OPTIONAL PURE / NUMBER THEORY

# Divisibility Tests and Number Bases

Worked solutions 11 questions · 74 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545

**Instructions**

Edexcel FP2 (9FM0/4A): numbers written in bases other than ten are not examined. Questions 1, 5 and 6 can be left out, as can any other part set in a base other than ten.

## Questions


| | | |
|---|---|---|
| Question 1 | *3 marks* | 2 |
| Question 2 | *5 marks* | 3 |
| Question 3 | *6 marks* | 4 |
| Question 4 | *9 marks* | 6 |
| Question 5 | *4 marks* | 7 |
| Question 6 | *4 marks* | 8 |
| Question 7 | *7 marks* | 9 |
| Question 8 | *9 marks* | 11 |
| Question 9 | *7 marks* | 13 |
| Question 10 | *8 marks* | 14 |
| Question 11 | *12 marks* | 15 |

**1.** The number $N$ is written as B07A in base-12 form, where the digits A and B represent ten and eleven respectively.
By first finding $N$ in decimal (base-10) form, express $N$ in base-8 form. **[3]**

**Solution**

In base 12 the place values are $12^3 = 1728$, $12^2 = 144$, 12 and 1. With B $= 11$ and A $= 10$, the decimal value is

$$\begin{aligned} N &= 11 \times 12^3 + 0 \times 12^2 + 7 \times 12 + 10 \\ &= 19\,008 + 0 + 84 + 10 \\ &= 19\,102 \end{aligned}$$

To convert to base 8, divide repeatedly by 8 and record the remainders:

$$\begin{aligned} 19\,102 &= 8 \times 2387 + 6 \\ 2387 &= 8 \times 298 + 3 \\ 298 &= 8 \times 37 + 2 \\ 37 &= 8 \times 4 + 5 \\ 4 &= 8 \times 0 + 4 \end{aligned}$$

The last remainder is the leading digit, so reading the remainders from the bottom up gives

$$N = 45236_8$$

**2.** Without performing any division, explain why $n = 20\,240\,616$ is divisible by 264 but not by 792. **[5]**

**Solution**

Factorising both divisors:

$$264 = 2^3 \times 3 \times 11 = 8 \times 3 \times 11, \qquad 792 = 2^3 \times 3^2 \times 11 = 8 \times 9 \times 11$$

Each of these factors is now checked against $n$ with a divisibility test.

**Divisibility by 8:** since $1000 = 8 \times 125$, every multiple of 1000 is divisible by 8. A number is therefore divisible by 8 exactly when the number formed by its last three digits is. The last three digits of $n$ form 616, and

$$616 = 640 - 24 = 8 \times 80 - 8 \times 3 = 8 \times 77$$

so $n$ is divisible by 8.

**Divisibility by 3 and by 9:** each power of ten is one more than a multiple of 9, since $10^k - 1 = 99\ldots9$. So a number leaves the same remainder as its digit sum on division by 9, and also on division by 3. The digit sum of $n$ is

$$2 + 0 + 2 + 4 + 0 + 6 + 1 + 6 = 21 = 3 \times 7$$

This is a multiple of 3, so $n$ is divisible by 3. It lies strictly between 18 and 27, so it is not a multiple of 9, and $n$ is not divisible by 9.

**Divisibility by 11:** since $10 = 11 - 1$, an even power of ten is one more than a multiple of 11 and an odd power of ten is one less than a multiple of 11. So a number is divisible by 11 exactly when its alternating digit sum, starting with a plus sign at the units digit, is. For $n$ this sum is

$$6 - 1 + 6 - 0 + 4 - 2 + 0 - 2 = 11$$

so $n$ is divisible by 11.

**Conclusion:** no two of 8, 3 and 11 have a common factor other than 1, and $n$ is divisible by each of them, so $n$ is divisible by their product $8 \times 3 \times 11 = 264$.

If $n$ were divisible by 792, then, since $792 = 9 \times 88$, $n$ would also be divisible by 9. This contradicts the digit-sum test, so $n$ is not divisible by 792.

