 = -3$, no real roots Periodic with period 3 for all other a	**E1**	States period 3, oe, with a reason that the terms are not all equal, such as $u_n^2 = -3$ having no real roots. Condone the excluded values, and convergence, not mentioned.
		[6]	
7(b)	$u_1 = 5,\ u_2 = \frac{1}{3},\ u_3 = -2$ $\sum u_n = 33\left(5 + \frac{1}{3} - 2\right) + 5$	**M1**	Uses the period to write the sum as complete cycles of their three terms from $a = 5$, with the terms left over. Condone a wrong number of cycles.
	115	**A1**	
		[2]	
	Total for Question 7	**8**	

Question	Scheme	Marks	Notes
8(a)	$3 = k + \frac{6}{3} \Rightarrow k = \ldots$	**M1**	Substitutes $u_n = u_{n+1} = 3$, or sets $u_2 = 3$, in the recurrence relation.
	$k = 1$	**A1**	
		[2]	
8(b)	$u_2 = k + 2$ $u_3 = k + \frac{6}{k+2} = 3$	**M1**	Uses the recurrence relation twice to form an equation in k alone from $u_3 = 3$. Condone slips in u_2.
	$k(k+2) + 6 = 3(k+2) \Rightarrow k^2 - k = 0$	**A1**	Any correct quadratic equation with the fractions cleared; terms need not be collected.
	$k(k-1) = 0 \Rightarrow k = \ldots$	**dM1**	Solves their quadratic in k by any method, leading to $k = \ldots$
	$k = 0$ or $k = 1$ Rejects $k = 1$, the constant sequence: $k = 0$	**A1**	$k = 0$ only, with $k = 1$ rejected as giving the constant sequence or $u_2 = u_1$. Checking that $k = 0$ gives $3, 2, 3, 2, \ldots$ is not needed.
		[4]	
8(c)	$u_3 = \frac{k^2+2k+6}{k+2}$ $u_4 = k + \frac{6(k+2)}{k^2+2k+6} = 3$	**M1 A1**	**M1** Uses the recurrence relation to form an equation in k alone from $u_4 = 3$. Condone slips in u_3. **A1** Any correct equation in k alone; fractions need not be cleared.
	$k^3 - k^2 + 6k - 6 = 0$ $\Rightarrow (k-1)(k^2+6) = 0 \Rightarrow k = \ldots$	**dM1**	Clears the fractions to reach a cubic in k and factorises it, or solves it by calculator. Condone slips in expanding.
	$k^2 + 6 \neq 0$ for real k, so $k = 1$ only	**A1**	$k = 1$ as the only real value, with a reason, such as $k^2 + 6 > 0$ or the other roots $\pm i\sqrt{6}$ seen.
	$k = 1$ gives the constant sequence $3, 3, 3, \ldots$ so no value of k exists	**E1**	States that their only real value $k = 1$ gives the constant sequence, and concludes that no value of k exists. Condone no mention of $k = -2$ or of nonzero denominators.
		[5]	
Total for Question 8		**11**	

Question	Scheme	Marks	Notes
9(a)	$\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a three-term auxiliary equation, condoning sign slips, and solves it by any method, a calculator included.
	$\lambda = 3,\ -2$ $(u_n =)\ A \times 3^n + B(-2)^n$	**A1**	Any letters for the constants. Needs $(-2)^n$, its brackets seen or implied by later work.
	$A + B = 3$ $3A - 2B = 4$	**M1**	Substitutes $n = 0$ and $n = 1$ into their general solution, of the form $A\alpha^n + B\beta^n$, to form two equations in the constants. Condone slips.
	$5A = 10 \Rightarrow A = \ldots,\ B = \ldots$	**dM1**	Solves their two equations to find a value for each constant, by any method. Condone arithmetic slips.
	$u_n = 2 \times 3^n + (-2)^n$	**A1**	oe, such as $2(3^n) + (-2)^n$; $A = 2$, $B = 1$ alone does not score.
		[5]	
9(b)(i)	$\frac{u_n}{u_{n-1}} = 1 + \frac{6u_{n-2}}{u_{n-1}}$	**M1**	Divides every term of $(*)$ by u_{n-1}, or substitutes $(*)$ into $v_n = \frac{u_n}{u_{n-1}}$. Condone one slip.
	$\frac{u_{n-2}}{u_{n-1}} = \frac{1}{v_{n-1}}$ $\Rightarrow v_n = 1 + \frac{6}{v_{n-1}}$	**A1 AG**	Correct divided equation leading without error to the printed result; replacing both quotients in one step is accepted. No reason for $u_{n-1} \neq 0$ is needed.
		[2]	
9(b)(ii)	Replaces v_n and v_{n-1} by L: $v_n v_{n-1} = v_{n-1} + 6 \Rightarrow L^2 = L + 6$	**M1**	Replaces both v_n and v_{n-1} by one letter in the result of (b)(i), in either form. $L = 1 + \frac{6}{L}$ is enough.
	$L^2 - L - 6 = 0 \Rightarrow (L - 3)(L + 2) = 0$ $\Rightarrow L = 3$ or $L = -2$	**A1 AG**	Correct quadratic solved to both values, with factorising, the formula or a check of both roots shown. Any letter for the limit.
		[2]	
9(c)	$v_n =$ "$\frac{2\times3^n+(-2)^n}{2\times3^{n-1}+(-2)^{n-1}}$"	**M1**	Writes v_n or v_{n+1} as a quotient of consecutive terms of their part (a) answer, of the form $A\alpha^n + B\beta^n$.

