OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Transformations of the Complex Plane

Worked solutions 12 questions · 115 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

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1. The transformation T from the z-plane to the w-plane is given by

$$w = iz^2$$

(a) Describe fully the geometric effect of T. **[3]**

The region R in the z-plane is given by

$$\left\{ z \in \mathbb{C} \;\middle|\; |z| < 2,\ -\frac{\pi}{6} < \arg z < \frac{\pi}{6} \right\}$$

(b) On an Argand diagram, sketch the image of R under T. Mark the radius of its curved boundary and the arguments of its straight boundaries. **[2]**

Solution

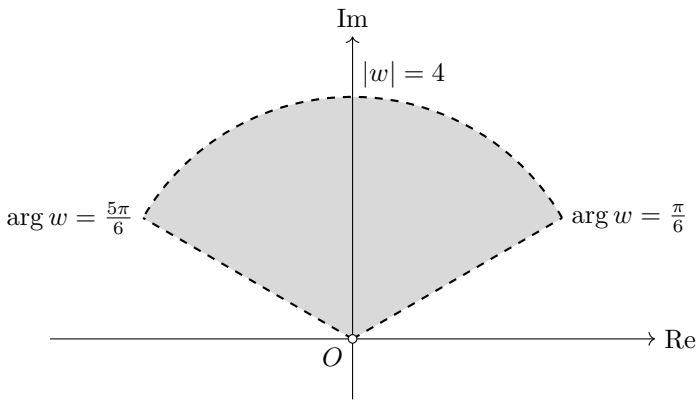
(a) Since $|i| = 1$ and $\arg i = \frac{\pi}{2}$, multiplying moduli and adding arguments gives

$$|w| = |i|\,|z|^2 = |z|^2$$

$$\arg w = \arg i + 2\arg z = 2\arg z + \frac{\pi}{2}$$

So T first squares the modulus of z and doubles its argument, which is the mapping $z \mapsto z^2$. It then rotates the result through $\frac{\pi}{2}$ $(90°)$ anticlockwise about the origin, which is multiplication by i.

(b) The sketch should look like:



For z in R, $0 < |z| < 2$ and $-\frac{\pi}{6} < \arg z < \frac{\pi}{6}$. Using part (a):

$$|w| = |z|^2 \quad \Longrightarrow \quad 0 < |w| < 4$$

$$\arg w = 2\arg z + \frac{\pi}{2} \quad \Longrightarrow \quad \frac{\pi}{6} < \arg w < \frac{5\pi}{6}$$

The image is the sector of radius 4 between the half-lines $\arg w = \frac{\pi}{6}$ and $\arg w = \frac{5\pi}{6}$, symmetric about the positive imaginary axis. The inequalities are strict, so all of its boundaries are excluded and are drawn dashed.

2. A transformation from the z-plane to the w-plane is given by

$$w = \frac{2z + i}{z - 1}, \qquad z \neq 1$$

The transformation maps points on the line $\text{Re}(z) = 1$ in the z-plane, other than the point $z = 1$, onto points on the line ℓ in the w-plane.
Given that $w = u + iv$, where u and v are real, find an equation of ℓ in the form $au + bv = c$, where a, b and c are integers. **[5]**

Solution

Rearrange to make z the subject:

$$w(z - 1) = 2z + i$$

$$wz - 2z = w + i$$

$$z = \frac{w + i}{w - 2}$$

Substituting $w = u + iv$:

$$z = \frac{u + (v + 1)i}{(u - 2) + iv}$$

Multiply the numerator and denominator by $(u - 2) - iv$:

$$z = \frac{\big(u + (v + 1)i\big)\big((u - 2) - iv\big)}{(u - 2)^2 + v^2}$$

The real part of the numerator is $u(u - 2) + v(v + 1)$, so

$$\text{Re}(z) = \frac{u(u - 2) + v(v + 1)}{(u - 2)^2 + v^2}$$

Points on the line $\text{Re}(z) = 1$ satisfy

$$u(u - 2) + v(v + 1) = (u - 2)^2 + v^2$$

$$u^2 - 2u + v^2 + v = u^2 - 4u + 4 + v^2$$

$$2u + v = 4$$

So an equation of ℓ is $2u + v = 4$, which is in the form $au + bv = c$ with $a = 2$, $b = 1$, $c = 4$.