**3.** Let $N$ be the number 25 107 381.

**(a)** Use a standard divisibility test to show that $N$ is a multiple of 9. **[2]**

**(b)** A student uses the following test for divisibility by 13.

> "Throw away" multiples of 13 that appear within a block of one, two or three consecutive digits of the test number, replacing that block by the remainder, written with the same number of digits.
> Stop when the number obtained is one of $0, 1, 2, \ldots, 12$.
> The test number is only divisible by 13 if that obtained number is 0.

For example, for the number $N$, they first "throw away" $247 = 13 \times 19$ from the left-hand block "251" of three digits, leaving the number $N_1 = 00\,407\,381$. At the second stage, they "throw away" $403 = 13 \times 31$ from the block "407" of $N_1$, leaving $N_2 = 00\,004\,381$; and so on, until a number from 0 to 12 is obtained.

- Justify the validity of this process.
- Continue the student's test to show that $13 \mid N$.

**[3]**

**(c)** Given that $N = 9 \times 2\,789\,709$, explain why $13 \mid 2\,789\,709$. **[1]**

**Solution**

**(a)** Since $10 \equiv 1 \pmod 9$, every power $10^k \equiv 1 \pmod 9$. So a number is congruent modulo 9 to the sum of its digits. For $N$ the digit sum is

$$2 + 5 + 1 + 0 + 7 + 3 + 8 + 1 = 27 = 9 \times 3$$

So $N \equiv 27 \equiv 0 \pmod 9$, and $N$ is a multiple of 9.

**(b)** **Validity:** suppose the rightmost digit of the chosen block is in the $10^m$ place, and the student throws away $13k$ from that block, where $k$ is a non-negative integer with $13k$ no larger than the block's value. Replacing the block by its remainder (padded with zeros to the same number of digits) leaves every other digit unchanged, so the whole number decreases by

$$13k \times 10^m$$

This is a multiple of 13, so the new number is congruent to the old one modulo 13: it leaves the same remainder on division by 13. Each stage leaves a non-negative whole number, and the number decreases strictly whenever $k \geq 1$, so the process must stop. The final number $r$, with $0 \leq r \leq 12$, therefore satisfies $N \equiv r \pmod{13}$, so $r$ is the remainder when $N$ is divided by 13. Hence $13 \mid N$ exactly when $r = 0$.

**Continuing the test:** starting from $N_2 = 00\,004\,381$, take the block 438:

$$438 = 13 \times 33 + 9$$

Throwing away $429 = 13 \times 33$ replaces the block 438 by 009, leaving

$$N_3 = 00\,000\,091$$

Now take the block 091. Since $91 = 13 \times 7$, throwing away 91 replaces the block by 000, leaving

$$N_4 = 00\,000\,000 = 0$$

The number obtained is 0, so $13 \mid N$.

**(c)** From part (b), $13 \mid N = 9 \times 2\,789\,709$. Since 13 is prime and does not divide 9, $13 \mid 2\,789\,709$.

**4.** **(a)** Making your reasoning clear and using modular arithmetic, show that $40^{11} + 4$ is divisible by 7. **[3]**

**(b)** The following 8-digit number is written with the digits $a$ and $b$ in five of its places, where each box containing the same letter holds the same digit.

$$\boxed{a}1\boxed{b}\boxed{a}9\boxed{b}\boxed{a}2$$

(i) Given that the number is divisible by 11, determine the value of the digit $a$. **[2]**

(ii) Given further that the number is divisible by both 8 and 9, determine the value of the digit $b$. **[4]**

## Solution

**(a)** Since $40 = 42 - 2$ and $42 = 6 \times 7$, $40 \equiv -2 \pmod 7$, so

$$40^{11} \equiv (-2)^{11} = -2^{11} \pmod 7$$

Now $2^3 = 8 \equiv 1 \pmod 7$, so

$$\begin{aligned} 2^{11} &= \left(2^3\right)^3 \times 2^2 \\ &\equiv 1^3 \times 4 \\ &\equiv 4 \pmod 7 \end{aligned}$$

Therefore

$$40^{11} + 4 \equiv -4 + 4 = 0 \pmod 7$$

so $40^{11} + 4$ is divisible by 7, as required.

