OPTIONAL PURE / RECURRENCE RELATIONS

2nd Order Recurrence Relations

Mark scheme 12 questions · 124 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

Question	Marks by part	Marks	Page
Question 1		5	4
Question 2		5	5
Question 3	(a) 5 (b) 4	9	6
Question 4	(a) 2 (b)(i) 2 (b)(ii) 3 (c) 3 (d) 2	12	7
Question 5	(a) 5 (b) 3	8	9
Question 6	(a) 3 (b) 3	6	10
Question 7	(a) 7 (b) 6	13	11
Question 8	(a) 9 (b) 2	11	13
Question 9	(a) 4 (b)(i) 3 (b)(ii) 2 (b)(iii) 4 (c) 2	15	14
Question 10	(a) 1 (b)(i) 3 (b)(ii) 2 (b)(iii) 2 (c)(i) 1 (c)(ii) 2	11	16
Question 11	(a) 2 (b) 8 (c) 2 (d) 2	14	18
Question 12	(a) 6 (b) 3 (c) 1 (d) 5	15	20
	Total	**124**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$\lambda^2 - \lambda - 12 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a three-term auxiliary equation, condoning sign slips, and solves it by any method, including a calculator.
	$[\lambda = 4,\ -3]$ $(u_n =)\ A \times 4^n + B(-3)^n$	**A1**	Any letters for the constants. Condone -3^n if later working treats it as $(-3)^n$.
	$4A - 3B = 11$ $16A + 9B = 23$	**M1**	Uses two known terms, normally u_1 and u_2, in their general solution of the form $A\alpha^n + B\beta^n$, giving two equations in A and B. Condone slips.
	$28A = 56 \Rightarrow A = \ldots,\ B = \ldots$ $[A = 2,\ B = -1]$	**dM1**	Solves their two equations to reach values of both constants, by any method, including a calculator.
	$u_n = 2 \times 4^n - (-3)^n$	**A1**	Any correct form, such as $2^{2n+1} - (-3)^n$. The brackets round -3 are needed.
	Total for Question 1	**5**	

Question	Scheme	Marks	Notes
2	$4\lambda^2 - 4\lambda + 1 = 0 \Rightarrow \lambda = \ldots \ \left[= \frac{1}{2}\right]$	**M1**	Forms the auxiliary equation and solves it by any method. Condone sign slips in the equation.
	$(H_n =) \ (A + Bn)\left(\frac{1}{2}\right)^n$	**A1**	oe, such as $A\left(\frac{1}{2}\right)^n + Bn\left(\frac{1}{2}\right)^n$ or $(A + Bn)2^{-n}$, with any letters for the constants.
	$n = 0$: $A = 5$ $n = 1$: $\frac{1}{2}(A + B) = 4$	**M1**	Substitutes $n = 0$ and $n = 1$ into their general solution of the form $(A + Bn)\lambda^n$ or $A\alpha^n + B\beta^n$ to form two equations. Condone slips.
	$5 + B = 8 \Rightarrow B = \ldots \ [= 3]$	**dM1**	Solves their two equations to reach values for both constants.
	$H_n = (5 + 3n)\left(\frac{1}{2}\right)^n$	**A1**	oe, such as $(5 + 3n)2^{-n}$ or $\frac{5+3n}{2^n}$.
	Total for Question 2	**5**	

