OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Further Loci in the Argand Diagram

Worked solutions 12 questions · 105 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

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1. Sketch on an Argand diagram the region defined by

$$\left\{z \in \mathbb{C} \;\middle|\; \frac{\pi}{6} \leq \arg(z-1-i) < \frac{2\pi}{3}\right\} \cap \left\{z \in \mathbb{C} \;\middle|\; |z-1-3i| \leq 2\right\}$$

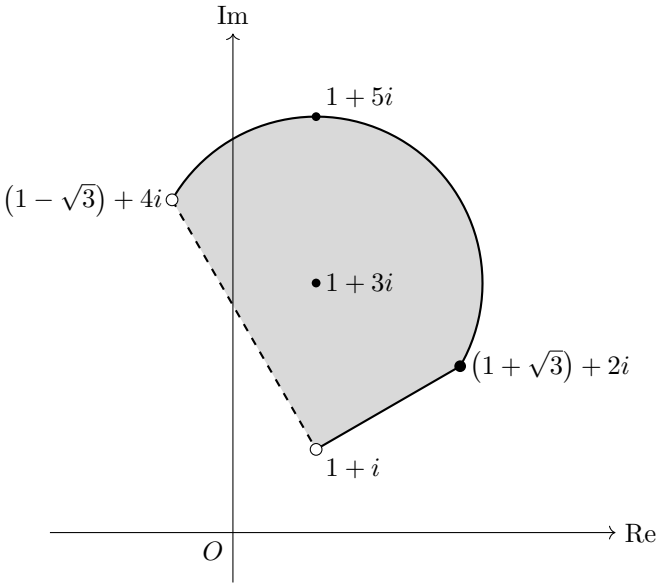
On your sketch

- shade the part of the diagram that is included in the region
- use solid lines to show the parts of the boundary that are included in the region
- use dashed lines to show the parts of the boundary that are not included in the region

[4]

Solution

The sketch should look like:



The locus $\arg(z-1-i) = \theta$ is a half-line starting at the point $(1,1)$ and making an angle θ with the positive real direction. The point $(1,1)$ itself is not included, since $\arg 0$ is undefined, so it is shown as an open circle. The half-line at angle $\frac{\pi}{6}$ is included (solid), and the half-line at angle $\frac{2\pi}{3}$ is not included (dashed).

The set $|z-1-3i| \leq 2$ is the disc with centre $(1,3)$ and radius 2, with its circle included (solid). Since $(1,3)$ is a distance 2 directly above $(1,1)$, the circle passes through $(1,1)$ as its lowest point.

Points on the half-line at angle $\frac{\pi}{6}$ are $x = 1+\frac{\sqrt{3}}{2}t$, $y = 1+\frac{1}{2}t$ with $t > 0$. Substituting into $(x-1)^2+(y-3)^2 = 4$:

$$\begin{aligned} \tfrac{3}{4}t^2 + \left(\tfrac{1}{2}t - 2\right)^2 &= 4 \\ t^2 - 2t &= 0 \\ t &= 2 \end{aligned}$$

so this half-line meets the circle at $(1+\sqrt{3},\, 2)$, which is included in the region.

Points on the half-line at angle $\frac{2\pi}{3}$ are $x = 1-\frac{1}{2}t$, $y = 1+\frac{\sqrt{3}}{2}t$ with $t > 0$. Substituting into $(x-1)^2+(y-3)^2 = 4$:

$$\begin{aligned} \tfrac{1}{4}t^2 + \left(\tfrac{\sqrt{3}}{2}t - 2\right)^2 &= 4 \\ t^2 - 2\sqrt{3}\,t &= 0 \\ t &= 2\sqrt{3} \end{aligned}$$

so this half-line meets the circle at $(1-\sqrt{3},\,4)$. This point lies on the excluded half-line, so it is not in the region and is shown as an open circle.

The centre $(1,3)$ is directly above $(1,1)$, at argument $\frac{\pi}{2}$ from it, which lies between $\frac{\pi}{6}$ and $\frac{2\pi}{3}$. So the region is the part of the disc between the two half-lines that contains the centre. It is bounded by the solid line segment from $(1,1)$ to $(1+\sqrt{3},\,2)$, the dashed line segment from $(1,1)$ to $(1-\sqrt{3},\,4)$, and the solid arc of the circle through $(1,5)$. The midpoint of $(1+\sqrt{3},\,2)$ and $(1-\sqrt{3},\,4)$ is the centre $(1,3)$, so this arc is a semicircle.