**(b)** (i) Since $10 \equiv -1 \pmod{11}$, $10^k \equiv (-1)^k \pmod{11}$, so a number is congruent modulo 11 to its alternating digit sum, starting with a plus sign at the units digit. Reading the digits from the units digit, they are $2, a, b, 9, a, b, 1, a$, so the alternating digit sum is

$$2 - a + b - 9 + a - b + 1 - a = -6 - a$$

The number is divisible by 11, so $a + 6 \equiv 0 \pmod{11}$. Since $a$ is a digit, $6 \leq a + 6 \leq 15$, and the only multiple of 11 in this range is 11. Hence

$$a + 6 = 11 \implies a = 5$$

(ii) With $a = 5$, the digits of the number are $5, 1, b, 5, 9, b, 5, 2$.

**Divisibility by 9:** since $10 \equiv 1 \pmod 9$, the number is congruent modulo 9 to the sum of its digits:

$$5 + 1 + b + 5 + 9 + b + 5 + 2 = 27 + 2b$$

Since $27 \equiv 0 \pmod 9$, this requires $2b \equiv 0 \pmod 9$. As 2 and 9 have no common factor other than 1, 9 divides $b$, so $b = 0$ or $b = 9$.

**Divisibility by 8:** since $1000 = 8 \times 125 \equiv 0 \pmod 8$, the number is congruent modulo 8 to the number formed by its last three digits, $100b + 52$.

If $b = 0$, this is $52 = 6 \times 8 + 4$, which is not divisible by 8.

If $b = 9$, this is $952 = 8 \times 119$, which is divisible by 8.

Hence $b = 9$, and the number is 51 959 952.

**5.** Let $f(n)$ denote the base-$n$ number $10101_n$, where $n \geq 2$.

**(a)** By first writing $f(n)$ as a polynomial in $n$ and comparing it with $(n^2+1)^2$, express $f(n)$ as the product of two quadratic factors in $n$. **[2]**

**(b)** Deduce that $f(n)$ is composite for every integer $n \geq 2$, and that $10101_n$ is divisible by $111_n$ in every base $n \geq 2$. **[2]**

**Solution**

**(a)** The digits of $10101_n$ have place values $n^4, n^3, n^2, n, 1$, so

$$f(n) = 1 \times n^4 + 0 \times n^3 + 1 \times n^2 + 0 \times n + 1 = n^4 + n^2 + 1$$

Comparing with $(n^2+1)^2 = n^4 + 2n^2 + 1$ and using the difference of two squares:

$$\begin{aligned} f(n) &= (n^2+1)^2 - n^2 \\ &= \big(n^2 + 1 + n\big)\big(n^2 + 1 - n\big) \\ &= (n^2 + n + 1)(n^2 - n + 1) \end{aligned}$$

**(b)** For an integer $n \geq 2$, each factor is an integer, and

$$n^2 - n + 1 = n(n-1) + 1 \geq 2 \times 1 + 1 = 3$$

$$n^2 + n + 1 \geq 4 + 2 + 1 = 7$$

So $f(n)$ is the product of two integers, each greater than 1. Hence $f(n)$ has a factor other than 1 and itself, so it is composite for every integer $n \geq 2$.

In base $n$, the place values of $111_n$ are $n^2, n, 1$, so

$$111_n = n^2 + n + 1$$

Using part (a):

$$10101_n = f(n) = (n^2 + n + 1)(n^2 - n + 1) = 111_n \times (n^2 - n + 1)$$

Since $n^2 - n + 1$ is a positive integer, $10101_n$ is an integer multiple of $111_n$, so $10101_n$ is divisible by $111_n$ in every base $n \geq 2$.