Question	Scheme	Marks	Notes
3(a)	$3\lambda^2 - 5\lambda - 2 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms the auxiliary equation, condoning sign slips, and solves it by factorising, the formula, completing the square or a calculator.
	$\lambda = 2,\ -\frac{1}{3}$ $A \times 2^n + B\left(-\frac{1}{3}\right)^n$	**A1**	Correct complementary function, with any letters for the two constants.
	$u_n = a\left(\frac{1}{2}\right)^n \Rightarrow 3a\left(\frac{1}{2}\right)^{n+2}$ $= 5a\left(\frac{1}{2}\right)^{n+1} + 2a\left(\frac{1}{2}\right)^n - 15\left(\frac{1}{2}\right)^n$	**M1**	Substitutes a particular solution of the form $a\left(\frac{1}{2}\right)^n$, any letter, into the full recurrence relation. Condone index and sign slips.
	$\frac{3a}{4} = \frac{5a}{2} + 2a - 15 \Rightarrow a = \ldots$	**dM1**	Obtains an equation in a free of n, by cancelling $\left(\frac{1}{2}\right)^n$ or taking a value of n, and solves it. Condone arithmetic slips.
	$a = 4$ $u_n = A \times 2^n + B\left(-\frac{1}{3}\right)^n + 4\left(\frac{1}{2}\right)^n$	**A1**	Correct general solution with both constants, oe such as 2^{2-n} for the last term. Condone a missing $u_n =$.
		[5]	
3(b)	Convergent, so no 2^n term: $A = 0$	**B1**	Coefficient of 2^n stated as zero, soi by a solution with no 2^n term. No reason needed.
	$u_0 = 7 \Rightarrow B + 4 = 7 \Rightarrow B = \ldots\ [= 3]$ $k = u_1 =$ "3" $\left(-\frac{1}{3}\right) + 4\left(\frac{1}{2}\right)$	**M1**	Uses $u_0 = 7$ and $u_1 = k$, in either order, in their general solution with $A = 0$, of the form $B\beta^n + a\left(\frac{1}{2}\right)^n$, to reach a value for k.
	$k = 1$	**A1**	
	Limit is 0	**B1**	Accept $u_n \to 0$. Not earned if their solution keeps a 2^n term.
		[4]	
	Total for Question 3	**9**	

Question	Scheme	Marks	Notes
4(a)	$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a quadratic auxiliary equation, in any letter, and solves it by any method. Condone sign slips.
	$u_n = A + B \times 2^n$	**A1**	oe such as $A \times 1^n + B \times 2^n$, with any letters for the two constants.
		[2]	
4(b)(i)	$p(n+2) + q - 3\big(p(n+1) + q\big) + 2(pn + q)$	**M1**	Substitutes $pn + q$, $p(n+1) + q$ and $p(n+2) + q$ into $(*)$ or its left-hand side and attempts to simplify. Condone sign and bracket slips.
	$= -p$, with no term in n $-p = 2 - 6n$ cannot hold for all n $\Rightarrow$ no solution of the form $u_n = pn + q$	**A1 AG**	Correct substitution leading without error to $-p$, a reason (no term in n, or $0 = -6$ from coefficients of n) and the conclusion. Accept a contradiction from two values of n.
		[2]	
4(b)(ii)	Tries $u_n = an^2 + bn$: $a(n+2)^2 + b(n+2) - 3a(n+1)^2 - 3b(n+1)$ $+ 2an^2 + 2bn = 2 - 6n$	**M1**	Substitutes a trial of the form $an^2 + bn$, with or without a constant term, into $(*)$. Condone slips in expanding. Not for a linear trial.
	$-2an + (a - b) = 2 - 6n$ $\Rightarrow a = \ldots,\ b = \ldots\ [a = 3,\ b = 1]$	**dM1**	Equates coefficients of n and the constant terms, from their left-hand side simplified to be linear in n, to find values for both constants.
	$u_n = A + B \times 2^n + 3n^2 + n$	**A1ft**	$3n^2 + n$ added to their complementary function from part (a), which must have two arbitrary constants. Accept any letters for the constants.
		[3]	
4(c)	$n = 0:\ A + B = 3$ $n = 1:\ A + 2B + \text{“4”} = 5$	**M1**	Uses $u_0 = 3$ and $u_1 = 5$ in their general solution of $(*)$, which must have two arbitrary constants, to form two equations. Condone slips.
	$\Rightarrow A = \ldots,\ B = \ldots\ [A = 5,\ B = -2]$	**dM1**	Solves their two equations, by any method, to find values for both constants.
	$u_n = 5 - 2^{n+1} + 3n^2 + n$	**A1**	oe such as $5 - 2 \times 2^n + 3n^2 + n$.
		[3]	

