OPTIONAL PURE / NUMBER THEORY

Divisibility Tests and Number Bases

Mark scheme 11 questions · 74 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545

Questions

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	Total	**74**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$N = 11 \times 12^3 + 0 \times 12^2 + 7 \times 12 + 10$ $= \ldots$ $[= 19\,102]$	**M1**	Uses place values 12^3, 12^2, 12, 1 with B = 11 and A = 10. Condone arithmetic slips.
	"19 102" $= 8 \times 2387 + 6$ $2387 = 8 \times 298 + 3, \ \ldots$	**M1**	Divides their decimal value repeatedly by 8, recording remainders, or subtracts multiples of powers of 8. Condone arithmetic slips.
	$N = 45236_8$	**A1**	Condone the base subscript omitted.
	Total for Question 1	**3**	

Question	Scheme	Marks	Notes
2	$264 = 8 \times 3 \times 11$ $792 = 8 \times 9 \times 11$	**B1**	Both factorisations, or the prime factorisations $2^3 \times 3 \times 11$ and $2^3 \times 3^2 \times 11$.
	$616 = 8 \times \ldots$ $2 + 0 + 2 + 4 + 0 + 6 + 1 + 6 = \ldots$ $6 - 1 + 6 - 0 + 4 - 2 + 0 - 2 = \ldots$	**M1**	Applies a valid test for each of 8, 3 and 11, such as last three digits, digit sum, alternating digit sum. Condone arithmetic slips. Not for dividing n itself.
	$616 = 8 \times 77$, digit sum $21 = 3 \times 7$ alternating digit sum $= 11$	**A1**	All three correct, each stated to satisfy its test.
	21 not a multiple of $9 \Rightarrow 9 \nmid n$	**A1**	
	No two of 8, 3, 11 have a common factor other than 1 $\Rightarrow 264 \mid n$ $9 \mid 792$ and $9 \nmid n \Rightarrow 792 \nmid n$	**E1**	States both conclusions: $264 \mid n$ citing that no two of 8, 3, 11 have a common factor other than 1 (oe, such as coprime), and $792 \nmid n$ citing $9 \mid 792$ with $9 \nmid n$.
	Total for Question 2	**5**	

Question	Scheme	Marks	Notes
3(a)	$2+5+1+0+7+3+8+1=\ldots$	**M1**	Sums the digits of N to test for divisibility by 9. Condone one arithmetic slip.
	$=27=9\times 3\Rightarrow 9 \mid N$	**A1 AG**	Digit sum 27, stated as a multiple of 9, with a conclusion that N is a multiple of 9.
		[2]	
3(b)	Each stage subtracts $13k\times 10^m$ $\Rightarrow$ remainder modulo 13 unchanged	**E1**	States that each stage subtracts a multiple of 13 times a power of 10, so the remainder on division by 13 is unchanged. Termination need not be discussed.
	$438=13\times 33+9\Rightarrow N_3=00\,000\,091$ $91=13\times 7\Rightarrow N_4=\ldots$	**M1**	Applies the test to N_2, throwing away multiples of 13 from its blocks and continuing towards a number from 0 to 12. Any valid choice of blocks; condone one arithmetic slip.
	$N_4=0\Rightarrow 13 \mid N$	**A1 AG**	Correct stages reaching 0, with a conclusion that $13 \mid N$.
		[3]	
3(c)	$13 \mid 9\times 2\,789\,709$, 13 prime, $13\nmid 9$ $\Rightarrow 13 \mid 2\,789\,709$	**E1**	States that 13 is prime and does not divide 9, or that 13 and 9 have no common factor other than 1, so $13 \mid 2\,789\,709$.
		[1]	
	Total for Question 3	**6**	