**6.** In decimal (base-10) form, the number $N$ is 17 091.

**(a)** Express $N$ in binary (base-2) form. **[1]**

**(b)** Using the binary form of $N$, show that $N$ is divisible by 9, explaining why your method is valid. **[3]**

**Solution**

**(a)** Writing 17 091 as a sum of powers of 2, starting from the largest power not exceeding it:

$$\begin{aligned} 17\,091 &= 16\,384 + 512 + 128 + 64 + 2 + 1 \\ &= 2^{14} + 2^9 + 2^7 + 2^6 + 2^1 + 2^0 \end{aligned}$$

So the digits in positions $14, 9, 7, 6, 1, 0$ are 1 and all others are 0:

$$N = 100001011000011_2$$

**(b)** Since $2^3 = 8$, split the binary digits into groups of three, starting from the right:

$$N = 100\ 001\ 011\ 000\ 011_2$$

This is valid because a group of three bits $b_2b_1b_0$ in positions $3k+2$, $3k+1$, $3k$ contributes

$$\left(b_2 \times 2^2 + b_1 \times 2 + b_0\right) \times 2^{3k} = d \times 8^k$$

where $d = 4b_2 + 2b_1 + b_0$ is a whole number from 0 to 7. So each group is one digit of $N$ in base 8. The groups $100, 001, 011, 000, 011$ give the digits $4, 1, 3, 0, 3$, so

$$N = 4 \times 8^4 + 1 \times 8^3 + 3 \times 8^2 + 0 \times 8 + 3$$

Now $8 = 9 - 1$, so $8 \equiv -1 \pmod 9$, and therefore $8^k \equiv (-1)^k \pmod 9$ for every integer $k \geq 0$. This means that, modulo 9, a number is congruent to the alternating sum of its base-8 digits, starting with a plus sign at the units digit. Hence

$$\begin{aligned} N &\equiv 4 \times (-1)^4 + 1 \times (-1)^3 + 3 \times (-1)^2 + 0 \times (-1) + 3 \pmod 9 \\ &\equiv 3 - 0 + 3 - 1 + 4 \pmod 9 \\ &\equiv 9 \pmod 9 \\ &\equiv 0 \pmod 9 \end{aligned}$$

The remainder when $N$ is divided by 9 is 0, so $N$ is divisible by 9, as required.

**7.** A three-digit positive integer $N$ has hundreds digit $a$, tens digit $b$ and units digit $c$, so that $N = 100a+10b+c$. The integer $N$ has the following properties:

- $N$ is divisible by 11
- the sum of the digits of $N$ is odd
- $N$ is divisible by 12

**(a)** Use the first two properties to show that $a - b + c = 11$. **[2]**

**(b)** Hence determine all possible values of $N$, showing your reasoning clearly. **[5]**

**Solution**

**(a)** Since $100 = 99 + 1$ and $10 = 11 - 1$, the number can be rearranged as

$$N = 100a + 10b + c = 11(9a + b) + (a - b + c)$$

As $11(9a + b)$ is a multiple of 11, $N$ is divisible by 11 exactly when $a - b + c$ is divisible by 11. This is the divisibility test for 11.

The hundreds digit satisfies $1 \leq a \leq 9$, and $0 \leq b \leq 9$, $0 \leq c \leq 9$. The smallest value of $a - b + c$ is $1 - 9 + 0$ and the largest is $9 - 0 + 9$, so

$$-8 \leq a - b + c \leq 18$$

The only multiples of 11 in this range are 0 and 11, so $a - b + c = 0$ or $a - b + c = 11$.

The digit sum can be written as

$$a + b + c = (a - b + c) + 2b$$

Since $2b$ is even and $a + b + c$ is odd, $a - b + c$ must be odd. This rules out 0, so

$$a - b + c = 11$$

as required.

**(b)** Since $12 = 3 \times 4$ and 3 and 4 have no common factor other than 1, $N$ is divisible by 12 exactly when it is divisible by both 3 and 4.

**Divisibility by 3:** $N$ is divisible by 3 exactly when its digit sum is. Using part (a):

$$a + b + c = (a - b + c) + 2b = 11 + 2b$$

The condition $3 \mid (11 + 2b)$ becomes

$$\begin{aligned} 11 + 2b &\equiv 0 \pmod 3 \\ 2 + 2b &\equiv 0 \pmod 3 \\ 2(b + 1) &\equiv 0 \pmod 3 \end{aligned}$$

Since 3 is prime and does not divide 2, this needs $b + 1 \equiv 0 \pmod 3$, so $b \equiv 2 \pmod 3$. As $b$ is a digit, $b = 2$, 5 or 8.

