OPTIONAL PURE / NUMBER THEORY

Linear and Simultaneous Congruences

Mark scheme 10 questions · 75 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545

Questions

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	Total	**75**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$17 \times 11 = 187 \equiv 2 \pmod{37}$	**B1**	
		[1]	
1(b)	$11 \times 17x \equiv 11 \times 5 \pmod{37}$ $\Rightarrow$ "2"$x \equiv \ldots \pmod{37}$	**M1**	Multiplies both sides by 11 and replaces 11×17 by their value from part (a).
	$2x \equiv 55 \equiv 18 \pmod{37}$	**A1**	$2x \equiv 55$ or $2x \equiv 18 \pmod{37}$.
	$2x \equiv 18 \Rightarrow x \equiv \ldots \pmod{37}$	**M1**	Divides their congruence by 2, adding 37 if needed, or multiplies by an inverse of 2 modulo 37. Condone arithmetic slips.
	$x \equiv 9 \pmod{37}$	**A1**	In the required form only: $x \equiv 46 \pmod{37}$, or $x = 9$ alone, scores A0.
		[4]	
Total for Question 1		**5**	

Question	Scheme	Marks	Notes
2(a)	$\text{hcf}(24, 54) = 6$	**B1**	Or 3 found as a common factor of 24 and 54.
	$24x - 54m$ is a multiple of 6, but $6 \nmid 20$ so no solutions	**E1**	States that 6, or 3, divides 24 and 54 but not 20, and concludes there are no solutions.
		[2]	
2(b)(i)	$\text{hcf}(15, 40) = 5$, so need $5 \mid 2k + 1$	**M1**	States that 5 divides $2k + 1$, or lists the odd multiples of 5 as possible values of $2k + 1$.
	$2k + 1 \equiv 0 \pmod 5 \Rightarrow k \equiv 2 \pmod 5$	**A1**	oe, such as $2k + 1 = 5, 15, \ldots, 55$.
	$k = 2,\ 7,\ 12,\ 17,\ 22,\ 27$	**A1**	All six values and no others.
		[3]	
2(b)(ii)	5 solutions modulo 40 for each k	**B1**	
	$\text{hcf}(15, 40) = 5$, which divides $2k + 1$ or, with $2k + 1 = 5q$, $y \equiv 3q \pmod 8$ $\Rightarrow y \equiv 3q,\ 3q + 8,\ \ldots,\ 3q + 32 \pmod{40}$	**E1**	Gives a reason: the number of solutions equals $\text{hcf}(15, 40) = 5$, or one solution modulo 8 gives five distinct solutions modulo 40.
		[2]	
	Total for Question 2	**7**	

Question	Scheme	Marks	Notes
3(a)	$13^{-1} \equiv \ldots$ $[= 12]$ $(\text{mod } 31)$ $x \equiv$ "12" $\times 6 \equiv \ldots$ $(\text{mod } 31)$	**M1**	Finds an inverse of 13 modulo 31 by any method and multiplies 6 by it, or solves by another valid method such as adding multiples of 31. Condone arithmetic slips.
	$x \equiv 10$ $(\text{mod } 31)$	**A1**	oe, such as $x = 31k + 10$. Not 10 alone.
		[2]	
3(b)	$6x - 4 = 16k \Rightarrow 3x \equiv 2$ $(\text{mod } 8)$ $x \equiv$ "3" $\times 2 \equiv \ldots$ $(\text{mod } 8)$	**M1**	Rewrites the first congruence modulo 8 and solves it by any method, otherwise deduces the parity of its solutions, or substitutes $x = 12n + 7$ into it. Condone arithmetic slips.
	$x \equiv 6$ $(\text{mod } 8)$, so x is even	**A1**	Correct congruence with x even stated, or x even justified directly, as from $3x = 8k + 2$, or $8n \equiv 10$ $(\text{mod } 16)$ oe.
	$x = 12n + 7 = 2(6n + 3) + 1$, so x is odd No integer is both even and odd: no solution	**A1 AG**	Shows the second congruence gives x odd, or another incompatibility such as $x \equiv 2$ and $x \equiv 3$ $(\text{mod } 4)$, and concludes no solution, with no errors.
		[3]	
Total for Question 3		**5**	