3. The complex numbers $z = x + iy$ and $w = u + iv$, where x, y, u and v are real, are represented by the points P and Q respectively in Argand diagrams, where

$$w = z^2 + 2z$$

(a) Show that $u = x^2 - y^2 + 2x$ and obtain an expression for v in terms of x and y. **[4]**

(b) The point P moves along the line $x = 1$. Find a Cartesian equation of the locus of Q, giving your answer in the form $v^2 = a + bu$, where a and b are constants. **[4]**

Solution

(a) Substitute $z = x + iy$ into $w = z^2 + 2z$ and expand:

$$\begin{aligned} w &= (x + iy)^2 + 2(x + iy) \\ &= x^2 + 2ixy + i^2y^2 + 2x + 2iy \\ &= x^2 + 2ixy - y^2 + 2x + 2iy \\ &= \left(x^2 - y^2 + 2x\right) + (2xy + 2y)i \end{aligned}$$

Since $w = u + iv$ with u and v real, equating real parts gives

$$u = x^2 - y^2 + 2x$$

as required. Equating imaginary parts gives

$$v = 2xy + 2y = 2y(x + 1)$$

(b) On the line $x = 1$, substituting $x = 1$ into the results of part (a):

$$\begin{aligned} u &= 1^2 - y^2 + 2 \times 1 = 3 - y^2 \\ v &= 2y(1 + 1) = 4y \end{aligned}$$

Eliminate y. From the second equation, $y = \frac{v}{4}$, so

$$u = 3 - \left(\frac{v}{4}\right)^2 = 3 - \frac{v^2}{16}$$

Multiplying by 16 and rearranging:

$$16u = 48 - v^2 \quad \Longrightarrow \quad v^2 = 48 - 16u$$

so the locus of Q has equation $v^2 = 48 - 16u$, which is in the form $v^2 = a + bu$ with $a = 48$, $b = -16$.

4. A transformation from the z-plane to the w-plane is given by

$$w = \frac{(1+i)z + 2 - i}{z - 1}, \qquad z \neq 1$$

The circle C in the z-plane has equation $|z| = 2$.

(a) Show that C is mapped to a circle D in the w-plane, and find a Cartesian equation for D, stating its centre and radius. **[7]**

(b) Sketch C and D on separate Argand diagrams. **[2]**

Solution

(a) Rearranging to make z the subject:

$$\begin{aligned} w(z-1) &= (1+i)z + 2 - i \\ wz - (1+i)z &= w + 2 - i \\ z &= \frac{w+2-i}{w-(1+i)} \end{aligned}$$

On C, $|z| = 2$. Taking the modulus of both sides gives

$$\frac{|w+2-i|}{|w-(1+i)|} = 2 \quad \Longrightarrow \quad |w+2-i| = 2\,|w-(1+i)|$$

Let $w = u + iv$. Then $w + 2 - i = (u+2) + (v-1)i$ and $w - (1+i) = (u-1) + (v-1)i$. Squaring both sides of the modulus equation gives

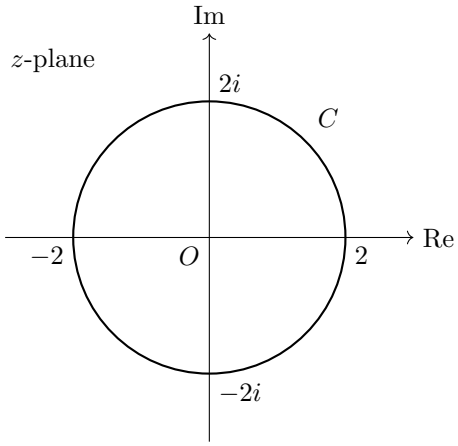
$$(u+2)^2 + (v-1)^2 = 4\left[(u-1)^2 + (v-1)^2\right]$$

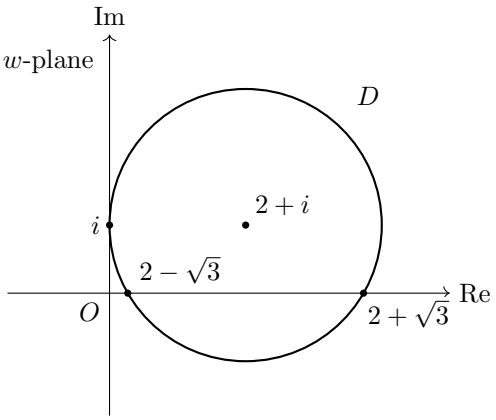
Expanding and simplifying:

$$\begin{aligned} u^2 + 4u + 4 + (v-1)^2 &= 4u^2 - 8u + 4 + 4(v-1)^2 \\ 0 &= 3u^2 - 12u + 3(v-1)^2 \\ 0 &= u^2 - 4u + (v-1)^2 \\ (u-2)^2 + (v-1)^2 &= 4 \end{aligned}$$

This is the equation of a circle. So C is mapped to the circle D with Cartesian equation $(u-2)^2 + (v-1)^2 = 4$, which has centre $2 + i$ and radius 2.

(b) The sketches should look like:





C is the circle with centre O and radius 2, meeting the axes at ± 2 and $\pm 2i$.

D is the circle with centre $2+i$ and radius 2. The real part of the centre equals the radius, so D touches the imaginary axis at i. Setting $v=0$ gives $(u-2)^2+1=4$, so $u=2\pm\sqrt{3}$, and D crosses the real axis at $2-\sqrt{3}$ and $2+\sqrt{3}$.