**Divisibility by 4:** 100 is a multiple of 4, so $N$ is divisible by 4 exactly when the number formed by its last two digits, $10b + c$, is divisible by 4.

**Restriction from $a$:** from part (a), $a = 11 + b - c$. The condition $a \leq 9$ gives

$$11 + b - c \leq 9 \quad \Longrightarrow \quad c \geq b + 2$$

and then $a \geq 1$ holds automatically, since $c \leq 9$ gives $a \geq b + 2 \geq 2$.

Conversely, any digits with $a - b + c = 11$ give $N$ divisible by 11, by the rearrangement in part (a), and an odd digit sum $11 + 2b$. So the three properties hold exactly when $a - b + c = 11$, $b \in \{2, 5, 8\}$, $4 \mid (10b + c)$ and $c \geq b + 2$.

**Case $b = 2$:** $c \geq 4$, and $20 + c$ is divisible by 4 for $c = 4$ or $c = 8$:

$$c = 4: \quad a = 11 + 2 - 4 = 9, \quad N = 924$$

$$c = 8: \quad a = 11 + 2 - 8 = 5, \quad N = 528$$

**Case $b = 5$:** $c \geq 7$, so the last two digits are 57, 58 or 59, none of which is divisible by 4. There are no solutions.

**Case $b = 8$:** $c \geq 10$, which is impossible for a digit. There are no solutions.

The possible values are

$$N = 528 \quad \text{or} \quad N = 924$$

**8.** For integers $a$ and $b$, with $a \geq 0$ and $0 \leq b \leq 99$, the numbers $M$ and $N$ are defined by

$$M = 100a + b \quad \text{and} \quad N = a + 4b$$

**(a)** Find the integer $k$, with $1 \leq k \leq 18$, such that $M - kN$ is a multiple of 19 for all such integers $a$ and $b$. Hence show that $19 \mid M$ if and only if $19 \mid N$. **[5]**

**(b)** Demonstrate step-by-step how an algorithm based on the result of part (a) can be used to show that $3\,088\,230\,953\,541$ is a multiple of 19. **[4]**

## Solution

**(a)** Writing $M - kN$ in terms of $a$ and $b$ gives

$$M - kN = 100a + b - k(a + 4b) = (100 - k)a + (1 - 4k)b$$

For this to be a multiple of 19 for all such $a$ and $b$, it must be one when $a = 1$, $b = 0$ and when $a = 0$, $b = 1$. So both $100 - k$ and $1 - 4k$ must be multiples of 19.

Since $100 - k = 95 + (5 - k)$ and $95 = 5 \times 19$, the number $100 - k$ is a multiple of 19 exactly when $5 - k$ is. With $1 \leq k \leq 18$, we have $-13 \leq 5 - k \leq 4$, and the only multiple of 19 in this range is 0, so $k = 5$. Then $1 - 4k = 1 - 20 = -19$, which is also a multiple of 19. Hence

$$k = 5$$

With $k = 5$, for all $a$ and $b$:

$$M - 5N = 95a - 19b = 19(5a - b)$$

so $M = 5N + 19(5a - b)$, where $5a - b$ is an integer.

If $19 \mid N$, then both $5N$ and $19(5a - b)$ are multiples of 19, so their sum $M$ is a multiple of 19.

Conversely, if $19 \mid M$, then

$$5N = M - 19(5a - b)$$

is a multiple of 19. Since 19 is prime and does not divide 5, it must divide $N$, so $19 \mid N$.

Therefore $19 \mid M$ if and only if $19 \mid N$, as required.

**(b)** Any whole number can be written as $M = 100a + b$, with $b$ its last two digits (so $0 \leq b \leq 99$) and $a \geq 0$ the number formed by the remaining digits. By part (a), $M$ is a multiple of 19 if and only if $N = a + 4b$ is. So repeatedly remove the last two digits, multiply them by 4 and add the result to the number that is left. Each new number is a multiple of 19 exactly when the previous one is.