Question	Scheme	Marks	Notes
4(a)	$83 = 2 \times 30 + 23, \quad 30 = 1 \times 23 + 7$ $23 = 3 \times 7 + 2, \quad 7 = 3 \times 2 + 1$	**M1**	Carries out at least two successive divisions with remainder, each divisor divided by the previous remainder. Condone arithmetic slips.
	$2 = 2 \times 1 + 0$ Last non-zero remainder 1, so hcf $= 1$	**A1 AG**	All divisions correct and a conclusion that the highest common factor is 1; stopping at remainder 1 accepted.
		[2]	
4(b)	$1 = 7 - 3 \times 2 = 7 - 3(23 - 3 \times 7)$	**M1**	Writes 1 from their division with remainder 1 and substitutes for the previous remainder, or writes two successive remainders as combinations of 30 and 83. Condone sign slips.
	$= 10 \times 30 - 13 \times 23$ $= 10 \times 30 - 13(83 - 2 \times 30)$	**dM1**	Substitutes for each remainder in turn, or continues their combinations down to remainder 1, reaching 1 as a combination of 30 and 83 only. Condone arithmetic slips.
	$p = 36, \ q = -13$	**A1**	Or any correct pair, $p = 36 + 83k$, $q = -13 - 30k$.
		[3]	
4(c)	$30 \times$ "36" $\equiv 1 \pmod{83}$ $\Rightarrow x \equiv$ "36" $\times 11 \equiv \ldots \pmod{83}$	**M1**	Multiplies 11 by their p from (b) as the inverse of 30 modulo 83.
	$x \equiv 64 \pmod{83}$	**A1**	Required form only; $x \equiv 396$ or $x \equiv -19$ scores A0.
		[2]	
4(d)	$83 \times$ ("-13") $\equiv 1 \pmod{30}$ $\Rightarrow y \equiv$ "-13" $\times 4 \equiv \ldots \pmod{30}$	**M1**	Multiplies 4 by their q from (b), or a value congruent to it modulo 30, as the inverse of 83 modulo 30.
	$y \equiv 8 \pmod{30}$	**A1**	Required form only; $y \equiv -52$ or $y \equiv -22$ scores A0.
		[2]	
	Total for Question 4	**9**	

Question	Scheme	Marks	Notes
5(a)	$89 = 4 \times 21 + 5, \quad 21 = 4 \times 5 + 1$ $1 = 21 - 4(89 - 4 \times 21) = \ldots$	**M1**	Attempts an inverse of 21 modulo 89, by the Euclidean algorithm with back substitution or any valid method, or solves $21x = 4 + 89k$ otherwise, which with $x = 68 + 89k$ earns all four marks. Condone arithmetic slips.
	$1 = 17 \times 21 - 4 \times 89 \Rightarrow 17 \times 21 \equiv 1 \pmod{89}$	**A1**	Inverse 17, soi.
	$x \equiv$ "17" $\times 4 \equiv \ldots \pmod{89}$	**M1**	Multiplies both sides by their inverse.
	$x = 68 + 89k, \ k \in \mathbb{Z}$	**A1**	Accept $x \equiv 68 \pmod{89}$ oe. Not 68 alone.
		[4]	
5(b)(i)	$68 \equiv -21 \Rightarrow 21x \equiv -8 \pmod{89}$ $x \equiv$ "17" $\times (-8) \equiv \ldots \pmod{89}$	**M1**	Solves for x by any valid method, such as relating 68 to -21 or to 4×17 and using part (a), or an inverse of 68 modulo 89. Condone arithmetic slips.
	$x \equiv 42 \pmod{89}$	**A1**	oe, such as $x = 42 + 89k$. Not 42 alone.
		[2]	
5(b)(ii)	Splits by $178 = 2 \times 89$: $21x \equiv 9 \pmod{89}, \quad 21x \equiv 9 \pmod{2}$	**M1**	Splits into congruences modulo 89 and 2, tests both residues modulo 178 after reducing modulo 89, or uses an inverse of 21 modulo 178. All three marks with $x \equiv 153 \pmod{178}$.
	$x \equiv 64 \pmod{89}$, x odd	**A1**	Both results, or $x \equiv 64 \pmod{89}$ with 64 rejected by a correct test.
	$x = 64 + 89(2m + 1) \Rightarrow x \equiv 153 \pmod{178}$	**A1**	oe, such as $x = 153 + 178m$. Not 153 alone.
		[3]	
	Total for Question 5	**9**	