2. A complex number $z = x + iy$ is represented by the point P in an Argand diagram, where

$$|z + 3| = 2|z - 3i|$$

(a) Show that the locus of P is a circle, and find its centre and radius. **[6]**

(b) Hence find the greatest and least values of $|z - 4 - i|$. **[3]**

Solution

(a) With $z = x + iy$, we have $z + 3 = (x + 3) + iy$ and $z - 3i = x + (y - 3)i$. Squaring both sides of the equation gives

$$(x + 3)^2 + y^2 = 4\left[x^2 + (y - 3)^2\right]$$

Expanding and collecting terms:

$$\begin{aligned} x^2 + 6x + 9 + y^2 &= 4x^2 + 4y^2 - 24y + 36 \\ 0 &= 3x^2 + 3y^2 - 6x - 24y + 27 \\ 0 &= x^2 + y^2 - 2x - 8y + 9 \end{aligned}$$

Completing the square in x and in y:

$$\begin{aligned} (x - 1)^2 - 1 + (y - 4)^2 - 16 + 9 &= 0 \\ (x - 1)^2 + (y - 4)^2 &= 8 \end{aligned}$$

This is the equation of a circle, so the locus of P is a circle with centre $(1, 4)$, representing $1 + 4i$, and radius $\sqrt{8} = 2\sqrt{2}$.

(b) Since $|z - 4 - i| = |z - (4 + i)|$, it is the distance from P to the point $A(4, 1)$. Let $C(1, 4)$ be the centre of the circle. Then

$$CA = \sqrt{(4 - 1)^2 + (1 - 4)^2} = \sqrt{18} = 3\sqrt{2}$$

Since $3\sqrt{2} > 2\sqrt{2}$, the point A lies outside the circle. The greatest and least distances from A to a point on the circle occur where the line through A and C meets the circle: the far point is one radius beyond C, and the near point is one radius short of C. Hence

$$\text{greatest value of } |z - 4 - i| = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} \approx 7.07$$

$$\text{least value of } |z - 4 - i| = 3\sqrt{2} - 2\sqrt{2} = \sqrt{2} \approx 1.41$$

3. A complex number z is represented by the point P in an Argand diagram, where

$$|z-2+7i| = 3|z-2-i|$$

(a) Show that the locus of P is a circle, and find its centre and radius. **[6]**

(b) Sketch on an Argand diagram the locus of P as z varies. **[2]**

(c) On your sketch, shade the region whose points satisfy both

$$|z-2+7i| \geq 3|z-2-i| \quad \text{and} \quad 0 \leq \arg z \leq \frac{\pi}{4}$$ **[2]**

Solution

(a) Let $z = x+iy$. Then $z-2+7i = (x-2)+(y+7)i$ and $z-2-i = (x-2)+(y-1)i$, so the equation becomes