Starting from $3\,088\,230\,953\,541$:

$$3\,088\,230\,953\,541 \to 30\,882\,309\,535 + 4 \times 41 = 30\,882\,309\,699$$

$$30\,882\,309\,699 \to 308\,823\,096 + 4 \times 99 = 308\,823\,492$$

$$308\,823\,492 \to 3\,088\,234 + 4 \times 92 = 3\,088\,602$$

$$3\,088\,602 \to 30\,886 + 4 \times 2 = 30\,894$$

$$30\,894 \to 308 + 4 \times 94 = 684$$

$$684 \to 6 + 4 \times 84 = 342$$

The final number is

$$342 = 18 \times 19$$

which is a multiple of 19. Working back up the chain, each number in it is a multiple of 19, so $3\,088\,230\,953\,541$ is a multiple of 19, as required.

**9.** For each positive integer $n$, let $R_n$ denote the integer whose decimal representation consists of $n$ digits, each equal to 1; for example, $R_4 = 1111$.
Use the standard test for divisibility by 11 to prove the following statements.

**(a)** $R_{22}$ is divisible by 11, but $R_{21}$ is not divisible by 11. **[3]**

**(b)** $R_{22}$ is divisible by 121. **[4]**

**Solution**

The standard test for divisibility by 11 is that an integer with digits $a_k \dots a_2 a_1 a_0$ is divisible by 11 if and only if its alternating digit sum, starting with a plus sign at the units digit, is divisible by 11. This sum is

$$a_0 - a_1 + a_2 - a_3 + \cdots$$

**(a)** $R_{22}$ has 22 digits, each equal to 1. Its alternating digit sum splits into 11 pairs:

$$(1-1) + (1-1) + \cdots + (1-1) = 11 \times 0 = 0$$

Since 0 is divisible by 11, $R_{22}$ is divisible by 11.

$R_{21}$ has 21 digits, each equal to 1. The digits $a_0$ to $a_{19}$ form 10 pairs, and the leading digit $a_{20}$ carries a plus sign, so the alternating digit sum is

$$(1-1) + (1-1) + \cdots + (1-1) + 1 = 10 \times 0 + 1 = 1$$

Since 1 is not divisible by 11, $R_{21}$ is not divisible by 11.

**(b)** Let

$$N = 1 + 10^2 + 10^4 + \cdots + 10^{20}$$

The decimal representation of $N$ is $10101 \dots 0101$, with 21 digits. The digit is 1 in each of the 11 places $10^0, 10^2, \dots, 10^{20}$, and 0 in each of the 10 places $10^1, 10^3, \dots, 10^{19}$.

Multiplying by 11 gives

$$\begin{aligned} 11N &= 10N + N \\ &= \left(10 + 10^3 + \cdots + 10^{21}\right) + \left(1 + 10^2 + \cdots + 10^{20}\right) \\ &= 1 + 10 + 10^2 + 10^3 + \cdots + 10^{21} \\ &= R_{22} \end{aligned}$$

The last line holds because $R_{22}$ has the digit 1 in each place from $10^0$ to $10^{21}$.

Now apply the test to $N$. The digits with a plus sign are $a_0, a_2, \dots, a_{20}$, which are 11 ones. The digits with a minus sign are $a_1, a_3, \dots, a_{19}$, which are 10 zeros. So the alternating digit sum of $N$ is

$$11 \times 1 - 10 \times 0 = 11$$

This is divisible by 11, so $N = 11m$ for some integer $m$. Hence

$$R_{22} = 11N = 11 \times 11m = 121m$$

so $R_{22}$ is divisible by 121, as required.