Question 4 continued

Question	Scheme	Marks	Notes
4(d)	$u_2 = 11,\ u_3 = 19$ $u_4 = 25,\ u_5 = 21\ [< u_4]$	**M1**	Calculates at least three consecutive terms beyond u_1 from their part (c), soi by their values. Condone arithmetic slips.
	Greatest value is 25 $[= u_4]$	**A1**	25 stated as the greatest value, with $u_3 = 19$, $u_4 = 25$ and $u_5 = 21$ seen. 25 with no supporting values scores no marks.
		[2]	
	Total for Question 4	**12**	

Question	Scheme	Marks	Notes
5(a)	$\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a three-term auxiliary equation, condoning sign slips, and solves it by any method, a calculator included.
	$[\lambda = 3,\ -2]$ $A \times 3^n + B(-2)^n$	**A1**	Correct complementary function, with any two letters for the constants. May be seen first within the general solution.
	Tries $u_n = an \times 3^n$ $a(n+2)3^{n+2} - a(n+1)3^{n+1} - 6an \times 3^n$ $= 15 \times 3^n$	**M1**	Substitutes a trial solution of the form $an \times 3^n$ or $(an + b) \times 3^n$ into all three terms of the recurrence relation. Condone sign and index slips.
	$a\,[9(n+2) - 3(n+1) - 6n] = 15 \Rightarrow a = \ldots$	**dM1**	Divides out 3^n, or substitutes a value of n, and solves a linear equation for a. Condone arithmetic slips.
	$a = 1$ $u_n = A \times 3^n + B(-2)^n + n \times 3^n$	**A1**	Correct general solution with two arbitrary constants, oe such as $(n + A) \times 3^n + B(-2)^n$.
		[5]	
5(b)	$A + B = 1$ $9A + 4B + 18 = 2(3A - 2B + 3)$	**M1**	Uses both conditions in their general solution, a complementary function with two arbitrary constants plus a particular solution, to form two equations in A and B. Condone slips.
	$A + B = 1,\ 3A + 8B = -12$ $\Rightarrow A = \ldots,\ B = \ldots\ [A = 4,\ B = -3]$	**dM1**	Solves their two linear equations to find values of both A and B, by any method, a calculator included.
	$u_n = (n+4) \times 3^n - 3(-2)^n$	**A1**	oe, such as $4 \times 3^n - 3(-2)^n + n \times 3^n$. Need not be simplified.
		[3]	
	Total for Question 5	**8**	

Question	Scheme	Marks	Notes
6(a)	$\lambda = 2\sqrt{3}\left(\cos\frac{5\pi}{6} \pm i\sin\frac{5\pi}{6}\right) = \ldots$	**M1**	Uses $r = 2\sqrt{3}$ and $\theta = \frac{5\pi}{6}$ to write the complex roots, in any form. Implied by $k_1 = -2r\cos\theta$ with $k_2 = r^2$.
	$\lambda =$ "$-3 \pm \sqrt{3}\,i$" $\Rightarrow (\lambda+3)^2 + 3 = 0$	**dM1**	Forms a quadratic from their roots of the form $p \pm qi$, or uses their sum and product for k_1 and k_2. Condone sign slips.
	$\lambda^2 + 6\lambda + 12 = 0$ $k_1 = 6,\ k_2 = 12$	**A1**	
		[3]	
6(b)	$2\sqrt{3}\left(-\frac{\sqrt{3}}{2}A + \frac{1}{2}B\right) = 3$ $12\left(\frac{1}{2}A - \frac{\sqrt{3}}{2}B\right) = 6$	**M1**	Substitutes $n = 1$ and $n = 2$ into the general solution, with values for the trigonometric terms, to form two equations in A and B. Condone slips in the values.
	$-3A + \sqrt{3}\,B = 3,\ A - \sqrt{3}\,B = 1$ $\Rightarrow A = \ldots,\ B = \ldots$	**dM1**	Solves their two linear equations in A and B to reach a value for each, by any method. Condone arithmetic slips.
	$A = -2,\ B = -\sqrt{3}$	**A1**	Both values, B in any exact form such as $-\frac{3}{\sqrt{3}}$.
		[3]	
	Total for Question 6	**6**	

