OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Sets of Numbers and Irrationality Proofs

Worked solutions 10 questions · 64 marks

EXAM BOARDS OCR MEI Extra Pure Y435

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1. The sets A and B are defined by

$$A = \{x \in \mathbb{Z} \mid x^2 < 10\}, \qquad B = \{x \in \mathbb{N} \mid x \text{ is a factor of } 12\}$$

(a) List the elements of A. **[1]**

(b) Find $A \cap B$ and $A \cup B$. **[2]**

(c) For each of the numbers

$$-4, \qquad \frac{3}{7}, \qquad \sqrt{11}, \qquad 2 - 5i$$

state the smallest of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to which it belongs. **[2]**

(d) Give a counterexample to show that the set of irrational numbers is not closed under multiplication. **[1]**

Solution

(a) Since $3^2 = 9 < 10$ but $4^2 = 16 > 10$, the integers with $x^2 < 10$ are those with $-3 \leq x \leq 3$, so

$$A = \{-3, -2, -1, 0, 1, 2, 3\}$$

(b) The positive factors of 12 are 1, 2, 3, 4, 6 and 12, so

$$B = \{1, 2, 3, 4, 6, 12\}$$

The elements in both sets are those of B that are at most 3:

$$A \cap B = \{1, 2, 3\}$$

Adding the elements of B that are not already in A gives

$$A \cup B = \{-3, -2, -1, 0, 1, 2, 3, 4, 6, 12\}$$

(c) -4 is a negative integer, so it is not a natural number:

$$-4 \in \mathbb{Z}$$

$\frac{3}{7}$ is a ratio of two integers but is not itself an integer:

$$\frac{3}{7} \in \mathbb{Q}$$

11 is not a perfect square, so $\sqrt{11}$ is irrational, but it is a real number:

$$\sqrt{11} \in \mathbb{R}$$

$2 - 5i$ has a non-zero imaginary part, so it is not real:

$$2 - 5i \in \mathbb{C}$$

(d) $\sqrt{2}$ is irrational, but

$$\sqrt{2} \times \sqrt{2} = 2$$

which is rational. So the product of two irrational numbers need not be irrational, and the set of irrational numbers is not closed under multiplication.

2. In this question you may assume that if p and q are distinct prime numbers and $p^\alpha = q^\beta$, where $\alpha, \beta \in \mathbb{Z}$, then $\alpha = 0$ and $\beta = 0$.

(a) Prove by contradiction that $\log_3 12 \notin \mathbb{Q}$. **[3]**

(b) Hence prove that $\log_3 2 \notin \mathbb{Q}$. **[2]**

Solution

(a) Assume, for contradiction, that $\log_3 12 \in \mathbb{Q}$. Then $\log_3 12 = \frac{m}{n}$ for some integers m and n with $n \neq 0$. By the definition of a logarithm:

$$3^{\frac{m}{n}} = 12$$

Raising both sides to the power n, and writing $12 = 2^2 \times 3$:

$$3^m = 12^n$$

$$3^m = 2^{2n} \times 3^n$$

$$3^{m-n} = 2^{2n}$$

Here 2 and 3 are distinct primes, and $2n$ and $m - n$ are integers, so by the given result $2n = 0$ and $m - n = 0$. This gives $n = 0$, which contradicts $n \neq 0$. Hence the assumption is false, and $\log_3 12 \notin \mathbb{Q}$.

(b) Using the laws of logarithms:

$$\log_3 12 = \log_3 3 + \log_3 4 = 1 + 2\log_3 2$$

Assume, for contradiction, that $\log_3 2 \in \mathbb{Q}$, so $\log_3 2 = \frac{r}{s}$ for some integers r and s with $s \neq 0$. Then

$$\log_3 12 = 1 + \frac{2r}{s} = \frac{s + 2r}{s}$$

Since $s + 2r$ and s are integers with $s \neq 0$, this means $\log_3 12 \in \mathbb{Q}$, which contradicts part (a). Hence $\log_3 2 \notin \mathbb{Q}$.

3. The numbers a, b, c and d are rational, and

$$a + b\sqrt{3} = c + d\sqrt{3}$$

Given that $\sqrt{3}$ is irrational, prove that $a = c$ and $b = d$. **[3]**

Solution

Assume, for contradiction, that $b \neq d$.