Question	Scheme	Marks	Notes
6(a)	$5 \times 7 \equiv 1 \pmod{17}$ $\Rightarrow x \equiv 7 \times 8 = 56 \equiv \ldots \pmod{17}$	**M1**	Uses an inverse of 5 modulo 17, or any valid method such as adding multiples of 17 to 8 until divisible by 5. Condone arithmetic slips.
	$x \equiv 5 \pmod{17}$	**A1**	In the required form only: $x \equiv 56 \pmod{17}$, or $x = 5$ alone, scores A0.
		[2]	
6(b)	$9 \times 18 \equiv 1 \pmod{23}$ $\Rightarrow x \equiv 18 \times 4 = 72 \equiv \ldots \pmod{23}$	**M1**	Rewrites $9x \equiv 4 \pmod{23}$ in the form $x \equiv b$ by an inverse of 9 or any valid method. Condone arithmetic slips.
	$x \equiv 3 \pmod{23}$	**A1**	
	$x =$ "5" $+ 17k$: "5" $+ 17k \equiv$ "3" $\pmod{23}$ $\Rightarrow 17k \equiv 21 \pmod{23} \Rightarrow k \equiv \ldots$	**M1**	Combines two congruences of the form $x \equiv a$, by substitution, listing terms or the Chinese remainder theorem, and attempts a common value. Condone arithmetic slips.
	$k \equiv 8 \pmod{23}$ $\Rightarrow x \equiv 141 \pmod{391}$	**A1**	oe, such as $x = 141 + 391m$.
	$1000 < 141 + 391m < 2000 \Rightarrow m = 3,\ 4$ $x = 1314,\ 1705$	**A1**	Both values and no others.
		[5]	
	Total for Question 6	**7**	

Question	Scheme	Marks	Notes
7(a)	$n \equiv 3x + 8y \pmod{17}$	**B1**	oe, including coefficients replaced by values congruent modulo 17, such as $3x - 9y$.
		[1]	
7(b)	$3(45 - y) + 8y \equiv 5 \pmod{17}$	**M1**	Uses $x + y = 45$ to eliminate one variable in their congruence and sets it congruent to 5. Condone sign slips.
	$5y \equiv 6 \pmod{17}$	**A1**	oe, such as $5y \equiv -11$, or $5x \equiv 15$ if y is eliminated.
	$7 \times 5y \equiv 7 \times 6 \pmod{17} \Rightarrow y \equiv \ldots$	**M1**	Solves their linear congruence by any valid method: an inverse of the coefficient, trial of residues, or writing $5y = 6 + 17t$.
	$y \equiv 8 \pmod{17}$	**A1**	oe, or $x \equiv 3 \pmod{17}$ if y was eliminated.
	$y = 8 + 17k,\ 0 \leq y \leq 45 \Rightarrow y = \ldots$	**M1**	Adds multiples of 17 to their residue for y (or x, then $y = 45 - x$), restricted to $0 \leq y \leq 45$, listing at least two values.
	$y = 8,\ 25,\ 42$	**A1**	All three values and no others.
		[6]	
Total for Question 7		**7**	

Question	Scheme	Marks	Notes
8(a)	$2x \equiv 3 \pmod 7$: $\text{hcf}(2,7) = 1$ $5x \equiv 2 \pmod{12}$: $\text{hcf}(5,12) = 1$ $\Rightarrow$ each has exactly one solution	**E1**	States 2 is coprime to 7 and 5 to 12, or that each has an inverse, so each congruence has exactly one solution modulo its modulus.
	$\text{hcf}(7,12) = 1 \Rightarrow$ Chinese remainder theorem $\Rightarrow$ exactly one solution modulo 84	**E1**	States 7 and 12 are coprime and concludes one solution modulo 84 by the Chinese remainder theorem. Not for $84 = 7 \times 12$ alone.
		[2]	
8(b)	$2 \times 4 \equiv 1 \pmod 7$ $\Rightarrow x \equiv 4 \times 3 \equiv \ldots \pmod 7$	**M1**	Solves either congruence for x by an inverse, testing residues or any valid method. Condone arithmetic slips.
	$x \equiv 5 \pmod 7$	**A1**	
	$x \equiv 10 \pmod{12}$	**A1**	
	"10" $+ 12k \equiv$ "5" $\pmod 7$ $\Rightarrow 5k \equiv 2 \Rightarrow k \equiv \ldots \pmod 7$	**M1**	Combines their two congruences, substituting the general form of one into the other and solving, or listing values of one tested in the other. Condone arithmetic slips.
	$x \equiv 82 \pmod{84}$	**A1**	Or $x = 82 + 84m$, oe.
	$x = 82,\ 166,\ 250,\ 334,\ 418$	**A1**	All five values and no others.
		[6]	
Total for Question 8		**8**	