Question	Scheme	Marks	Notes
7(a)	$\lambda^2 - 2\lambda + 4 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a three-term auxiliary equation, condoning sign slips, and solves it by the formula, completing the square or a calculator.
	$\lambda = 1 \pm \sqrt{3}\,i$	**A1**	Both roots, oe; need not be simplified.
	$\lvert\lambda\rvert = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \ldots$ $\arg\lambda = \arctan\sqrt{3} = \ldots$	**M1**	Attempts the modulus and argument of one of their roots, which must be of the form $p \pm qi$ with $q \neq 0$. Condone the ratio inverted. Implied by 2 and $\frac{\pi}{3}$.
	$u_n = 2^n\left(A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right)\right)$	**A1**	Correct general solution in 2^n and $\frac{n\pi}{3}$, oe, including a complex exponential form.
	$n = 0:\ A = 1$ $n = 1:\ 2\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) = 4$	**M1**	Substitutes $n = 0$ and $n = 1$ into their general solution with two arbitrary constants to form two equations. Condone slips in the trigonometric values.
	$1 + \sqrt{3}\,B = 4 \Rightarrow B = \ldots$	**dM1**	Solves their two equations to find values for both constants.
	$u_n = 2^n\left(\cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\sin\left(\frac{n\pi}{3}\right)\right)$	**A1**	Any exact real form, oe such as $2^{n+1}\cos\left(\frac{(n-1)\pi}{3}\right)$ or with $\frac{3}{\sqrt{3}}$ for $\sqrt{3}$. Not for the constants alone.
		[7]	
7(b)	$v_n = (2a)^n\left(\cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\sin\left(\frac{n\pi}{3}\right)\right)$	**M1**	Writes v_n as $(2a)^n$ times the trigonometric factor of their u_n, which must contain 2^n. Implied by $\left(\frac{2}{3}\right)^n$, 1^n or $\left(\frac{4}{3}\right)^n$, or by any case correctly described.
	$a = \frac{1}{3}:\ v_n = \left(\frac{2}{3}\right)^n(\ldots)$ $\left(\frac{2}{3}\right)^n \to 0 \Rightarrow v_n \to 0$	**E1**	States that v_n converges to 0, oe such as "tends to zero". Not for "converges" with no limit. Oscillation need not be mentioned.
	$a = \frac{1}{2}:\ v_n = 1,\ 2,\ 1,\ -1,\ -2,\ -1,\ \ldots$ periodic	**E1**	States that v_n is periodic or repeats, oe; implied by a stated period. The values need not be listed. Not for "oscillates" or "does not converge" alone.
	period 6	**B1**	

Question 7 continued

Question	Scheme	Marks	Notes
	$a = \frac{2}{3}: \ \lvert v_n \rvert \geq \left(\frac{4}{3}\right)^n \to \infty$ diverges	**E1**	States that the size of the terms increases without limit, or that v_n diverges, oe such as "unbounded". Condone "tends to infinity" for this mark.
	terms keep changing sign $\Rightarrow v_n \not\to \pm\infty$	**E1**	States that the terms keep changing sign, oe such as "oscillates"; condone "alternates". Need not say every three terms.
		[6]	
	Total for Question 7	**13**	