Question 9 continued

Question	Scheme	Marks	Notes
	$v_n = \dfrac{2+\left(-\frac{2}{3}\right)^n}{\frac{2}{3}-\frac{1}{2}\left(-\frac{2}{3}\right)^n}$	**dM1**	Divides numerator and denominator by the same power of 3, or of their base of greater modulus, such as 3^n or 3^{n-1}, or argues that those terms dominate. Condone coefficient slips.
	$\left(-\frac{2}{3}\right)^n \to 0$ as $n \to \infty$ $\Rightarrow \lim_{n\to\infty} v_n = \frac{2+0}{\frac{2}{3}-0} = 3$	**A1 AG**	Correct quotient in powers of $-\frac{2}{3}$, their tending to 0 stated or shown as zeros, and the limit 3 reached without error. $\left\|-\frac{2}{3}\right\| < 1$ need not be quoted.
		[3]	
9(d)	Sets the coefficient of 3^n to zero: $w_n = D(-2)^n$ $D = 5 \Rightarrow k = \ldots$	**M1**	Uses a solution with no 3^n term, $w_n = D(-2)^n$ or a common ratio of -2, with $w_0 = 5$ to find k.
	$k = -10$	**A1**	
		[2]	
	Total for Question 9	**14**	

Question	Scheme	Marks	Notes
10(a)	$n = 1$: $\ln a_1 = \ln 1 = 0$ $\left(1 - 2^0\right)\ln 2 = 0$	**B1**	Both sides evaluated as 0 when $n = 1$. Condone the comparison $0 \leq 0$ unstated.
	Assume $\ln a_k \leq \left(1 - 2^{k-1}\right)\ln 2$ $\ln a_{k+1} = 2\ln a_k - \ln(2 + a_k)$	**M1**	Assumes the result for $n = k$ and uses the recurrence to write $\ln a_{k+1}$, or a_{k+1}, in terms of a_k. Condone a slip in a log law.
	$2\ln a_k \leq 2\left(1 - 2^{k-1}\right)\ln 2$ $\ln(2 + a_k) > \ln 2$ $\Rightarrow \ln a_{k+1} < \ldots$	**dM1**	Uses the assumption and $\ln(2 + a_k) > \ln 2$ in their expression for $\ln a_{k+1}$ in terms of a_k to form an inequality. Condone sign and coefficient slips.
	$\ln a_{k+1} < \left(2 - 2^k\right)\ln 2 - \ln 2$	**A1**	Correct unsimplified bound, oe. Accept $\leq$ for $<$.
	$\ln a_{k+1} < \left(1 - 2^k\right)\ln 2$ $= \left(1 - 2^{(k+1)-1}\right)\ln 2$	**A1**	Reaches $\left(1 - 2^k\right)\ln 2$ oe, recognisably the $n = k + 1$ form, with $<$ or $\leq$.
	True for $n = 1$ True for $n = k \Rightarrow$ true for $n = k + 1$ So true for all positive integers n	**E1**	States all three points, that it is true for $n = 1$, that true for $n = k$ implies true for $n = k + 1$, and so true for all positive integers n.
		[6]	
10(b)	$\ln a_n - \ln a_{n+1} = \ln(2 + a_n) - \ln a_n$ $\ln(2 + a_n) > \ln 2$ $-\ln a_n \geq \left(2^{n-1} - 1\right)\ln 2$	**M1**	Uses the recurrence to reach $\ln(2 + a_n) - \ln a_n$ oe, then part (a) and $\ln(2 + a_n) > \ln 2$ to bound it. Condone sign slips.
	$\ln a_n - \ln a_{n+1} > \ln 2 + \left(2^{n-1} - 1\right)\ln 2$ $= 2^{n-1}\ln 2$	**A1 AG**	Correct inequality from part (a) and the given fact, leading without error to the printed result with strict inequality. Combined steps accepted.
	$0 < \frac{a_{n+1}}{a_n} < \frac{1}{2^{2^{n-1}}} \to 0$ $\Rightarrow \lim \frac{a_{n+1}}{a_n} = 0$	**E1**	States the limit 0, as the ratio is below $2^{-2^{n-1}}$ oe, which tends to 0, or as $\ln\left(\frac{a_{n+1}}{a_n}\right) \to -\infty$. Condone positivity unstated. Not for 0 alone.
		[3]	
	Total for Question 10	**9**	