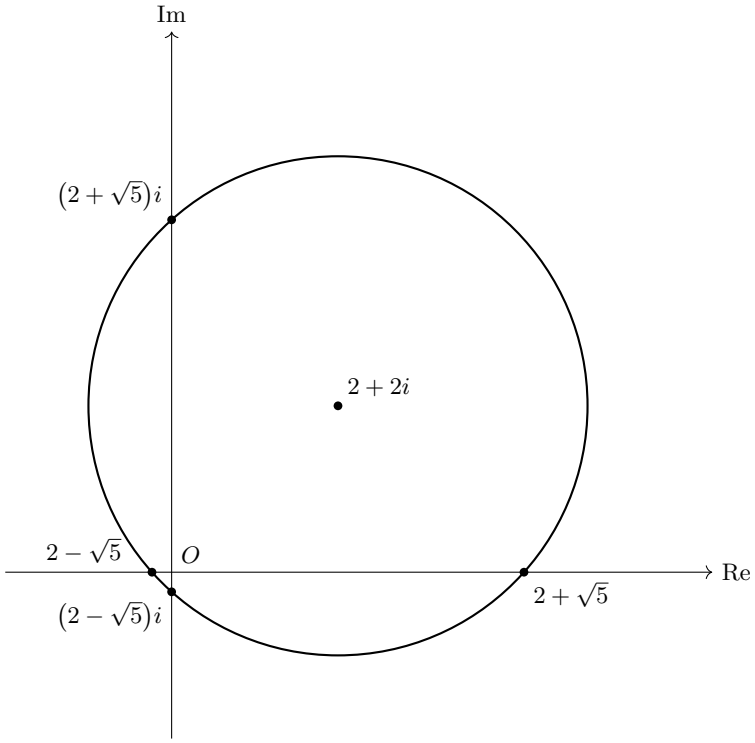
$$|(x-2)+(y+7)i| = 3|(x-2)+(y-1)i|$$

Squaring both sides gives

$$\begin{aligned}
(x-2)^2+(y+7)^2 &= 9\left[(x-2)^2+(y-1)^2\right] \\
(x-2)^2+y^2+14y+49 &= 9(x-2)^2+9y^2-18y+9 \\
0 &= 8(x-2)^2+8y^2-32y-40 \\
0 &= (x-2)^2+y^2-4y-5 \\
0 &= (x-2)^2+(y-2)^2-4-5 \\
(x-2)^2+(y-2)^2 &= 9
\end{aligned}$$

This is the equation of a circle, so the locus of P is a circle with centre $(2,2)$, representing $2+2i$, and radius 3.

(b) The sketch should look like:



The locus is the circle with centre $2+2i$ and radius 3. The distance from O to the centre is $\sqrt{2^2+2^2} = 2\sqrt{2} < 3$, so the origin lies inside the circle.

Setting $y = 0$ in $(x-2)^2 + (y-2)^2 = 9$ gives $(x-2)^2 = 5$, so the circle meets the real axis at $x = 2 \pm \sqrt{5}$. By symmetry in the line $y = x$, setting $x = 0$ gives $y = 2 \pm \sqrt{5}$ on the imaginary axis.

(c) Following the working of part (a) with $\geq$ in place of $=$ (squaring is valid since both sides are non-negative):

$$\begin{aligned}(x-2)^2 + (y+7)^2 &\geq 9\big[(x-2)^2 + (y-1)^2\big] \\ 0 &\geq 8(x-2)^2 + 8(y-2)^2 - 72 \\ (x-2)^2 + (y-2)^2 &\leq 9\end{aligned}$$

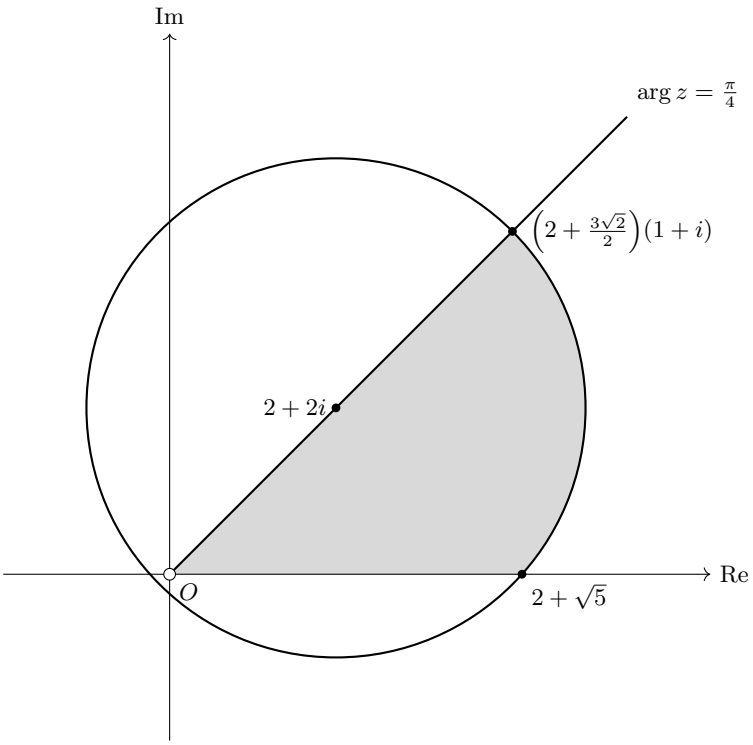
So $|z-2+7i| \geq 3|z-2-i|$ describes the points on or inside the circle.

The condition $0 \leq \arg z \leq \frac{\pi}{4}$ describes the points between the positive real axis and the half-line from O at angle $\frac{\pi}{4}$, which is $y = x$, $x > 0$. This half-line passes through the centre $(2, 2)$. It meets the circle where

$$2(x-2)^2 = 9 \quad \Longrightarrow \quad x = 2 + \frac{3\sqrt{2}}{2}$$

so at the point $\left(2 + \frac{3\sqrt{2}}{2}\right)(1+i) \approx 4.12 + 4.12i$. The positive real axis meets the circle at $2+\sqrt{5} \approx 4.24$. All the boundaries are included, except the origin itself, since $\arg 0$ is undefined.