Question	Scheme	Marks	Notes
8(a)	$\lambda^2 + 2\lambda + 4 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a three-term auxiliary equation and solves it by the formula with their values, completing the square or a calculator. Condone sign slips in the equation.
	$\lambda = -1 \pm \sqrt{3}\,i$	**A1**	
	$\lvert\lambda\rvert = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \ldots$ $\arg\lambda = \pi - \arctan\sqrt{3} = \ldots$	**M1**	Finds the modulus and an argument of a complex root, soi, or writes the complementary function in powers of the roots. Condone the ratio inverted and the wrong quadrant.
	$2^n\left(A\cos\left(\frac{2n\pi}{3}\right) + B\sin\left(\frac{2n\pi}{3}\right)\right)$	**A1**	Any correct form, such as with $\frac{4n\pi}{3}$. May be seen within the general solution.
	$T_n = an + b$ $(an + 2a + b) + 2(an + a + b) + 4(an + b) = 7n + 25$	**M1**	Substitutes a particular solution of the form $an + b$ into all three terms of the recurrence relation. Condone slips in T_{n+1} and T_{n+2}.
	$7a = 7,\ 4a + 7b = 25 \Rightarrow a = \ldots,\ b = \ldots$	**dM1**	Equates the coefficients of n and the constant terms in their equation and solves for both a and b.
	$a = 1,\ b = 3$	**A1**	
	$A + \text{“3”} = 5$ $2\left(-\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) + \text{“4”} = 8$ $\Rightarrow A = \ldots,\ B = \ldots$	**M1**	Uses $T_0 = 5$ and $T_1 = 8$ in their general solution, a complementary function with two constants plus their $an + b$, and solves for both constants.
	$T_n = 2^n\left(2\cos\left(\frac{2n\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{2n\pi}{3}\right)\right) + n + 3$	**A1**	oe, such as $2^{n+2}\cos\left(\frac{(2n-1)\pi}{3}\right) + n + 3$. Need not be simplified.
		[9]	
8(b)	$T_{16} = \text{“}2^{16}\left(2\cos\frac{32\pi}{3} + 2\sqrt{3}\sin\frac{32\pi}{3}\right) + 19\text{”}$ $= 2^{16}(-1 + 3) + 19$	**M1**	Substitutes $n = 16$ into their part (a) expression, complementary function included, and evaluates, or iterates the recurrence relation to T_{16}. Implied by a value correct for their expression.
	$T_{16} = 131\,091$	**A1**	
		[2]	
	Total for Question 8	**11**	

Question	Scheme	Marks	Notes
9(a)	$16\lambda^2 - 24\lambda + 9 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms the auxiliary equation and solves it by any method. Condone another letter for λ.
	$\lambda = \frac{3}{4}$ $(A + Bn)\left(\frac{3}{4}\right)^n$	**A1**	Correct complementary function, oe such as $A\left(\frac{3}{4}\right)^n + Bn\left(\frac{3}{4}\right)^n$, with any two letters other than c for the constants.
	$u_n = p \Rightarrow 16p - 24p + 9p = c$	**M1**	Substitutes a constant for all three terms of the recurrence relation, as a particular solution. Any letter. Condone sign slips.
	$p = c$ $u_n = (A + Bn)\left(\frac{3}{4}\right)^n + c$	**A1**	Correct general solution, oe. Condone a missing $u_n =$.
		[4]	
9(b)(i)	$A - 6 = 30$ $\frac{3}{4}(A + B) - 6 = 60$	**M1**	Uses $u_0 = 30$ and $u_1 = 60$ in their general solution with $c = -6$, which must have two arbitrary constants, to form two equations. Condone slips.
	$A = \ldots,\ B = \ldots\ [A = 36,\ B = 52]$	**dM1**	Solves their two equations to obtain values for both constants.
	$u_n = (36 + 52n)\left(\frac{3}{4}\right)^n - 6$	**A1**	The particular solution, oe. Condone a missing $u_n =$.
		[3]	
9(b)(ii)	$u_2 =$ "$140\left(\frac{3}{4}\right)^2 - 6$" $= \ldots$ $u_3 =$ "$192\left(\frac{3}{4}\right)^3 - 6$" $= \ldots$ $u_4 =$ "$244\left(\frac{3}{4}\right)^4 - 6$" $= \ldots$	**M1**	Calculates at least two of u_2, u_3 and u_4 from their part (b)(i) solution in n, or from the recurrence relation with $c = -6$.
	$u_2 = 72.75,\ u_3 = 75,\ u_4 \approx 71.2$ Largest number of containers $= 75$	**A1**	Needs 75 with $u_2 = 72.75$ and $u_4 =$ awrt 71.2 seen, or these rounded to 73 and 71. Not for 75 alone.
		[2]	
9(b)(iii)	$u_{17} =$ "$920\left(\frac{3}{4}\right)^{17} - 6$" $= \ldots$ $u_{18} =$ "$972\left(\frac{3}{4}\right)^{18} - 6$" $= \ldots$	**M1**	Calculates terms beyond the peak from their part (b)(i) solution in n, or from the recurrence relation, seeking the first negative term. Accept $u_n = 0$ solved numerically.
	$u_{17} \approx 0.916,\ u_{18} \approx -0.520$	**A1**	Both values, awrt 0.92 and awrt -0.52, or rounded to 1 and -1.