Question	Scheme	Marks	Notes
4(a)	$40 \equiv -2, \quad 2^3 \equiv 1 \pmod 7$ $40^{11} \equiv -\left(2^3\right)^3 \times 2^2 \equiv \ldots$	**M1**	Finds 40 modulo 7, then reduces 40^{11} by any valid route, such as a power congruent to ± 1, repeated squaring or $(-2)^{11}$. Condone sign slips.
	$40^{11} \equiv -4 \ [\equiv 3] \pmod 7$	**A1**	-4 or 3, oe.
	$40^{11} + 4 \equiv -4 + 4 = 0 \pmod 7$, so divisible by 7	**A1 AG**	Correct congruences leading without error to $0 \pmod 7$, with a conclusion of divisibility by 7.
		[3]	
4(b)(i)	$2 - a + b - 9 + a - b + 1 - a = -6 - a$ $a + 6 \equiv 0 \pmod{11} \Rightarrow a = \ldots$	**M1**	Forms the alternating digit sum of all eight digits, either sign convention, and sets it to a multiple of 11. Condone one slip.
	$a = 5$	**A1**	
		[2]	
4(b)(ii)	"5" $+ 1 + b +$ "5" $+ 9 + b +$ "5" $+ 2 = 27 + 2b$ $9 \mid 2b \Rightarrow b = \ldots$	**M1**	Sums all eight digits with their a and sets the sum divisible by 9. Condone one arithmetic slip.
	$b = 0$ or $b = 9$	**A1**	Both values, and no others.
	$b = 0$: $52 = 6 \times 8 + 4$, not divisible by 8 $b = 9$: $952 = 8 \times 119$	**M1**	Tests the last three digits, or the whole number, for divisibility by 8, for their candidate values of b or for all b.
	$b = 9$	**A1**	cso, with $b = 0$ rejected.
		[4]	
Total for Question 4		**9**	

Question	Scheme	Marks	Notes
5(a)	$f(n) = n^4 + n^2 + 1 = (n^2+1)^2 - n^2$ $= \ldots$	**M1**	Writes $f(n) = n^4 + n^2 + 1$ as $(n^2+1)^2 - n^2$ and attempts to factorise it as a difference of two squares. Condone sign slips.
	$f(n) = (n^2 + n + 1)(n^2 - n + 1)$	**A1**	Factors in either order, oe.
		[2]	
5(b)	$n^2 - n + 1 = n(n-1) + 1 \geq 3, \quad n^2 + n + 1 \geq 7$ $\Rightarrow f(n)$ composite	**E1**	Shows both factors are integers greater than 1 for $n \geq 2$, with a valid reason for $n^2 - n + 1$, and concludes $f(n)$ is composite.
	$111_n = n^2 + n + 1$ $10101_n = 111_n \times (n^2 - n + 1)$	**E1**	States $111_n = n^2 + n + 1$ and uses (a) to write 10101_n as 111_n times an integer, concluding divisibility.
		[2]	
	Total for Question 5	**4**	

Question	Scheme	Marks	Notes
6(a)	$N = 100001011000011_2$	**B1**	
		[1]	
6(b)	$N = 100\ 001\ 011\ 000\ 011_2$ $= 41303_8$	**M1**	Splits their bits from the right into threes and converts each group to a base-8 digit, or uses another valid grouping such as sixes. Condone one conversion slip.
	$2^{3k} = 8^k$, so each group of three bits is a base-8 digit $8 \equiv -1 \pmod 9 \Rightarrow 8^k \equiv (-1)^k \pmod 9$	**E1**	States that each group of three bits is a base-8 digit and that $8 \equiv -1 \pmod 9$ gives the alternating digit sum, oe for another valid grouping. Both points needed.
	$N \equiv 3 - 0 + 3 - 1 + 4 = 9 \equiv 0 \pmod 9$ so N is divisible by 9	**A1 AG**	cso. Correct alternating digit sum from correct digits, shown to be 0 (mod 9), with a conclusion that N is divisible by 9.
		[3]	
	Total for Question 6	**4**	

Question	Scheme	Marks	Notes
7(a)	$N = 11(9a+b)+(a-b+c)$ $-8 \le a-b+c \le 18$ $\Rightarrow a-b+c = 0$ or 11	**M1**	Uses the test for 11, or the rearrangement, with digit ranges to restrict $a-b+c$ to 0 or 11. Condone the bounds not stated exactly.
	$a+b+c = (a-b+c)+2b$ odd $\Rightarrow a-b+c$ odd $\Rightarrow a-b+c = 11$	**A1 AG**	Relates the digit sum's parity to $a-b+c$, rejects 0 and reaches the printed result without error.
		[2]	
7(b)	$a+b+c = 11+2b$ $3 \mid (11+2b) \Rightarrow b = \ldots$	**M1**	Uses divisibility by 3 of the digit sum, written in b using part (a), and attempts its b values by any means.
	$b = 2,\ 5,\ 8$	**A1**	All three, and no others.
	$4 \mid (10b+c)$ $a = 11+b-c \le 9 \Rightarrow c \ge b+2$	**M1**	Tests $10b+c$ for divisibility by 4, with $a = 11+b-c$ a digit, for at least one of their b values.
	$b = 2:\ c = 4$ or 8 $\Rightarrow N = 924$ or 528	**A1**	One of 528 or 924.
	$b = 5$: $57, 58, 59$ not divisible by 4 $b = 8$: $c \ge 10$, impossible $N = 528$ or 924 only	**A1**	Both values and no others, with $b = 5$ and $b = 8$ rejected with a reason.
		[5]	
Total for Question 7		**7**	

