OPTIONAL PURE / NUMBER THEORY

Divisibility Tests and Number Bases

Questions by topic 11 questions · 74 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545

Instructions

Edexcel FP2 (9FM0/4A): numbers written in bases other than ten are not examined. Questions 1, 5 and 6 can be left out, as can any other part set in a base other than ten.

Questions

Question 1	*3 marks*	2
Question 2	*5 marks*	3
Question 3	*6 marks*	4
Question 4	*9 marks*	5
Question 5	*4 marks*	6
Question 6	*4 marks*	7
Question 7	*7 marks*	8
Question 8	*9 marks*	9
Question 9	*7 marks*	10
Question 10	*8 marks*	11
Question 11	*12 marks*	12

1. The number N is written as B07A in base-12 form, where the digits A and B represent ten and eleven respectively.
By first finding N in decimal (base-10) form, express N in base-8 form. **[3]**

2. Without performing any division, explain why $n = 20\,240\,616$ is divisible by 264 but not by 792. **[5]**

3. Let N be the number $25\,107\,381$.

(a) Use a standard divisibility test to show that N is a multiple of 9. **[2]**

(b) A student uses the following test for divisibility by 13.

> "Throw away" multiples of 13 that appear within a block of one, two or three consecutive digits of the test number, replacing that block by the remainder, written with the same number of digits.
> Stop when the number obtained is one of $0, 1, 2, \ldots, 12$.
> The test number is only divisible by 13 if that obtained number is 0.

For example, for the number N, they first "throw away" $247 = 13 \times 19$ from the left-hand block "251" of three digits, leaving the number $N_1 = 00\,407\,381$. At the second stage, they "throw away" $403 = 13 \times 31$ from the block "407" of N_1, leaving $N_2 = 00\,004\,381$; and so on, until a number from 0 to 12 is obtained.

- Justify the validity of this process.
- Continue the student's test to show that $13 \mid N$. **[3]**

(c) Given that $N = 9 \times 2\,789\,709$, explain why $13 \mid 2\,789\,709$. **[1]**

4. **(a)** Making your reasoning clear and using modular arithmetic, show that $40^{11} + 4$ is divisible by 7. **[3]**

(b) The following 8-digit number is written with the digits a and b in five of its places, where each box containing the same letter holds the same digit.

$$\boxed{a}1\boxed{b}\boxed{a}9\boxed{b}\boxed{a}2$$

(i) Given that the number is divisible by 11, determine the value of the digit a. **[2]**

(ii) Given further that the number is divisible by both 8 and 9, determine the value of the digit b. **[4]**

5. Let $f(n)$ denote the base-n number 10101_n, where $n \geq 2$.

(a) By first writing $f(n)$ as a polynomial in n and comparing it with $(n^2+1)^2$, express $f(n)$ as the product of two quadratic factors in n. **[2]**

(b) Deduce that $f(n)$ is composite for every integer $n \geq 2$, and that 10101_n is divisible by 111_n in every base $n \geq 2$. **[2]**

6. In decimal (base-10) form, the number N is 17 091.

(a) Express N in binary (base-2) form. **[1]**

(b) Using the binary form of N, show that N is divisible by 9, explaining why your method is valid. **[3]**

7. A three-digit positive integer N has hundreds digit a, tens digit b and units digit c, so that $N = 100a+10b+c$. The integer N has the following properties:

- N is divisible by 11
- the sum of the digits of N is odd
- N is divisible by 12

(a) Use the first two properties to show that $a - b + c = 11$. **[2]**

(b) Hence determine all possible values of N, showing your reasoning clearly. **[5]**

8. For integers a and b, with $a \geq 0$ and $0 \leq b \leq 99$, the numbers M and N are defined by

$$M = 100a + b \quad \text{and} \quad N = a + 4b$$

(a) Find the integer k, with $1 \leq k \leq 18$, such that $M - kN$ is a multiple of 19 for all such integers a and b. Hence show that $19 \mid M$ if and only if $19 \mid N$. **[5]**

(b) Demonstrate step-by-step how an algorithm based on the result of part (a) can be used to show that $3\,088\,230\,953\,541$ is a multiple of 19. **[4]**

9. For each positive integer n, let R_n denote the integer whose decimal representation consists of n digits, each equal to 1; for example, $R_4 = 1111$.
Use the standard test for divisibility by 11 to prove the following statements.

(a) R_{22} is divisible by 11, but R_{21} is not divisible by 11. **[3]**

(b) R_{22} is divisible by 121. **[4]**

10. **(a)** The number N is written in base 10 as a string of identical blocks of four digits, each block having the digits a, b, $9-a$ and $9-b$, in that order, where a and b are integers such that $1 \leq a < 10$ and $0 \leq b < 10$. For example, with $a = 3$, $b = 7$ and three blocks, $N = 376\,237\,623\,762$.
Use standard divisibility tests to show that N is always divisible by 99. **[4]**

(b) The number M is written in base n, where $n > 11$, as a string of identical blocks of four digits, each block having the digits c, d, $n-1-c$ and $n-1-d$, in that order, where c and d are integers such that $1 \leq c < n$ and $0 \leq d < n$.
Show that M is always divisible by two different numbers of the form $k_1 n + k_2$, where k_1 and k_2 are integers to be determined for each. **[4]**

11. The four-digit number n is written $abcd$ in base 10, where a, b, c and d are its digits. It has the following properties.

(1) $n \equiv 3 \pmod{13}$
(2) n is divisible by 11
(3) $a + d = b + c$
(4) n is divisible by 3
(5) the digit c is even

(a) Use property (1) to explain why $d - a - 4b - 3c \equiv 3 \pmod{13}$. **[2]**

(b) Use properties (2) and (3) to show that $a = b$ and $c = d$. **[4]**

(c) Deduce that $c \equiv 4(a - 2) \pmod{13}$. **[2]**

(d) Hence, using properties (4) and (5), determine the number n, verifying that it is unique. You must make your reasoning clear. **[4]**