Question 9 continued

Question	Scheme	Marks	Notes
	$n = 18 \Rightarrow$ June 2021	**A1**	Needs the month and year. Not for $n = 18$ or "month 18" alone.
	The number of containers cannot be negative.	**E1**	States that the model gives a negative number of containers, which is impossible, oe. Not for "the model breaks down" or $u_{18} < 0$ alone.
		[4]	
9(c)	$u_n = (A + Bn)\left(\frac{3}{4}\right)^n + 24$ $\left(\frac{3}{4}\right)^n \to 0$ as $n \to \infty$	**M1**	Takes the limit of their solution with $c = 24$, constants found or not, its terms in $\left(\frac{3}{4}\right)^n$ tending to 0; soi by $u_n \to 24$ written from it. Not for setting $u_{n+2} = u_{n+1} = u_n = k$ alone.
	$u_n \to 24 \Rightarrow k = 24$	**A1**	Concludes $u_n \to 24$ or $k = 24$. No comment on $n\left(\frac{3}{4}\right)^n$ or on rounding is needed.
		[2]	
	Total for Question 9	**15**	

Question	Scheme	Marks	Notes
10(a)	$\alpha+\beta=4,\ \alpha\beta=1$	**B1**	Both values.
		[1]	
10(b)(i)	$S_2=(\alpha+\beta)^2-2\alpha\beta$ $=$ "4"$^2-2\times$ "1"	**M1**	Writes $\alpha^2+\beta^2$ in terms of $\alpha+\beta$ and $\alpha\beta$ and substitutes their values from (a). Condone a sign or coefficient slip. Accept oe, such as $\left(2\pm\sqrt{3}\right)^2$ expanded and added.
	$S_4=\left(\alpha^2+\beta^2\right)^2-2(\alpha\beta)^2$ $=$ "14"$^2-2\times$ "1"^{2}	**M1**	Writes $\alpha^4+\beta^4$ in terms of $\alpha^2+\beta^2$ and $\alpha\beta$, or of $\alpha+\beta$ and $\alpha\beta$, and substitutes their values. Condone a sign or coefficient slip. Accept oe, such as $\left(2\pm\sqrt{3}\right)^4$ expanded and added.
	$S_2=14,\ S_4=194$	**A1**	Both values.
		[3]	
10(b)(ii)	$\alpha^2=4\alpha-1\Rightarrow\alpha^{n+2}=4\alpha^{n+1}-\alpha^n$ $\beta^2=4\beta-1\Rightarrow\beta^{n+2}=4\beta^{n+1}-\beta^n$	**M1**	Uses the quadratic for at least one root, multiplied by that root's nth power; or expands $(\alpha+\beta)\left(\alpha^{n+1}+\beta^{n+1}\right)$ oe. Condone sign slips.
	$\alpha^{n+2}+\beta^{n+2}=4\left(\alpha^{n+1}+\beta^{n+1}\right)-(\alpha^n+\beta^n)$ $\Rightarrow S_{n+2}=4S_{n+1}-S_n$	**A1 AG**	Correct equations in powers of both roots, combined to reach the printed relation without error. Combined steps accepted.
		[2]	
10(b)(iii)	$S_0=2,\ S_1=4$	**B1**	Both values. No comment that they are even is needed for this mark.
	$S_k=2p,\ S_{k+1}=2q$ $\Rightarrow S_{k+2}=4(2q)-2p=2(4q-p)$ $\Rightarrow S_n$ is an even integer for all $n\geq 0$	**E1**	Shows that if two consecutive terms are even integers then $4S_{k+1}-S_k$ is an even integer, and concludes for all $n\geq 0$. Needs the B1. Formal induction wording is not needed.
		[2]	
10(c)(i)	$\beta=2-\sqrt{3}>0\Rightarrow\beta^n>0$ $T_n=S_n-\beta^n<S_n$	**E1**	States that $\beta>0$ so $\beta^n>0$, and that T_n is S_n less this positive amount. Accept $\beta\approx 0.27$ as evidence. Not for $S_n=T_n+\beta^n$ alone.
		[1]	