Question	Scheme	Marks	Notes
8(a)	$M - kN = (100 - k)a + (1 - 4k)b$ $19 \mid (100 - k)$ and $19 \mid (1 - 4k) \Rightarrow k = \ldots$	**M1**	States that the coefficients of a and b in $M - kN$, in any form, must both be multiples of 19; congruences mod 19 accepted. Condone sign slips.
	$k = 5$	**A1**	
	$M - 5N = 95a - 19b = 19(5a - b)$	**M1**	Uses $k = 5$ to write $M - 5N$ as 19 times an integer expression, or $M \equiv 5N \pmod{19}$.
	$19 \mid N \Rightarrow 19 \mid \big(5N + 19(5a - b)\big) = M$	**A1**	
	$19 \mid M \Rightarrow 19 \mid 5N = M - 19(5a - b)$ Since 19 is prime and does not divide 5, $19 \mid N$	**A1 AG**	Needs 19 prime or coprime to 5, or an equivalent such as $N = 4M - 19(\ldots)$, and a conclusion covering both directions. A0 for dividing by 5 unjustified.
		[5]	
8(b)	$3\,088\,230\,953\,541 \rightarrow 30\,882\,309\,535 + 4 \times 41$ $= \ldots$	**M1**	Splits off the last two digits as b and forms $a + 4b$. Condone arithmetic slips.
	$= 30\,882\,309\,699$	**A1**	
	$\rightarrow 308\,823\,492 \rightarrow 3\,088\,602$ $\rightarrow 30\,894 \rightarrow 684$	**dM1**	Applies the step repeatedly to their results until a number is checked directly for divisibility by 19. Condone arithmetic slips.
	$684 \rightarrow 6 + 4 \times 84 = 342 = 18 \times 19$ so $3\,088\,230\,953\,541$ is a multiple of 19	**A1 AG**	cso. Ends at a number shown to be a multiple of 19, such as $684 = 36 \times 19$ or $342 = 18 \times 19$, with a conclusion about the original number.
		[4]	
	Total for Question 8	**9**	

Question	Scheme	Marks	Notes
9(a)	R_{22}: $(1-1)+(1-1)+\cdots+(1-1)=\ldots$	**M1**	Forms the alternating digit sum of R_{22}, by pairing or by counting 11 plus and 11 minus signs.
	$=0$, divisible by 11, so $11 \mid R_{22}$	**A1**	Sum 0 and conclusion that R_{22} is divisible by 11.
	R_{21}: $(1-1)+\cdots+(1-1)+1=1$ Not divisible by 11, so $11 \nmid R_{21}$	**A1 AG**	Sum 1 from 11 plus and 10 minus signs, or pairs with the leading 1 left over, and conclusion that R_{21} is not divisible by 11.
		[3]	
9(b)	Factorises: $R_{22}=11N$ $N=1+10^2+\cdots+10^{20}=10101\ldots0101$	**M1**	Writes R_{22} as 11 times a number whose digits alternate 1 and 0, in any form, such as a sum of powers of 100. Condone its length miscounted.
	$11N=10N+N$ $=(10+10^3+\cdots+10^{21})$ $+(1+10^2+\cdots+10^{20})$ $=R_{22}$	**A1**	Correct justification that $11N=R_{22}$, by $10N+N$, long multiplication or another valid route.
	$11\times1-10\times0=\ldots$	**M1**	Applies the alternating digit sum to their N, with signs alternating from a plus at the units digit.
	$=11\Rightarrow N=11m$ $R_{22}=11\times11m=121m$	**A1 AG**	Sum 11, so $11 \mid N$, and conclusion $R_{22}=121m$ or $121 \mid R_{22}$, with no errors.
		[4]	
	Total for Question 9	**7**	