The shaded region is shown below:



4. The complex number z is represented on an Argand diagram by the point P, where

$$|z - 5 + i| = 2\,|z + 1 - 2i|$$

The locus of P is a circle with centre C.

(a) Find the complex number represented by C and the exact radius of the circle. **[6]**

The complex number w is represented on the same Argand diagram by the point Q, where

$$\arg(w + 7 + 3i) = \frac{\pi}{4}$$

The locus of Q meets the locus of P at two distinct points R and S.

(b) Find the complex numbers represented by R and S. **[4]**

(c) Find the area of triangle CRS. **[2]**

Solution

(a) Let $z = x + iy$. Then $z - 5 + i = (x - 5) + (y + 1)i$ and $z + 1 - 2i = (x + 1) + (y - 2)i$. Squaring both sides of the locus equation gives

$$(x - 5)^2 + (y + 1)^2 = 4\big[(x + 1)^2 + (y - 2)^2\big]$$

Expanding both sides:

$$\begin{aligned} x^2 + y^2 - 10x + 2y + 26 &= 4x^2 + 4y^2 + 8x - 16y + 20 \\ 0 &= 3x^2 + 3y^2 + 18x - 18y - 6 \\ 0 &= x^2 + y^2 + 6x - 6y - 2 \end{aligned}$$

Completing the square:

$$\begin{aligned} (x + 3)^2 - 9 + (y - 3)^2 - 9 - 2 &= 0 \\ (x + 3)^2 + (y - 3)^2 &= 20 \end{aligned}$$

So C represents the complex number $-3 + 3i$, and the radius of the circle is $\sqrt{20} = 2\sqrt{5}$.

(b) The locus of Q is the half-line starting at the point $(-7, -3)$ (not included) making an angle $\frac{\pi}{4}$ with the positive real direction. Its gradient is $\tan\frac{\pi}{4} = 1$, so with $w = x + iy$ it is

$$y + 3 = x + 7 \quad \Longrightarrow \quad y = x + 4, \qquad x > -7$$

Substituting $y = x + 4$ into the equation of the circle from part (a):

$$\begin{aligned} (x + 3)^2 + (x + 1)^2 &= 20 \\ 2x^2 + 8x + 10 &= 20 \\ x^2 + 4x - 5 &= 0 \\ (x + 5)(x - 1) &= 0 \end{aligned}$$

So $x = 1$ or $x = -5$. Both satisfy $x > -7$, so both lie on the half-line, and $y = x + 4$ gives $y = 5$ and $y = -1$ respectively.

So R and S represent $1 + 5i$ and $-5 - i$.

(c) Take $R(1, 5)$ and $S(-5, -1)$. By Pythagoras, the length of the chord RS is

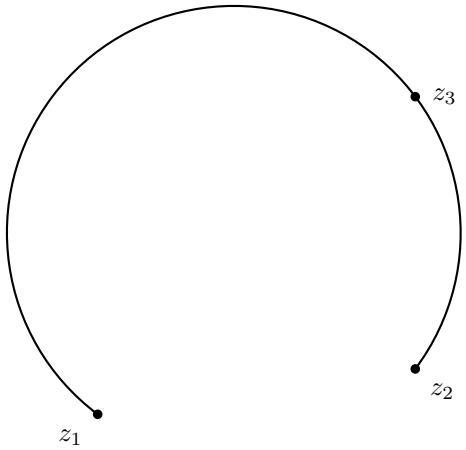
$$RS = \sqrt{6^2 + 6^2} = 6\sqrt{2}$$

The midpoint of RS is $M(-2, 2)$. The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord, so CM is the height of triangle CRS with base RS:

$$CM = \sqrt{(-2 + 3)^2 + (2 - 3)^2} = \sqrt{2}$$

Hence the area of triangle CRS is

$$\tfrac{1}{2} \times RS \times CM = \tfrac{1}{2} \times 6\sqrt{2} \times \sqrt{2} = 6$$



5. The diagram above shows a locus in the complex plane.
The locus is an arc of a circle from the point represented by $z_1 = -1 - 3i$ to the point represented by $z_2 = 6 + ai$, where a is a constant, $a \neq 4$.
Given that