**10.** **(a)** The number $N$ is written in base 10 as a string of identical blocks of four digits, each block having the digits $a$, $b$, $9-a$ and $9-b$, in that order, where $a$ and $b$ are integers such that $1 \leq a < 10$ and $0 \leq b < 10$. For example, with $a = 3$, $b = 7$ and three blocks, $N = 376\,237\,623\,762$.
Use standard divisibility tests to show that $N$ is always divisible by 99. **[4]**

**(b)** The number $M$ is written in base $n$, where $n > 11$, as a string of identical blocks of four digits, each block having the digits $c$, $d$, $n-1-c$ and $n-1-d$, in that order, where $c$ and $d$ are integers such that $1 \leq c < n$ and $0 \leq d < n$.
Show that $M$ is always divisible by two different numbers of the form $k_1 n + k_2$, where $k_1$ and $k_2$ are integers to be determined for each. **[4]**

## Solution

**(a)** Let $N$ consist of $m$ blocks. Since $1 \leq a \leq 9$ and $0 \leq b \leq 9$, the numbers $9-a$ and $9-b$ are also digits from 0 to 9.

**Divisibility by 9:** a number is divisible by 9 if and only if its digit sum is divisible by 9. Each block contributes

$$a + b + (9-a) + (9-b) = 18$$

so the digit sum of $N$ is $18m = 9 \times 2m$, which is divisible by 9. Hence 9 divides $N$.

**Divisibility by 11:** a number is divisible by 11 if and only if its alternating digit sum, starting with a plus sign at the units digit, is divisible by 11. Reading each block from its right-hand end, the digits are $9-b$, $9-a$, $b$, $a$. Each block has four digits, an even number, so every block starts with a + sign in the same position, and each block contributes

$$(9-b) - (9-a) + b - a = 0$$

The alternating digit sum of $N$ is therefore 0, which is divisible by 11. Hence 11 divides $N$.

Since 9 and 11 have no common factor other than 1, and both divide $N$, their product $9 \times 11 = 99$ divides $N$, as required.

**(b)** Let $M$ consist of $m$ blocks, and write $M = \sum x_r n^r$, where $x_r$ is the digit in position $r$ (the units digit being $x_0$). Each of $n-1-c$ and $n-1-d$ lies between 0 and $n-1$, so they are valid base-$n$ digits.

**Divisibility by $n-1$:** since $n^r - 1 = (n-1)\big(n^{r-1} + \cdots + n + 1\big)$, the difference $M - \sum x_r = \sum x_r(n^r - 1)$ is divisible by $n-1$. So $M$ is divisible by $n-1$ whenever its digit sum is. Each block contributes

$$c + d + (n-1-c) + (n-1-d) = 2(n-1)$$

so the digit sum is $2m(n-1)$, and $n-1$ divides $M$.

**Divisibility by $n+1$:** putting $n = -1$ in $n^r - (-1)^r$ gives 0, so by the factor theorem $n+1$ divides $n^r - (-1)^r$. Hence $M - \sum(-1)^r x_r$ is divisible by $n+1$, and $M$ is divisible by $n+1$ whenever its alternating digit sum is. As each block has four digits, the signs line up in every block, and reading each block from its right-hand end, it contributes

$$(n-1-d) - (n-1-c) + d - c = 0$$

so the alternating digit sum is 0, and $n+1$ divides $M$.

So $M$ is always divisible by $n-1$, with $k_1 = 1$, $k_2 = -1$, and by $n+1$, with $k_1 = 1$, $k_2 = 1$.

**11.** The four-digit number $n$ is written $abcd$ in base 10, where $a$, $b$, $c$ and $d$ are its digits. It has the following properties.

(1) $n \equiv 3 \pmod{13}$
(2) $n$ is divisible by 11
(3) $a + d = b + c$
(4) $n$ is divisible by 3
(5) the digit $c$ is even

**(a)** Use property (1) to explain why $d - a - 4b - 3c \equiv 3 \pmod{13}$. **[2]**

**(b)** Use properties (2) and (3) to show that $a = b$ and $c = d$. **[4]**

**(c)** Deduce that $c \equiv 4(a - 2) \pmod{13}$. **[2]**

**(d)** Hence, using properties (4) and (5), determine the number $n$, verifying that it is unique. You must make your reasoning clear. **[4]**

**Solution**

**(a)** Writing $n$ in base 10:

$$n = 1000a + 100b + 10c + d$$

Reducing each power of 10 modulo 13:

$$10 = 13 - 3 \equiv -3 \pmod{13}$$
$$100 = 104 - 4 \equiv -4 \pmod{13}$$
$$1000 = 13 \times 77 - 1 \equiv -1 \pmod{13}$$

Hence

$$n \equiv -a - 4b - 3c + d \pmod{13}$$

By property (1), $n \equiv 3 \pmod{13}$, so

$$d - a - 4b - 3c \equiv 3 \pmod{13}$$

as required.