5. The transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{5z - p}{z + i}, \qquad z \neq -i, \quad p \in \mathbb{R}$$

The point representing $2 + 2i$ is invariant under T.
Find the value of p, and find the complex number represented by the other point that is invariant under T. **[4]**

Solution

A point is invariant under T when $w = z$, so the invariant points satisfy

$$z = \frac{5z - p}{z + i}$$

$$z(z + i) = 5z - p$$

$$z^2 + (i - 5)z + p = 0$$

Since $2 + 2i$ is invariant, it satisfies this equation. Working out the separate terms:

$$(2 + 2i)^2 = 4 + 8i + 4i^2 = 8i$$

$$(i - 5)(2 + 2i) = 2i + 2i^2 - 10 - 10i = -12 - 8i$$

Substituting $z = 2 + 2i$ then gives

$$8i + (-12 - 8i) + p = 0 \quad \Longrightarrow \quad p = 12$$

The equation is now $z^2 + (i - 5)z + 12 = 0$. Let its roots be $\alpha = 2 + 2i$ and β. For $az^2 + bz + c = 0$, the sum of the roots is

$$\alpha + \beta = -\frac{b}{a}$$

Here $a = 1$ and $b = i - 5$, so

$$(2 + 2i) + \beta = -(i - 5) = 5 - i$$

$$\beta = (5 - i) - (2 + 2i) = 3 - 3i$$

So $p = 12$, and the other invariant point represents the complex number $3 - 3i$.

6. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+1}{z-i}, \qquad z \neq i$$

The circle with equation $|z-1| = 2$ is mapped by T onto the curve C.

(a) Show that every point w on C satisfies

$$|w-(1+i)| = \sqrt{2}\,|w-1| \qquad \textbf{[4]}$$

(b) Hence show that C is a circle, and find its centre and radius. **[6]**

The region $|z-1| < 2$ in the z-plane is mapped by T onto the region R in the w-plane.

(c) Shade the region R on an Argand diagram. **[2]**

Solution

(a) Rearrange to make z the subject:

$$\begin{aligned} w(z-i) &= z+1 \\ wz - z &= 1 + iw \\ z &= \frac{1+iw}{w-1} \end{aligned}$$

Here $w \neq 1$, since $w = 1$ would need $z + 1 = z - i$. Then

$$z - 1 = \frac{1+iw-(w-1)}{w-1} = \frac{(i-1)w+2}{w-1}$$

Since $(i-1)(1+i) = i + i^2 - 1 - i = -2$, the numerator factorises as

$$(i-1)w + 2 = (i-1)w - (i-1)(1+i) = (i-1)\big(w-(1+i)\big)$$

Taking moduli, with $|i-1| = \sqrt{2}$:

$$|z-1| = \frac{|i-1|\,|w-(1+i)|}{|w-1|} = \frac{\sqrt{2}\,|w-(1+i)|}{|w-1|}$$

For z on the circle $|z-1| = 2$:

$$\frac{\sqrt{2}\,|w-(1+i)|}{|w-1|} = 2 \quad \Longrightarrow \quad |w-(1+i)| = \frac{2}{\sqrt{2}}\,|w-1| = \sqrt{2}\,|w-1|$$

as required.

(b) Let $w = u + iv$. Squaring the result of part (a):

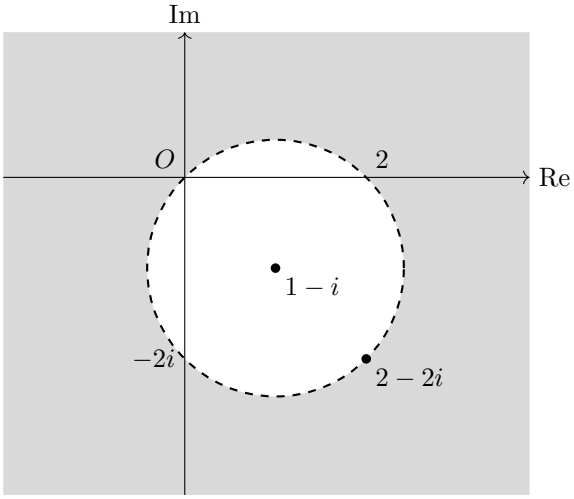
$$\begin{aligned} (u-1)^2 + (v-1)^2 &= 2\big[(u-1)^2 + v^2\big] \\ (u-1)^2 + v^2 - 2v + 1 &= 2(u-1)^2 + 2v^2 \\ 0 &= (u-1)^2 + v^2 + 2v - 1 \\ 0 &= (u-1)^2 + (v+1)^2 - 2 \end{aligned}$$

so

$$(u-1)^2+(v+1)^2=2$$

This is the equation of a circle, so C is a circle with centre $1-i$ and radius $\sqrt{2}$.