Question 10 continued

Question	Scheme	Marks	Notes
10(c)(ii)	$0 < \beta < 1 \Rightarrow 0 < \beta^n < 1$ $S_n - 1 < T_n < S_n$	**E1**	States that $0 < \beta^n < 1$, so T_n lies between the integers $S_n - 1$ and S_n oe. A reason for rounding to the nearest integer needs $\beta^n < \frac{1}{2}$.
	Round T_n up to the next integer	**B1**	Up to the next integer, or to the nearest integer, oe such as "round up to a whole number".
		[2]	
	Total for Question 10	**11**	

Question	Scheme	Marks	Notes
11(a)	$x_{n+2} = x_{n+1} + 2y_{n+1} + 4$ $\quad = x_{n+1} + 2(3x_n + 2y_n - 1) + 4$ $2y_n = x_{n+1} - x_n - 4$	**M1**	Uses both relations to eliminate y_{n+1} and y_n, reaching an equation in x_{n+2}, x_{n+1} and x_n only. Condone sign and arithmetic slips.
	$x_{n+2} = x_{n+1} + 6x_n + (2x_{n+1} - 2x_n - 8) + 2$ $\Rightarrow x_{n+2} - 3x_{n+1} - 4x_n = -6$	**A1 AG**	Correct unsimplified equation in x_{n+2}, x_{n+1} and x_n only, leading without error to the printed relation; combined steps accepted.
		[2]	
11(b)	$\lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms a three-term auxiliary equation and solves it by any method, a calculator included. Condone sign slips in the equation.
	$\lambda = 4,\ -1$ $A \times 4^n + B(-1)^n$	**A1**	Correct complementary function, with any letters for the constants.
	$x_n = k \Rightarrow k - 3k - 4k = -6 \Rightarrow k = \ldots$	**M1**	Substitutes a constant, $x_n = k$, into the relation of part (a) and solves for k. Condone sign slips.
	$x_n = A \times 4^n + B(-1)^n + 1$	**A1**	Correct general solution, soi by their equations in A and B.
	$x_2 = 3 + 2a + 4 = 7 + 2a$	**B1**	Need not be simplified.
	$4A - B + 1 = 3$ $16A + B + 1 =$ "$7 + 2a$"	**M1**	Uses $x_1 = 3$ and their x_2, an expression in a, in their general solution with two arbitrary constants to form two equations. Condone slips.
	$20A = 8 + 2a \Rightarrow A = \ldots,\ B = \ldots$	**dM1**	Solves their two equations, reaching expressions in a for both constants. Condone arithmetic slips.
	$x_n = \frac{a+4}{10} \times 4^n + \frac{2a-2}{5}(-1)^n + 1$	**A1**	oe; the coefficients need not be simplified.
		[8]	
11(c)	"$\frac{a+4}{10}$" $\times 4^5 +$ "$\frac{2a-2}{5}$"$(-1)^5 + 1 = 1023$ $\Rightarrow a = \ldots$	**M1**	Forms an equation in a from $x_5 = 1023$, by $n = 5$ in their part (b) expression containing a or by repeated use of the relations, and solves it.
	$a = 6$	**A1**	
		[2]	

Question 11 continued

Question	Scheme	Marks	Notes
11(d)	$x_n =$ "$4^n + 2(-1)^n + 1$" $y_n = \frac{1}{2}(x_{n+1} - x_n - 4) = \ldots$	**M1**	Rearranges the first relation for y_n and substitutes their x_{n+1} and x_n from part (b) with their a. Condone sign slips.
	$y_n = \frac{3}{2} \times 4^n - 2(-1)^n - 2$	**A1**	oe, such as $\frac{1}{2}(3 \times 4^n - 4(-1)^n - 4)$.
		[2]	
	Total for Question 11	**14**	