Question	Scheme	Marks	Notes
9(a)	$270 = 2 \times 114 + 42$	**M1**	Divides 270 by 114, giving a quotient and a remainder. Condone arithmetic slips.
	$114 = 2 \times 42 + 30, \quad 42 = 1 \times 30 + 12$ $30 = 2 \times 12 + 6, \quad 12 = 2 \times 6 + 0$	**M1**	Divides each divisor by its remainder in turn, continuing to a remainder of zero. Condone arithmetic slips.
	$h = 6$	**A1**	
		[3]	
9(b)	$6 = 30 - 2 \times 12 = 30 - 2 \times (42 - 30)$	**M1**	Writes their h from the division giving it as a remainder and substitutes for the remainder before it, or writes two successive remainders as combinations of 270 and 114.
	$6 = 3 \times (114 - 2 \times 42) - 2 \times 42$ $= 3 \times 114 - 8 \times (270 - 2 \times 114)$	**dM1**	Substitutes for each remainder in turn, or continues their combinations down to their h, reaching a combination of 270 and 114 only. Condone arithmetic slips.
	$270 \times (-8) + 114 \times 19 = 6$ $\Rightarrow a = -8, \ b = 19$	**A1**	Or any other correct pair, such as $a = 11$, $b = -26$.
		[3]	
9(c)	$270x - 24 = 114k \Rightarrow 45x - 4 = 19k$	**M1**	Divides the coefficient of x (270, or 42 after reducing), 24 and the modulus 114 by their h, as a congruence or an equation in k.
	$45x \equiv 4 \pmod{19}$	**A1**	oe, such as $7x \equiv 4 \pmod{19}$.
	$45 \times (-8) + 19 \times 19 = 1$ $\Rightarrow x \equiv -8 \times 4 \equiv \ldots \pmod{19}$	**M1**	Uses their part (b) divided by their h for an inverse of their coefficient modulo their reduced modulus, and multiplies their reduced congruence by it.
	$x \equiv 6 \pmod{19}$	**A1**	oe, such as $x \equiv -32 \pmod{19}$.
	$x = 6, \ 25, \ 44, \ 63, \ 82, \ 101$	**A1**	All six values and no others in $0 \leq x < 114$.
		[5]	
	Total for Question 9	**11**	

Question	Scheme	Marks	Notes
10	$7 \times 4x \equiv 7 \times 3 \pmod 9$ $\Rightarrow x \equiv 3 \pmod 9$	**M1 A1**	**M1** Multiplies $4x \equiv 3 \pmod 9$ or $3x \equiv 2 \pmod{13}$ by an inverse, or solves either by any valid means such as trial. Condone an arithmetic slip.
	$9 \times 3x \equiv 9 \times 2 \pmod{13}$ $\Rightarrow x \equiv 5 \pmod{13}$	**A1**	
	$x = 7 + 11k \Rightarrow 7 + 11k \equiv 3 \pmod 9$ $\Rightarrow 2k \equiv 5 \pmod 9 \Rightarrow k \equiv \ldots$	**M1**	Combines any two of the congruences, as by writing one as $x = a + mk$, substituting into another and solving for k. Condone arithmetic slips.
	$k \equiv 7 \pmod 9 \Rightarrow x = 84 + 99j$	**A1**	oe, such as $x \equiv 84 \pmod{99}$, or $x \equiv 57 \pmod{117}$ or $x \equiv 18 \pmod{143}$ for the other pairs.
	"$84 + 99j$" $\equiv$ "5" $\pmod{13}$ $\Rightarrow 8j \equiv 12 \pmod{13} \Rightarrow j \equiv \ldots$	**dM1**	Substitutes their combined form into the remaining congruence and solves for the parameter. Condone arithmetic slips.
	$j \equiv 8 \pmod{13} \Rightarrow x = 1287n + 876$	**A1**	In the required form only: $x \equiv 876 \pmod{1287}$ alone, or r outside $0 \leq r < 1287$, scores A0.
	Total for Question 10	**7**	