Collecting the rational terms on one side and the terms in $\sqrt{3}$ on the other gives

$$d\sqrt{3} - b\sqrt{3} = a - c \quad \Longrightarrow \quad (d-b)\sqrt{3} = a - c$$

Since $b \neq d$, $d - b \neq 0$, so it is possible to divide by $d - b$:

$$\sqrt{3} = \frac{a-c}{d-b}$$

Since a, b, c and d are rational, $a - c$ and $d - b$ are rational, and the quotient of two rational numbers with a non-zero denominator is rational. So $\sqrt{3}$ is rational, which contradicts the fact that $\sqrt{3}$ is irrational. Hence the assumption is false, and $b = d$.

Substituting $d = b$ into the original equation gives

$$a + b\sqrt{3} = c + b\sqrt{3} \quad \Longrightarrow \quad a = c$$

Therefore $a = c$ and $b = d$, as required.

4. The real number x is irrational, so that $x+3 \neq 0$. Use proof by contradiction to show that

$$\frac{2x-1}{x+3}$$

is also irrational. **[4]**

Solution

Assume, for contradiction, that $\frac{2x-1}{x+3}$ is rational. Then there are integers a and b, with $b \neq 0$, such that

$$\frac{2x-1}{x+3} = \frac{a}{b}$$

Since $x+3 \neq 0$, multiplying both sides by $b(x+3)$ and rearranging gives

$$b(2x-1) = a(x+3)$$

$$2bx - b = ax + 3a$$

$$(2b-a)x = 3a + b$$

If $2b-a=0$, then $a=2b$, and the last line becomes $0 = 6b + b = 7b$, so $b=0$. This contradicts $b \neq 0$, so $2b-a \neq 0$ and

$$x = \frac{3a+b}{2b-a}$$

Here $3a+b$ and $2b-a$ are integers, and $2b-a \neq 0$, so x is rational. This contradicts the fact that x is irrational.

Hence the assumption is false, and $\frac{2x-1}{x+3}$ is irrational.

5. **(a)** Prove by contradiction that the product of a non-zero rational number and an irrational number is always irrational. **[4]**

(b) Hence, given that $\sqrt{3}$ is irrational, prove that $\sqrt{75} - \sqrt{12}$ is irrational. **[2]**

Solution

(a) Let r be a non-zero rational number and x an irrational number. Since r is rational and non-zero, $r = \frac{m}{n}$ for some integers m and n with $m \neq 0$ and $n \neq 0$.

Assume, for contradiction, that rx is rational. Then $rx = \frac{p}{q}$ for some integers p and q with $q \neq 0$. Since $r \neq 0$, divide both sides by r:

$$\begin{aligned} x &= \frac{p}{q} \div \frac{m}{n} \\ &= \frac{pn}{qm} \end{aligned}$$

Here pn and qm are integers, and $qm \neq 0$ because $q \neq 0$ and $m \neq 0$. So x is a ratio of two integers with non-zero denominator, which means x is rational. This contradicts x being irrational, so the assumption is false. Therefore the product of a non-zero rational number and an irrational number is always irrational.

(b) Write each surd in terms of $\sqrt{3}$:

$$\begin{aligned} \sqrt{75} - \sqrt{12} &= \sqrt{25 \times 3} - \sqrt{4 \times 3} \\ &= 5\sqrt{3} - 2\sqrt{3} \\ &= 3\sqrt{3} \end{aligned}$$

Here 3 is a non-zero rational number and $\sqrt{3}$ is irrational, so by part (a) their product $3\sqrt{3}$ is irrational. Hence $\sqrt{75} - \sqrt{12}$ is irrational.

6. You may assume that $\sqrt{21}$ is irrational.

(a) Use proof by contradiction to prove that $\sqrt{3}+\sqrt{7}$ is irrational. **[3]**

(b) Hence prove that $\sqrt{7}-\sqrt{3}$ is irrational. **[2]**

Solution

(a) Assume, for contradiction, that $\sqrt{3}+\sqrt{7}$ is rational. Then there are integers a and b, with $b \neq 0$, such that

$$\sqrt{3}+\sqrt{7} = \frac{a}{b}$$

Squaring both sides gives

$$3 + 2\sqrt{3}\sqrt{7} + 7 = \frac{a^2}{b^2}$$

$$10 + 2\sqrt{21} = \frac{a^2}{b^2}$$

$$\sqrt{21} = \frac{a^2 - 10b^2}{2b^2}$$

Since a and b are integers, $a^2 - 10b^2$ and $2b^2$ are integers, with $2b^2 \neq 0$. So $\sqrt{21}$ is a quotient of two integers, which means $\sqrt{21}$ is rational. This contradicts the fact that $\sqrt{21}$ is irrational, so the assumption was false, and $\sqrt{3}+\sqrt{7}$ is irrational.