(c) The sketch should look like:



Following the working of part (a), $|z-1|<2$ becomes

$$\frac{\sqrt{2}\,|w-(1+i)|}{|w-1|}<2 \quad\Longrightarrow\quad |w-(1+i)|<\sqrt{2}\,|w-1|$$

since $|w-1|>0$. Both sides are non-negative, so squaring and repeating the algebra of part (b) with the inequality gives

$$0<(u-1)^2+(v+1)^2-2 \quad\Longrightarrow\quad (u-1)^2+(v+1)^2>2$$

So R is the region outside the circle C, with the circle itself not included, so it is drawn dashed. The circle has centre $1-i$ and radius $\sqrt{2}$, so it passes through 0, 2, $-2i$ and $2-2i$.

7. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+2i}{z-2}, \qquad z \neq 2$$

The transformation T maps the circle $|z| = 2$ in the z-plane onto the line ℓ in the w-plane.

(a) Find a Cartesian equation of the line ℓ. **[5]**

The circle $|z - a - bi| = c$ in the z-plane, where a, b and c are real constants with $c > 0$, is mapped by T onto the circle $|w - i| = 2$ in the w-plane.

(b) Find the exact values of a, b and c. **[7]**

Solution

(a) Make z the subject of the transformation:

$$w(z-2) = z + 2i$$
$$wz - z = 2w + 2i$$
$$z = \frac{2w+2i}{w-1}$$

The circle is $|z| = 2$, so

$$\left|\frac{2w+2i}{w-1}\right| = 2 \quad \Longrightarrow \quad 2|w+i| = 2|w-1| \quad \Longrightarrow \quad |w+i| = |w-1|$$

Let $w = u + iv$. Then $|u + (v+1)i| = |(u-1) + iv|$, and squaring both moduli gives

$$u^2 + (v+1)^2 = (u-1)^2 + v^2$$
$$u^2 + v^2 + 2v + 1 = u^2 - 2u + 1 + v^2$$
$$2v = -2u$$

Hence a Cartesian equation of ℓ is

$$v = -u$$

(b) The image circle is $|w - i| = 2$. Using $w = \frac{z+2i}{z-2}$:

$$w - i = \frac{z+2i}{z-2} - i$$
$$= \frac{z + 2i - i(z-2)}{z-2}$$
$$= \frac{(1-i)z + 4i}{z-2}$$

So $|w - i| = 2$ becomes

$$|(1-i)z + 4i| = 2|z-2|$$

Let $z = x + iy$. Then

$$(1-i)(x+iy) + 4i = x + iy - ix + y + 4i = (x+y) + (y - x + 4)i$$

Squaring both moduli:

$$(x+y)^2 + (y-x+4)^2 = 4\big[(x-2)^2 + y^2\big]$$

$$(x^2 + 2xy + y^2) + (x^2 - 2xy + y^2 - 8x + 8y + 16) = 4x^2 - 16x + 16 + 4y^2$$

$$2x^2 + 2y^2 - 8x + 8y + 16 = 4x^2 + 4y^2 - 16x + 16$$

$$0 = 2x^2 + 2y^2 - 8x - 8y$$

$$x^2 + y^2 - 4x - 4y = 0$$

$$(x-2)^2 + (y-2)^2 = 8$$

This is the circle with centre $2 + 2i$ and radius $\sqrt{8} = 2\sqrt{2}$, that is

$$|z - 2 - 2i| = 2\sqrt{2}$$

This is in the form $|z - a - bi| = c$, where $c > 0$, with $a = 2$, $b = 2$ and $c = 2\sqrt{2}$.

8. The complex numbers z and w are represented, respectively, by the points $P(x, y)$ and $Q(u, v)$ in Argand diagrams, where

$$w = \frac{z}{z - 2i}, \qquad z \neq 2i$$

(a) Show that

$$u = \frac{x^2 + y^2 - 2y}{x^2 + (y-2)^2}$$

and obtain an expression for v in terms of x and y. **[5]**

(b) The point P moves along the line $y = x$. Show that the locus of Q is a circle, and find its centre and its radius. **[7]**

Solution

(a) With $z = x + iy$, the denominator is $z - 2i = x + (y-2)i$. Multiplying the numerator and denominator by the conjugate of the denominator, $x - (y-2)i$:

$$\begin{aligned} w &= \frac{(x+iy)\big(x-(y-2)i\big)}{\big(x+(y-2)i\big)\big(x-(y-2)i\big)} \\ &= \frac{x^2 - x(y-2)i + xyi - y(y-2)i^2}{x^2 + (y-2)^2} \\ &= \frac{x^2 + y^2 - 2y + 2xi}{x^2 + (y-2)^2} \end{aligned}$$

Equating real parts:

$$u = \frac{x^2 + y^2 - 2y}{x^2 + (y-2)^2}$$

as required. Equating imaginary parts:

$$v = \frac{2x}{x^2 + (y-2)^2}$$

(b) First make z the subject:

$$\begin{aligned} w(z - 2i) = z \quad &\Longrightarrow \quad wz - z = 2iw \\ &\Longrightarrow \quad z = \frac{2iw}{w-1} \end{aligned}$$