- the point $z_3 = 6 + 4i$ also lies on the locus
- the centre of the circle has imaginary part equal to 1

(a) determine the complex number representing the centre of the circle and the value of a. **[3]**

(b) Hence determine a complex equation for the locus, giving any angles in the equation as positive values. **[3]**

Solution

(a) Let the centre of the circle be $x + i$, where $x \in \mathbb{R}$. The points z_1 and z_3 both lie on the circle, so they are the same distance from the centre:

$$|(-1 - 3i) - (x + i)|^2 = |(6 + 4i) - (x + i)|^2$$

$$(x + 1)^2 + 4^2 = (6 - x)^2 + 3^2$$

$$x^2 + 2x + 17 = x^2 - 12x + 45$$

$$14x = 28$$

$$x = 2$$

So the centre of the circle is $2 + i$. The radius is

$$r = |(-1 - 3i) - (2 + i)| = |-3 - 4i| = \sqrt{9 + 16} = 5$$

The point $z_2 = 6 + ai$ also lies on the circle, so $|(6 + ai) - (2 + i)| = 5$, which gives

$$4^2 + (a - 1)^2 = 25$$

$$(a - 1)^2 = 9$$

$$a = 4 \quad \text{or} \quad a = -2$$

Since $a \neq 4$, the centre is $2 + i$ and $a = -2$.

(b) Using part (a), $z_2 = 6 - 2i$. Relative to the centre $2 + i$:

$$z_1 - (2 + i) = -3 - 4i, \qquad z_2 - (2 + i) = 4 - 3i$$

The scalar product of the corresponding vectors is

$$\begin{pmatrix} -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} = -12 + 12 = 0$$

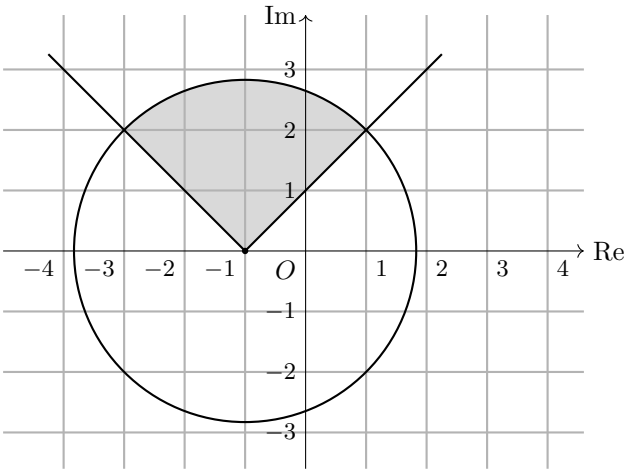
so the two radii are perpendicular, and the chord from z_1 to z_2 subtends an angle of $\frac{\pi}{2}$ at the centre. The locus is the major arc (it passes through z_3), and the angle at the centre is twice the angle at the circumference, so the chord subtends an angle of $\frac{\pi}{4}$ at every point z on the locus.

To find the order of the arguments, use the point $z_3 = 6 + 4i$ on the locus. Here $z_3 - z_2 = 6i$ and $z_3 - z_1 = 7 + 7i$, both of which have arguments that can be read directly:

$$\arg(6i) - \arg(7 + 7i) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

So the angle $\arg(z - z_2) - \arg(z - z_1)$ is the positive one, and the locus is

$$\arg\left(\frac{z - 6 + 2i}{z + 1 + 3i}\right) = \frac{\pi}{4}$$



6. The diagram above shows a shaded region in an Argand diagram.
The set of points, A, that lie within the shaded region, including its boundaries, is defined by

$$A = \{z \in \mathbb{C} \mid p \leq \arg(z - a) \leq q\} \cap \{z \in \mathbb{C} \mid |z - a| \leq r\}$$

where a is a real constant and p, q and r are positive constants.