(b) Using the difference of two squares:

$$\left(\sqrt{7}-\sqrt{3}\right)\left(\sqrt{7}+\sqrt{3}\right) = 7 - 3 = 4$$

Assume, for contradiction, that $\sqrt{7}-\sqrt{3}$ is rational, so $\sqrt{7}-\sqrt{3} = \frac{p}{q}$ for integers p and q, with $q \neq 0$. Since $\sqrt{7} > \sqrt{3}$, $p \neq 0$. Then

$$\sqrt{7}+\sqrt{3} = \frac{4}{\sqrt{7}-\sqrt{3}} = \frac{4q}{p}$$

which is a quotient of two integers with $p \neq 0$, so $\sqrt{3}+\sqrt{7}$ is rational. This contradicts part (a), so the assumption was false, and $\sqrt{7}-\sqrt{3}$ is irrational.

7. A student is attempting to prove that $\sqrt[3]{4}$ is irrational.
The first three lines of their proof are shown below.

Assume that $\sqrt[3]{4}$ is rational, so it can be written as $\sqrt[3]{4} = \frac{a}{b}$.	**Line 1**
Cubing both sides gives $4 = \frac{a^3}{b^3}$. Hence $a^3 = 4b^3$.	**Line 2**
Hence a^3 is even, so a is even, and $a = 2k$ for some integer k.	**Line 3**

(a) State the conditions on a and b that are required in **Line 1**. **[2]**

(b) In **Line 3**, the student uses the fact that if a^3 is even then a is even.
Prove that, for any integer n, if n^3 is even then n is even. **[3]**

(c) Complete the proof that $\sqrt[3]{4}$ is irrational. **[3]**

Solution

(a) a and b must be integers with $b \neq 0$.

Also, a and b must have no common factors other than 1, so that the fraction $\frac{a}{b}$ is in its lowest terms.

(b) Prove the contrapositive: for any integer n, if n is odd then n^3 is odd.

Let n be odd, so $n = 2m + 1$ for some integer m. Then

$$\begin{aligned} n^3 &= (2m+1)^3 \\ &= 8m^3 + 12m^2 + 6m + 1 \\ &= 2\big(4m^3 + 6m^2 + 3m\big) + 1 \end{aligned}$$

Since $4m^3 + 6m^2 + 3m$ is an integer, n^3 is odd.

So if n is odd then n^3 is odd. The contrapositive of this statement is equivalent to it, so for any integer n, if n^3 is even then n is even.

(c) Substituting $a = 2k$ into $a^3 = 4b^3$ gives

$$\begin{aligned} (2k)^3 &= 4b^3 \\ 8k^3 &= 4b^3 \\ b^3 &= 2k^3 \end{aligned}$$

Since k^3 is an integer, b^3 is even, so by part (b), b is even.

Hence a and b are both even, so they have a common factor of 2. This contradicts the assumption that a and b have no common factors other than 1.

Therefore the assumption that $\sqrt[3]{4}$ is rational is false, so $\sqrt[3]{4}$ is irrational.

8. In this question you may assume that every integer greater than 1 can be written as a product of primes in exactly one way, apart from the order of the factors.

(a) Let m be an integer with $m > 1$. Prove that every prime appears to an even power in the prime factorisation of m^2. **[3]**

(b) Using part (a) and by considering the power of 2 in the prime factorisations of both sides of a suitable equation, prove by contradiction that $\sqrt{18}$ is irrational.
Explain why considering the power of 3 instead would not lead to a contradiction. **[6]**

Solution

(a) Since $m > 1$, m can be written as a product of primes:

$$m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $n_1, n_2, \ldots, n_k$ are positive integers. Squaring gives

$$m^2 = (p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k})^2 = p_1^{2n_1} p_2^{2n_2} \cdots p_k^{2n_k}$$

This writes m^2 as a product of primes, and the prime factorisation of m^2 is unique, so this is the prime factorisation of m^2. Each of $p_1, p_2, \ldots, p_k$ appears to the power $2n_1, 2n_2, \ldots, 2n_k$ respectively, and each of these is even. Any other prime does not divide m^2, so it appears to the power 0, which is also even. Hence every prime appears to an even power in the prime factorisation of m^2.