This requires $w \neq 1$. With $w = u + iv$, the numerator is $2i(u + iv) = -2v + 2ui$ and the denominator is $(u-1) + iv$. Multiplying the numerator and denominator by $(u-1) - iv$:

$$\begin{aligned} z &= \frac{(-2v + 2ui)\big((u-1) - iv\big)}{(u-1)^2 + v^2} \\ &= \frac{-2v(u-1) + 2v^2 i + 2u(u-1)i + 2uv}{(u-1)^2 + v^2} \\ &= \frac{2v + 2(u^2 + v^2 - u)i}{(u-1)^2 + v^2} \end{aligned}$$

Equating real parts and imaginary parts:

$$x = \frac{2v}{(u-1)^2+v^2}, \qquad y = \frac{2(u^2+v^2-u)}{(u-1)^2+v^2}$$

Since P lies on $y = x$, the two expressions are equal. They share the non-zero denominator, so

$$\begin{aligned} 2(u^2+v^2-u) &= 2v \\ u^2-u+v^2-v &= 0 \\ \left(u-\tfrac{1}{2}\right)^2-\tfrac{1}{4}+\left(v-\tfrac{1}{2}\right)^2-\tfrac{1}{4} &= 0 \\ \left(u-\tfrac{1}{2}\right)^2+\left(v-\tfrac{1}{2}\right)^2 &= \tfrac{1}{2} \end{aligned}$$

This is the equation of a circle, so the locus of Q is a circle (the point $w = 1$, which lies on it, is not attained). Its centre is $\frac{1}{2}+\frac{1}{2}i$ and its radius is $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

9. The transformation T maps points from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$. The transformation T is given by

$$w = \frac{z-1}{z+i}, \qquad z \neq -i$$

The transformation T maps the line ℓ in the z-plane onto the line with equation $u = 1$ in the w-plane.

(a) Find a Cartesian equation of ℓ in terms of x and y. **[5]**

The transformation T maps the line with equation $y = x$ in the z-plane onto the curve C in the w-plane.

(b) (i) Show that C is a circle, and find its centre and its radius. **[5]**

(ii) Find the exact greatest value of $|w|$ for points w on C. **[2]**

Solution

(a) Make z the subject of the transformation:

$$w(z+i) = z - 1$$
$$wz - z = -1 - iw$$
$$z(w-1) = -(1+iw)$$
$$z = \frac{1+iw}{1-w}$$

Points on the line $u = 1$ have $w = 1 + iv$, with $v \neq 0$ because the denominator $1 - w$ cannot be zero. Substituting:

$$z = \frac{1+i(1+iv)}{1-(1+iv)}$$
$$= \frac{(1-v)+i}{-iv}$$

Multiplying the numerator and the denominator by i:

$$z = \frac{-1+(1-v)i}{v} = -\frac{1}{v} + \left(\frac{1}{v} - 1\right)i$$

Equating real parts and equating imaginary parts gives

$$x = -\frac{1}{v}, \qquad y = \frac{1}{v} - 1$$

Since $\frac{1}{v} = -x$, this gives $y = -x - 1$, so a Cartesian equation of ℓ is

$$x + y + 1 = 0$$

(b) (i) Using $z = \frac{1+iw}{1-w}$ from part (a) with $w = u + iv$:

$$z = \frac{1 + i(u+iv)}{1 - (u+iv)}$$

$$= \frac{(1-v) + iu}{(1-u) - iv}$$

$$= \frac{\big((1-v) + iu\big)\big((1-u) + iv\big)}{(1-u)^2 + v^2}$$

Expanding the numerator:

$$\text{real part} = (1-v)(1-u) - uv = 1 - u - v$$

$$\text{imaginary part} = (1-v)v + u(1-u) = u + v - u^2 - v^2$$

Equating real parts and equating imaginary parts gives

$$x = \frac{1-u-v}{(1-u)^2+v^2}, \qquad y = \frac{u+v-u^2-v^2}{(1-u)^2+v^2}$$

On the line $y = x$ the two numerators are equal, since the denominators are the same and non-zero:

$$u + v - u^2 - v^2 = 1 - u - v$$

$$u^2 - 2u + v^2 - 2v + 1 = 0$$

$$(u-1)^2 + (v-1)^2 = 1$$

So C is a circle with centre $1 + i$ and radius 1.

(ii) The point of C furthest from the origin lies on the line through the origin and the centre, on the far side of the centre. The distance from the origin to the centre is

$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

so the greatest value of $|w|$ is the distance to the centre plus the radius:

$$|w|_{\max} = \sqrt{2} + 1$$

10. A transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{z + 2i}{z - 1}, \qquad z \neq 1$$

The locus of points in the z-plane satisfying $|z - 3 - i| = |z - 1 + i|$ is mapped by T onto a circle C in the w-plane.