Question	Scheme	Marks	Notes
11	$I_n = \left[-\frac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_0^4$ $+ \frac{2n}{3}\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,\mathrm{d}x$	**M1**	Integrates I_n by parts, lowering the power of x by 1 and raising that of $4-x$ by 1, oe. Condone sign and coefficient slips.
	$I_n = \frac{2n}{3}\int_0^4 \left(4x^{n-1} - x^n\right)\sqrt{4-x}\,\mathrm{d}x$ $= \frac{2n}{3}\left(4I_{n-1} - I_n\right)$	**dM1**	Uses $(4-x)^{\frac{3}{2}} = (4-x)\sqrt{4-x}$, or $x = 4-(4-x)$, to reach an equation in I_n and I_{n-1} only. Condone sign and coefficient slips.
	$(2n+3)I_n = 8nI_{n-1}$	**A1**	Any correct relation, oe such as $I_n = \frac{8n}{2n+3}I_{n-1}$.
	$A_n = \frac{I_{n+1}}{I_n} \times \frac{I_{n+1}}{I_{n+2}}$ $= \text{“}\frac{8(n+1)}{2n+5}\text{”} \times \text{“}\frac{2n+7}{8(n+2)}\text{”}$	**M1**	Uses their relation, of the form $I_n = f(n)I_{n-1}$, for both $\frac{I_{n+1}}{I_n}$ and $\frac{I_{n+2}}{I_{n+1}}$, giving A_n in n only. Condone slips in replacing n. Division by I_n needs no justification.
	$A_n = \frac{(n+1)(2n+7)}{(n+2)(2n+5)}$ $\left[= 1 - \frac{3}{2n^2+9n+10}\right]$	**A1**	Any correct form in n only; need not be simplified.
	$A_{n+1} - A_n$ $= \text{“}\frac{3}{2n^2+9n+10} - \frac{3}{2n^2+13n+21}\text{”}$ $= \frac{\ldots}{(2n^2+9n+10)(2n^2+13n+21)}$	**M1**	Forms $A_{n+1} - A_n$ from their A_n in n only as a single fraction, or an equivalent comparison such as $\frac{A_{n+1}}{A_n}$ with 1, or $2n^2+9n+10$ increasing. Condone algebra slips.
	$A_{n+1} - A_n = \frac{3(4n+11)}{(2n^2+9n+10)(2n^2+13n+21)} > 0$ $\Rightarrow A_{n+1} > A_n$ for all $n \geq 0$	**A1**	Correct difference stated positive, or a correct equivalent comparison, with the conclusion $A_{n+1} > A_n$ or that the sequence increases. A0 if the conclusion is missing.
	$\frac{3}{2n^2+9n+10} \to 0$ as $n \to \infty$ $\Rightarrow A_n \to 1$, so $A = 1$	**A1**	$A = 1$ from a correct A_n, with a reason such as the fraction tending to 0 or division by n^2. Not for 1 alone. Needs the M1 for A_n, not the M1 above.
	Total for Question 11	**8**	