(b) Assume, for contradiction, that $\sqrt{18}$ is rational. Since $\sqrt{18} > 0$, it can be written as

$$\sqrt{18} = \frac{a}{b}$$

where a and b are positive integers. Squaring and multiplying by b^2:

$$a^2 = 18b^2 = 2 \times 3^2 \times b^2$$

Since $b \geq 1$, $a^2 \geq 18$, so $a > 1$.

Left-hand side: by part (a), with $m = a$, the power of 2 in the prime factorisation of a^2 is even.

Right-hand side: if $b = 1$, then $18b^2 = 2^1 \times 3^2$, and the power of 2 is 1. If $b > 1$, then by part (a), with $m = b$, the power of 2 in the prime factorisation of b^2 is even, say $2t$ for some integer $t \geq 0$. Then

$$18b^2 = 2 \times 3^2 \times b^2$$

contains the prime 2 to the power $1 + 2t$. In either case the power of 2 in the prime factorisation of $18b^2$ is odd.

So the integer $a^2 = 18b^2$ has a prime factorisation in which 2 appears to an even power and one in which 2 appears to an odd power. This contradicts the uniqueness of prime factorisation, so the assumption is false and $\sqrt{18}$ is irrational.

Now consider the power of 3 instead. By part (a), the power of 3 in the prime factorisation of a^2 is even. In $18b^2 = 2 \times 3^2 \times b^2$, the factor 3^2 contributes 2, and b^2 contributes an even power of 3 (by part (a) if $b > 1$, and 0 if $b = 1$), so the power of 3 in $18b^2$ is 2 plus an even number, which is even. Both sides have an even power of 3, so there is no contradiction. This happens because $18 = 2 \times 3^2$: the prime 3 appears in 18 to an even power, whereas 2 appears to the odd power 1.

9. Consider the cubic equation

$$x^3 - 3x - 1 = 0 \qquad (*)$$

(a) Suppose that $x = \frac{p}{q}$ is a root of $(*)$, where p and q are integers with $q > 0$ and $\frac{p}{q}$ is in its lowest terms. Show that q divides p^3 and that p divides q^3. Hence prove that $(*)$ has no rational roots. **[4]**

(b) Using the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, show that $x = 2\cos 20°$ is a root of $(*)$. Hence prove by contradiction that $\cos 20°$ is irrational. **[5]**

Solution

(a) Substituting $x = \frac{p}{q}$ into $(*)$ gives

$$\frac{p^3}{q^3} - \frac{3p}{q} - 1 = 0$$

Multiplying through by q^3:

$$p^3 - 3pq^2 - q^3 = 0$$

Rearranging in two ways:

$$p^3 = 3pq^2 + q^3 = q\left(3pq + q^2\right)$$

$$q^3 = p^3 - 3pq^2 = p\left(p^2 - 3q^2\right)$$

Since $3pq + q^2$ and $p^2 - 3q^2$ are integers, q divides p^3 and p divides q^3.

Since $\frac{p}{q}$ is in its lowest terms, p and q have no common prime factor. Suppose $q > 1$. Then q has a prime factor r, and since q divides p^3, r divides p^3. A prime that divides a product divides one of its factors, so r divides p. Then r is a common prime factor of p and q, which is a contradiction. Hence $q = 1$.

Now p divides $q^3 = 1$, so $p = 1$ or $p = -1$. The only possible rational roots are therefore $x = 1$ and $x = -1$. Substituting each into $(*)$:

$$x = 1: \quad 1^3 - 3 \times 1 - 1 = -3 \neq 0$$

$$x = -1: \quad (-1)^3 - 3 \times (-1) - 1 = 1 \neq 0$$

Neither is a root, so $(*)$ has no rational roots.

(b) Put $\theta = 20°$ in the identity. Since $\cos 60° = \frac{1}{2}$, this gives

$$4\cos^3 20° - 3\cos 20° = \frac{1}{2}$$

Substituting $x = 2\cos 20°$ into the left-hand side of $(*)$:

$$\begin{aligned} x^3 - 3x - 1 &= 8\cos^3 20° - 6\cos 20° - 1 \\ &= 2\left(4\cos^3 20° - 3\cos 20°\right) - 1 \\ &= 2 \times \frac{1}{2} - 1 \\ &= 0 \end{aligned}$$

So $x = 2\cos 20°$ is a root of $(*)$.