(a) Find the coordinates of the centre of C and the exact radius of C. **[9]**

The region $|z - 3 - i| < |z - 1 + i|$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) On a single Argand diagram, sketch the circle C and shade and label the region R. **[2]**

Solution

(a) Let $z = x + iy$. Squaring both sides of the locus equation:

$$\begin{aligned}(x-3)^2 + (y-1)^2 &= (x-1)^2 + (y+1)^2\\ -6x + 9 - 2y + 1 &= -2x + 1 + 2y + 1\\ -4x - 4y + 8 &= 0\\ x + y &= 2\end{aligned}$$

Now make z the subject of the transformation:

$$\begin{aligned}w(z-1) &= z + 2i\\ wz - z &= w + 2i\\ z &= \frac{w + 2i}{w - 1}\end{aligned}$$

With $w = u + iv$, this is

$$x + iy = \frac{u + (v+2)i}{(u-1) + iv}$$

Multiplying the numerator and denominator by $(u - 1) - iv$, the numerator becomes

$$\begin{aligned}\big(u + (v+2)i\big)\big((u-1) - iv\big) &= u(u-1) - uvi + (v+2)(u-1)i + v(v+2)\\ &= \big(u^2 - u + v^2 + 2v\big) + (2u - v - 2)i\end{aligned}$$

and the denominator becomes $(u - 1)^2 + v^2$. Equating real parts and equating imaginary parts:

$$x = \frac{u^2 - u + v^2 + 2v}{(u-1)^2 + v^2}, \qquad y = \frac{2u - v - 2}{(u-1)^2 + v^2}$$

Substituting into $x + y = 2$ and multiplying through by $(u - 1)^2 + v^2$:

$$\begin{aligned}\big(u^2 - u + v^2 + 2v\big) + (2u - v - 2) &= 2\big((u-1)^2 + v^2\big)\\ u^2 + u + v^2 + v - 2 &= 2u^2 - 4u + 2 + 2v^2\\ 0 &= u^2 - 5u + v^2 - v + 4\end{aligned}$$

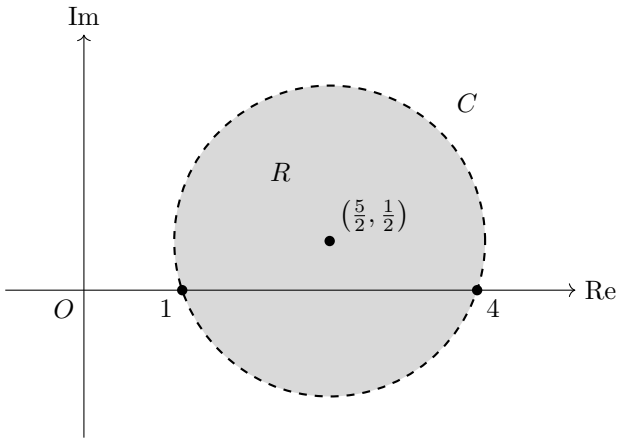
Completing the square:

$$\left(u-\tfrac{5}{2}\right)^2-\tfrac{25}{4}+\left(v-\tfrac{1}{2}\right)^2-\tfrac{1}{4}+4=0$$

$$\left(u-\tfrac{5}{2}\right)^2+\left(v-\tfrac{1}{2}\right)^2=\tfrac{5}{2}$$

So C has centre $\left(\frac{5}{2},\frac{1}{2}\right)$ and radius $\sqrt{\frac{5}{2}}=\frac{\sqrt{10}}{2}$.

(b) The sketch should look like:



By the same expansion as in part (a), the region $|z-3-i|<|z-1+i|$ gives

$$-4x-4y+8<0 \quad\Longrightarrow\quad x+y>2$$

Since $(u-1)^2+v^2>0$, multiplying through by it keeps the inequality the same way round, so the working of part (a) gives

$$u^2-5u+v^2-v+4<0 \quad\Longrightarrow\quad \left(u-\tfrac{5}{2}\right)^2+\left(v-\tfrac{1}{2}\right)^2<\tfrac{5}{2}$$

So R is the inside of C, with the circle itself not included, so it is drawn dashed. Setting $v=0$ in the equation of C gives $u^2-5u+4=0$, so C meets the real axis at $u=1$ and $u=4$.

11. A complex number z is represented by the point P in an Argand diagram, where

$$|z-1| = |z+1-2i|$$

(a) Find a Cartesian equation for the locus of P, and sketch this locus on an Argand diagram, marking the coordinates of the points where it meets the axes. **[3]**

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2z}{z+i}, \qquad z \neq -i$$

Given that T maps the locus of P onto a circle C in the w-plane,

(b) find the equation of C, giving your answer in the form

$$|w-(p+qi)| = r$$

where p, q and r are real constants to be found. **[8]**

Solution

(a) Let $z = x+iy$. Then $|z-1| = |z+1-2i|$ becomes

$$(x-1)^2 + y^2 = (x+1)^2 + (y-2)^2$$

Expanding both sides:

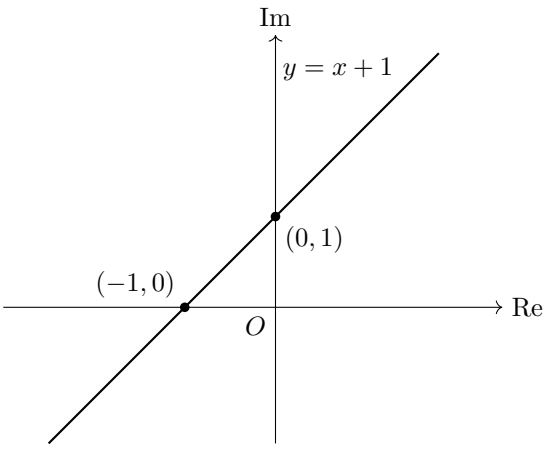
$$x^2 - 2x + 1 + y^2 = x^2 + 2x + 1 + y^2 - 4y + 4$$

$$4y = 4x + 4$$

$$y = x + 1$$

The locus is the perpendicular bisector of the points $(1,0)$ and $(-1,2)$. It meets the real axis where $y=0$, at $(-1,0)$, and the imaginary axis where $x=0$, at $(0,1)$.

The sketch should look like:



(b) Rearranging the transformation to make z the subject:

$$w(z+i) = 2z$$
$$wz + iw = 2z$$
$$iw = z(2-w)$$
$$z = \frac{iw}{2-w}$$

Substituting this into each side of the locus equation:

$$z - 1 = \frac{iw - (2-w)}{2-w} = \frac{(1+i)w - 2}{2-w}$$
$$z + 1 - 2i = \frac{iw + (1-2i)(2-w)}{2-w} = \frac{iw + 2 - w - 4i + 2iw}{2-w} = \frac{(-1+3i)w + 2 - 4i}{2-w}$$

Both fractions have the same denominator, so $|z-1| = |z+1-2i|$ becomes

$$|(1+i)w - 2| = |(-1+3i)w + 2 - 4i|$$

Taking out the coefficient of w on each side, with

$$\frac{2}{1+i} = \frac{2(1-i)}{2} = 1-i, \qquad \frac{2-4i}{-1+3i} = \frac{(2-4i)(-1-3i)}{10} = \frac{-14-2i}{10} = -\frac{7}{5} - \frac{1}{5}i$$

gives

$$|1+i|\,|w-(1-i)| = |-1+3i|\,\left|w - \left(\tfrac{7}{5} + \tfrac{1}{5}i\right)\right|$$
$$\sqrt{2}\,|w-(1-i)| = \sqrt{10}\,\left|w - \left(\tfrac{7}{5} + \tfrac{1}{5}i\right)\right|$$
$$|w-(1-i)| = \sqrt{5}\,\left|w - \left(\tfrac{7}{5} + \tfrac{1}{5}i\right)\right|$$

Let $w = u + iv$ and square both sides:

$$(u-1)^2 + (v+1)^2 = 5\left[\left(u - \tfrac{7}{5}\right)^2 + \left(v - \tfrac{1}{5}\right)^2\right]$$
$$u^2 + v^2 - 2u + 2v + 2 = 5u^2 + 5v^2 - 14u - 2v + 10$$
$$0 = 4u^2 + 4v^2 - 12u - 4v + 8$$
$$0 = u^2 + v^2 - 3u - v + 2$$

Completing the square:

$$\left(u - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \left(v - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + 2 = 0$$
$$\left(u - \tfrac{3}{2}\right)^2 + \left(v - \tfrac{1}{2}\right)^2 = \tfrac{1}{2}$$

This is a circle with centre $\frac{3}{2} + \frac{1}{2}i$ and radius $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so C has equation

$$\left|w - \left(\tfrac{3}{2} + \tfrac{1}{2}i\right)\right| = \frac{\sqrt{2}}{2}$$

This is in the form $|w - (p+qi)| = r$ with $p = \frac{3}{2}$, $q = \frac{1}{2}$, $r = \frac{\sqrt{2}}{2}$.

12. A transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{\left(-1 + \sqrt{3}\,i\right)(z - 3i)}{z + 3i}, \qquad z \neq -3i$$

(a) Show that the imaginary axis in the z-plane is mapped by T onto the line with equation $v = -\sqrt{3}\,u$ in the w-plane. **[3]**

(b) Show that the circle in the z-plane with equation $|z| = 3$ is mapped by T onto a line in the w-plane, stating clearly an equation for this line. **[5]**

The region R in the z-plane is defined by

$$\{z \in \mathbb{C} \mid |z| > 3\} \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$$

(c) Determine the image of R under T, giving your answer in the form $\{w \in \mathbb{C} \mid \alpha < \arg w < \beta\}$, where α and β are rational multiples of π, and sketch this image on an Argand diagram. **[6]**

Solution

(a) A point on the imaginary axis is $z = iy$, where $y \in \mathbb{R}$ and $y \neq -3$. Substituting gives

$$w = \frac{\left(-1 + \sqrt{3}\,i\right)(iy - 3i)}{iy + 3i} = \frac{\left(-1 + \sqrt{3}\,i\right)i(y - 3)}{i(y + 3)} = \left(-1 + \sqrt{3}\,i\right)\frac{y - 3}{y + 3}$$

Let $t = \frac{y-3}{y+3}$, which is real. Then $w = -t + \sqrt{3}\,t\,i$, so

$$u = -t, \qquad v = \sqrt{3}\,t$$

Eliminating t gives $v = -\sqrt{3}\,u$, so the imaginary axis is mapped onto the line $v = -\sqrt{3}\,u$, as required.