**(b)** Reducing each power of 10 modulo 11:

$$10 = 11 - 1 \equiv -1 \pmod{11}$$
$$100 = 99 + 1 \equiv 1 \pmod{11}$$
$$1000 = 11 \times 91 - 1 \equiv -1 \pmod{11}$$

Hence $n \equiv -a + b - c + d \pmod{11}$, and by property (2) this is 0, so

$$11 \mid \big((b - a) + (d - c)\big)$$

Property (3), $a + d = b + c$, rearranges to $d - c = b - a$, so

$$(b - a) + (d - c) = 2(b - a)$$

Therefore $11 \mid 2(b - a)$. Since 11 is prime and does not divide 2, it follows that $11 \mid (b - a)$. The digits satisfy $1 \le a \le 9$ and $0 \le b \le 9$, so

$$-9 \le b - a \le 8$$

and the only multiple of 11 in this range is 0. So $b - a = 0$, giving $a = b$. Then $d - c = b - a = 0$, so $c = d$.

**(c)** Substituting $b = a$ and $d = c$ into the congruence from part (a):

$$c - a - 4a - 3c \equiv 3 \pmod{13}$$

$$-5a - 2c \equiv 3 \pmod{13}$$

$$2c \equiv -5a - 3 \pmod{13}$$

Since $-5 \equiv 8$ and $-3 \equiv 10 \pmod{13}$, this becomes

$$2c \equiv 8a + 10 = 2(4a + 5) \pmod{13}$$

So $13 \mid 2(c - 4a - 5)$. Since 13 is prime and does not divide 2, $13 \mid (c - 4a - 5)$, and therefore

$$c \equiv 4a + 5 \equiv 4a + 5 - 13 = 4a - 8 = 4(a - 2) \pmod{13}$$

as required.

**(d)** By part (b), $n$ has digits $a, a, c, c$. Since $0 \le c \le 9 < 13$, part (c) means that $c$ must equal the remainder when $4(a - 2)$ is divided by 13, and that remainder must be at most 9. Working through $a = 1, 2, \ldots, 9$:

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $4(a-2)$ | $-4$ | 0 | 4 | 8 | 12 | 16 | 20 | 24 | 28 |
| remainder mod 13 | 9 | 0 | 4 | 8 | 12 | 3 | 7 | 11 | 2 |

The remainders 12 and 11 are not digits, so $a = 5$ and $a = 8$ are impossible. The possible pairs $(a, c)$ are

$$(1, 9),\ (2, 0),\ (3, 4),\ (4, 8),\ (6, 3),\ (7, 7),\ (9, 2)$$

Property (5) requires $c$ to be even, which leaves $(2, 0)$, $(3, 4)$, $(4, 8)$ and $(9, 2)$.

Property (4): a number is divisible by 3 exactly when its digit sum is. The digit sum of $n$ is $a + a + c + c = 2(a + c)$, and since 3 is prime and does not divide 2, this requires $3 \mid (a + c)$. The remaining pairs give

$$a + c = 2, \quad 7, \quad 12, \quad 11$$

respectively, and only 12 is a multiple of 3. So the only possibility is $a = b = 4$, $c = d = 8$, and every other candidate has been ruled out.

Conversely, 4488 has all five properties: $4488 = 13 \times 345 + 3$; $4488 = 11 \times 408$; $4 + 8 = 4 + 8$; $4488 = 3 \times 1496$; and $c = 8$ is even. Hence the number is unique:

$$n = 4488$$