Now assume, for contradiction, that $\cos 20^\circ$ is rational, so that $\cos 20^\circ = \frac{m}{n}$ for some integers m and n with $n \neq 0$. Then

$$2\cos 20^\circ = \frac{2m}{n}$$

is a quotient of integers, so it is rational. But $2\cos 20^\circ$ is a root of $(*)$, and by part (a), $(*)$ has no rational roots. This is a contradiction, so the assumption is false and $\cos 20^\circ$ is irrational.

10. You are given that $q \in \mathbb{Z}$ with $q \geq 1$ and that

$$S = \frac{1}{(2q+1)(2q+2)} + \frac{1}{(2q+1)(2q+2)(2q+3)(2q+4)} + \frac{1}{(2q+1)(2q+2)\cdots(2q+6)} + \cdots$$

(a) By considering a suitable geometric series, show that

$$S < \frac{1}{4q(q+1)}$$ **[3]**

(b) Deduce that $S \notin \mathbb{Z}$. **[2]**

You are also given that

$$\cosh 1 = \sum_{r=0}^{\infty} \frac{1}{(2r)!}$$

(c) Assume that $\cosh 1 = \frac{p}{q}$, where p and q are positive integers. By considering $(2q)! \cosh 1$ and using the result of part (b), prove by contradiction that $\cosh 1$ is irrational. **[4]**

Solution

(a) For $k \geq 1$, the kth term of S is

$$\frac{1}{(2q+1)(2q+2)\cdots(2q+2k)}$$

The denominator is a product of $2k$ factors, each at least $2q+1$, and the factor $2q+2$ is greater than $2q+1$, so

$$(2q+1)(2q+2)\cdots(2q+2k) > (2q+1)^{2k}$$

Hence each term of S is less than the corresponding term of the geometric series

$$\frac{1}{(2q+1)^2} + \frac{1}{(2q+1)^4} + \frac{1}{(2q+1)^6} + \cdots$$

whose first term and common ratio are both $\frac{1}{(2q+1)^2}$. The sum to infinity of a geometric series with first term a and common ratio ρ is

$$\frac{a}{1-\rho} \quad (|\rho| < 1)$$

The common ratio satisfies

$$0 < \frac{1}{(2q+1)^2} < 1$$

So this series converges, and its sum to infinity is

$$\frac{\dfrac{1}{(2q+1)^2}}{1-\dfrac{1}{(2q+1)^2}} = \frac{1}{(2q+1)^2-1}$$

$$= \frac{1}{4q^2+4q+1-1}$$

$$= \frac{1}{4q(q+1)}$$

Comparing the two series term by term gives

$$S < \frac{1}{4q(q+1)}$$

as required.

(b) Every term of S is positive, so $S > 0$. Since $q \geq 1$, $4q(q+1) \geq 4 \times 1 \times 2 = 8$, so by part (a)

$$0 < S < \frac{1}{8} < 1$$

There is no integer strictly between 0 and 1, so $S \notin \mathbb{Z}$.

(c) Since $\cosh 1 > 0$, assume, for contradiction, that $\cosh 1 = \frac{p}{q}$, where p and q are positive integers. Then

$$(2q)!\cosh 1 = \frac{(2q)!}{q} \times p = 2 \times (2q-1)! \times p$$

which is an integer.

Splitting the given series after the term with $r = q$ gives

$$(2q)!\cosh 1 = \sum_{r=0}^{q} \frac{(2q)!}{(2r)!} + \sum_{r=q+1}^{\infty} \frac{(2q)!}{(2r)!}$$

For $0 \leq r \leq q$, every factor of $(2r)!$ appears in $(2q)!$, so $\frac{(2q)!}{(2r)!}$ is an integer: it is 1 when $r = q$, and $(2r+1)(2r+2)\cdots(2q)$ when $r < q$. Hence the first sum is an integer, N say.

The terms of the second sum are

$$r = q+1: \quad \frac{(2q)!}{(2q+2)!} = \frac{1}{(2q+1)(2q+2)}$$

$$r = q+2: \quad \frac{(2q)!}{(2q+4)!} = \frac{1}{(2q+1)(2q+2)(2q+3)(2q+4)}$$

$$r = q+3: \quad \frac{(2q)!}{(2q+6)!} = \frac{1}{(2q+1)(2q+2)\cdots(2q+6)}$$

and so on, so the second sum is exactly S, with q a positive integer. Therefore

$$S = (2q)!\cosh 1 - N$$

is the difference of two integers, so $S \in \mathbb{Z}$. This contradicts part (b), so the assumption is false, and $\cosh 1$ is irrational.