Question	Scheme	Marks	Notes
10(a)	$a+b+(9-a)+(9-b)=\ldots$	**M1**	Adds the four digits of a block, or the digits of N with m blocks. Need not be simplified.
	$=18 \Rightarrow$ digit sum $=18m \Rightarrow 9 \mid N$	**A1**	Block sum 18, with $9 \mid N$ concluded.
	$(9-b)-(9-a)+b-a=\ldots$	**M1**	Forms the alternating digit sum of a block, noting that four-digit blocks keep the signs aligned. Condone the signs reversed throughout.
	$=0 \Rightarrow 11 \mid N$ 9, 11 no common factor other than $1 \Rightarrow 99 \mid N$	**A1 AG**	cso. Needs block sum 0, $11 \mid N$, and 9 and 11 stated to have no common factor other than 1 before concluding $99 \mid N$.
		[4]	
10(b)	$c+d+(n-1-c)+(n-1-d)=2(n-1)$	**M1**	Applies the digit-sum test for $n-1$, quoted or justified, summing a block's digits; or shows $(n-1)$ divides a block's value. Condone sign slips.
	$(n-1) \mid M$: $k_1=1$, $k_2=-1$	**A1**	Conclusion with $k_1=1$, $k_2=-1$, oe such as $k_1=-1$, $k_2=1$.
	$(n-1-d)-(n-1-c)+d-c=0$	**M1**	Applies the alternating digit-sum test for $n+1$, summing a block with alternating signs; or shows $(n+1)$ divides a block's value. Condone the signs reversed throughout.
	$(n+1) \mid M$: $k_1=1$, $k_2=1$	**A1**	Conclusion with $k_1=1$, $k_2=1$, oe.
		[4]	
	Total for Question 10	**8**	

Question	Scheme	Marks	Notes
11(a)	$n = 1000a + 100b + 10c + d$ $1000 \equiv -1,\ 100 \equiv -4,\ 10 \equiv -3 \pmod{13}$	**M1**	Finds the residues of 10, 100 and 1000 modulo 13 in the base-10 expansion of n. Any residues; condone one slip.
	$n \equiv -a - 4b - 3c + d \equiv 3 \pmod{13}$	**A1 AG**	All three residues correct, combined with $n \equiv 3$ to reach the printed congruence without error.
		[2]	
11(b)	$1000 \equiv -1,\ 100 \equiv 1,\ 10 \equiv -1 \pmod{11}$	**M1**	Finds the residues of the powers of 10 modulo 11, or uses the alternating digit sum test.
	$-a + b - c + d \equiv 0 \pmod{11}$	**A1**	oe, such as $11 \mid ((b - a) + (d - c))$.
	$d - c = b - a \Rightarrow 11 \mid 2(b - a)$	**M1**	Uses property (3) to replace one difference by the other, reaching 11 dividing a multiple of a single difference.
	$11 \mid (b - a),\ -9 \leq b - a \leq 8$ $\Rightarrow b - a = 0 \Rightarrow a = b,\ c = d$	**A1 AG**	Needs 11 prime or coprime to 2 to cancel, or an equivalent parity argument, a digit range for $b - a$, and both $a = b$ and $c = d$ stated.
		[4]	
11(c)	$c - a - 4a - 3c \equiv 3 \pmod{13}$	**M1**	Substitutes $b = a$ and $d = c$ into the part (a) congruence. Condone one sign slip.
	$2c \equiv -5a - 3 \equiv 2(4a + 5) \pmod{13}$ $\Rightarrow c \equiv 4a + 5 \equiv 4(a - 2) \pmod{13}$	**A1 AG**	Division by 2 justified, by 13 prime or multiplying by 7, reaching the printed result without error.
		[2]	
11(d)	$a = 1, \ldots, 9 \Rightarrow c \equiv 4(a - 2) \equiv \ldots \pmod{13}$	**M1**	Finds c as a residue of $4(a - 2)$ for each possible a, or a for each possible c, systematically. Condone one slip.
	$(a, c) = (1, 9), (2, 0), (3, 4), (4, 8),$ $(6, 3), (7, 7), (9, 2)$	**A1**	Correct pairs, rejecting $a = 5$ and $a = 8$; a correct list already restricted by property (4) or (5) accepted.
	c even: $(2, 0), (3, 4), (4, 8), (9, 2)$ $3 \mid (a + c)$: $a + c = 2, 7, 12, 11$	**M1**	Applies both c even and divisibility by 3, by digit sum or direct division, to their pairs.

Question 11 continued

Question	Scheme	Marks	Notes
	$a = b = 4,\ c = d = 8 \Rightarrow n = 4488$	**A1**	cso, with every other candidate shown to fail property (4) or (5).
		[4]	
	Total for Question 11	**12**	