Question	Scheme	Marks	Notes
12(a)	$2\lambda^2 - 7\lambda + 3 = 0 \Rightarrow \lambda = \ldots$	**M1**	Forms the auxiliary equation and solves it by any method, a calculator included. Condone one sign or coefficient slip in the equation.
	$\lambda = \frac{1}{2},\ 3$ $A\left(\frac{1}{2}\right)^n + B \times 3^n$	**A1**	Any letters for the two constants. Accept $A \times 2^{-n}$ oe, and the complementary function seen only within the general solution.
	$u_n = an^2 + bn + c$ $2\left[a(n+2)^2 + b(n+2) + c\right]$ $-7\left[a(n+1)^2 + b(n+1) + c\right]$ $+3\left(an^2 + bn + c\right) = 4n^2 + 6n - 3$	**M1**	Substitutes a quadratic trial solution with three unknown constants into the recurrence relation for u_n, u_{n+1} and u_{n+2}. Condone slips in expanding.
	$-2a = 4, \quad -6a - 2b = 6, \quad a - 3b - 2c = -3$ $\Rightarrow a = \ldots,\ b = \ldots,\ c = \ldots$	**dM1**	Equates coefficients of n^2, of n and the constant terms in their expansion, and solves for all three constants.
	$a = -2,\ b = 3$	**A1**	
	$c = -4$ $u_n = A\left(\frac{1}{2}\right)^n + B \times 3^n - 2n^2 + 3n - 4$	**A1**	Correct general solution, oe; $c = -4$ may be implied by it.
		[6]	
12(b)	$A + B - 4 = 5$ $\frac{1}{2}A + 3B - 2 + 3 - 4 = 4$	**M1**	Uses both initial conditions in their general solution, a complementary function with two arbitrary constants plus a particular solution, to form two equations. Condone arithmetic slips.
	$A + B = 9,\ A + 6B = 14 \Rightarrow A = \ldots,\ B = \ldots$	**dM1**	Solves their two equations, by any method, to find values for both constants.
	$u_n = 8\left(\frac{1}{2}\right)^n + 3^n - 2n^2 + 3n - 4$	**A1**	oe, such as 2^{3-n} for $8\left(\frac{1}{2}\right)^n$.
		[3]	
12(c)	$2u_2 - 7(4) + 3(5) = -3 \Rightarrow u_2 = 5$ $u_2 = 8\left(\frac{1}{2}\right)^2 + 3^2 - 8 + 6 - 4 = 5$	**B1**	Both values of 5 with substitutions shown, $n = 0$ in the recurrence relation and $n = 2$ in their part (b). The two equal values are enough; no conclusion in words is needed.
		[1]	

Question 12 continued

Question	Scheme	Marks	Notes
12(d)	$u_{n+1} = 8\left(\frac{1}{2}\right)^{n+1} + 3^{n+1}$ $\quad - 2(n+1)^2 + 3(n+1) - 4$ $v_n = u_{n+1} - 3u_n = \ldots$	**M1**	Replaces n by $n+1$ throughout their part (b) solution, which must have exponential and polynomial terms, and subtracts $3u_n$. Condone slips in expanding.
	$v_n = 4n^2 - 10n + 9 - 20\left(\frac{1}{2}\right)^n$	**A1**	Terms in 3^n cancelled and like terms collected, oe.
	$\frac{v_n}{n^p} = n^{2-p}\left(4 - \frac{10}{n} + \frac{9}{n^2} - \frac{20}{n^2}\left(\frac{1}{2}\right)^n\right)$	**M1**	Uses the dominant term of their v_n, which must be a power of n, to compare with n^p for large n. Implied by p equal to that power.
	$2 - p = 0 \Rightarrow p = 2$	**A1**	From a dominant term in n^2. No justification that other values of p fail is needed.
	$q = 4$	**A1**	From the dominant term $4n^2$; condone slips in the other terms of v_n.
		[5]	
	Total for Question 12	**15**	