(b) Make z the subject:

$$w(z + 3i) = \left(-1 + \sqrt{3}\,i\right)(z - 3i)$$

$$wz + 3iw = \left(-1 + \sqrt{3}\,i\right)z - 3i\left(-1 + \sqrt{3}\,i\right)$$

$$z\left(w + 1 - \sqrt{3}\,i\right) = -3i\left(w - 1 + \sqrt{3}\,i\right)$$

$$z = \frac{-3i\left(w - 1 + \sqrt{3}\,i\right)}{w + 1 - \sqrt{3}\,i}$$

Since $|-3i| = 3$, the condition $|z| = 3$ becomes

$$\frac{3\left|w - 1 + \sqrt{3}\,i\right|}{\left|w + 1 - \sqrt{3}\,i\right|} = 3 \quad \Longrightarrow \quad \left|w - \left(1 - \sqrt{3}\,i\right)\right| = \left|w - \left(-1 + \sqrt{3}\,i\right)\right|$$

This is the perpendicular bisector of the points $1-\sqrt{3}\,i$ and $-1+\sqrt{3}\,i$, which is a line. With $w = u+iv$, squaring both moduli gives

$$(u-1)^2 + \Big(v+\sqrt{3}\Big)^2 = (u+1)^2 + \Big(v-\sqrt{3}\Big)^2$$

$$u^2 - 2u + 1 + v^2 + 2\sqrt{3}\,v + 3 = u^2 + 2u + 1 + v^2 - 2\sqrt{3}\,v + 3$$

$$4\sqrt{3}\,v = 4u$$

So the circle $|z| = 3$ is mapped onto the line

$$v = \frac{u}{\sqrt{3}}$$

(c) Using the expression for z from part (b), the condition $|z| > 3$ becomes $\big|w - 1 + \sqrt{3}\,i\big| > \big|w + 1 - \sqrt{3}\,i\big|$. Squaring and expanding as in part (b):

$$u^2 - 2u + 1 + v^2 + 2\sqrt{3}\,v + 3 > u^2 + 2u + 1 + v^2 - 2\sqrt{3}\,v + 3$$

$$4\sqrt{3}\,v > 4u$$

$$v > \frac{u}{\sqrt{3}}$$

For $\mathrm{Re}(z) > 0$, multiply the numerator and denominator of z by $\big(w + 1 - \sqrt{3}\,i\big)^*$:

$$z = \frac{-3i\big[(u-1) + \big(v+\sqrt{3}\big)i\big]\big[(u+1) - \big(v-\sqrt{3}\big)i\big]}{\big|w + 1 - \sqrt{3}\,i\big|^2}$$

The product of the square brackets is

$$(u-1)(u+1) + \Big(v+\sqrt{3}\Big)\Big(v-\sqrt{3}\Big) + \Big[\Big(v+\sqrt{3}\Big)(u+1) - (u-1)\Big(v-\sqrt{3}\Big)\Big]i$$

$$= \big(u^2 + v^2 - 4\big) + 2\Big(\sqrt{3}\,u + v\Big)i$$

Multiplying by $-3i$ gives

$$z = \frac{6\big(\sqrt{3}\,u + v\big) - 3\big(u^2 + v^2 - 4\big)i}{\big|w + 1 - \sqrt{3}\,i\big|^2}$$

The denominator is positive, so

$$\mathrm{Re}(z) > 0 \quad\Longrightarrow\quad \sqrt{3}\,u + v > 0 \quad\Longrightarrow\quad v > -\sqrt{3}\,u$$

So the image of R is the set of points with $v > \frac{u}{\sqrt{3}}$ and $v > -\sqrt{3}\,u$, the region above both lines. Both lines pass through O. Since $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, the line $v = \frac{u}{\sqrt{3}}$ contains the half-line $\arg w = \frac{\pi}{6}$, and since $\tan\frac{2\pi}{3} = -\sqrt{3}$, the line $v = -\sqrt{3}\,u$ contains the half-line $\arg w = \frac{2\pi}{3}$. The point $w = i$ satisfies both inequalities, so the region is the sector between these half-lines containing the positive imaginary axis, with the boundaries and O excluded. The image of R is

$$\left\{ w \in \mathbb{C} \;\middle|\; \frac{\pi}{6} < \arg w < \frac{2\pi}{3} \right\}$$

so $\alpha = \frac{\pi}{6}$ and $\beta = \frac{2\pi}{3}$. The image is shaded in the diagram below, with the dashed boundary half-lines and O excluded.