(a) Write down the values of a, p, q and r. **[3]**

Given that $w = 3 - i$ and $z \in A$,

(b) find the minimum value of $|w - z|$, giving your answer in an exact simplified form. **[4]**

Solution

(a) The shaded sector has its vertex, which is also the centre of the circle, at the point -1 on the real axis. So

$$a = -1$$

The two edges of the sector start at -1 and pass through the grid points $1 + 2i$ and $-3 + 2i$. Relative to the vertex, these points are

$$(1 + 2i) - (-1) = 2 + 2i, \qquad (-3 + 2i) - (-1) = -2 + 2i$$

The point $2 + 2i$ lies in the first quadrant with $\arg(2 + 2i) = \arctan 1 = \frac{\pi}{4}$. The point $-2 + 2i$ lies in the second quadrant, so $\arg(-2 + 2i) = \pi - \arctan 1 = \frac{3\pi}{4}$. Hence

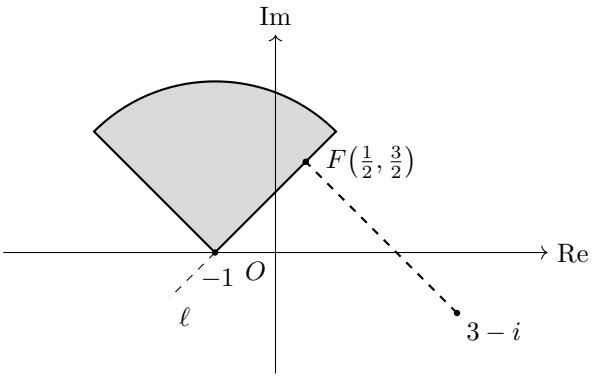
$$p = \frac{\pi}{4}, \qquad q = \frac{3\pi}{4}$$

The circle is centred at -1 and passes through $1 + 2i$, so its radius is

$$r = |2 + 2i| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

(b) Write $z = x + iy$. The value of $|w - z|$ is the distance between the points representing z and $w = 3 - i$

in the Argand diagram, so the minimum is the distance from $(3,-1)$ to the nearest point of A. The diagram below shows the region, the point $3-i$ and the nearest point F found below.



The edge $\arg(z+1)=\frac{\pi}{4}$ starts at $(-1,0)$ with gradient $\tan\frac{\pi}{4}=1$, so it lies on the line

$$\ell: \quad y=x+1$$

Every point of A has $\frac{\pi}{4}\le\arg(z+1)\le\frac{3\pi}{4}$, so lies on or above ℓ. For w, $y=-1$ while $x+1=4$, so w lies below ℓ. The line segment from w to any point of A therefore crosses ℓ, so $|w-z|$ is at least the perpendicular distance from w to ℓ.

The line through $(3,-1)$ perpendicular to ℓ has gradient -1:

$$y+1=-(x-3) \quad\Longrightarrow\quad y=-x+2$$

Intersecting with ℓ:

$$x+1=-x+2$$

$$x=\tfrac{1}{2}, \qquad y=\tfrac{3}{2}$$

So the foot of the perpendicular is F, representing $\frac{1}{2}+\frac{3}{2}i$. It lies in A, since F is on ℓ to the right of -1, giving $\arg\left(\frac{3}{2}+\frac{3}{2}i\right)=\frac{\pi}{4}$, and

$$\left|\tfrac{3}{2}+\tfrac{3}{2}i\right|=\sqrt{\tfrac{9}{4}+\tfrac{9}{4}}=\frac{3\sqrt{2}}{2}\le 2\sqrt{2}$$

Hence the minimum is attained at $z=\frac{1}{2}+\frac{3}{2}i$:

$$\begin{aligned}|w-z|_{\min}&=\left|(3-i)-\left(\tfrac{1}{2}+\tfrac{3}{2}i\right)\right|\\&=\left|\tfrac{5}{2}-\tfrac{5}{2}i\right|\\&=\sqrt{\tfrac{25}{4}+\tfrac{25}{4}}\\&=\frac{5\sqrt{2}}{2}\end{aligned}$$

7. The locus of points z satisfies

$$|z - 3a| = 2|z + ai|$$

where a is a non-zero integer.
The locus is a circle whose centre lies in the first quadrant and which encloses a region of area 40π.

(a) Find the value of a. **[8]**

(b) Find the exact least value of $|z|$ for points z on the circle. **[2]**

Solution

(a) Let $z = x + iy$. Then $z - 3a = (x - 3a) + iy$ and $z + ai = x + (y + a)i$, so squaring both sides of $|z - 3a| = 2|z + ai|$ gives

$$(x - 3a)^2 + y^2 = 4\left[x^2 + (y + a)^2\right]$$

$$x^2 - 6ax + 9a^2 + y^2 = 4x^2 + 4y^2 + 8ay + 4a^2$$

$$0 = 3x^2 + 3y^2 + 6ax + 8ay - 5a^2$$

$$0 = x^2 + y^2 + 2ax + \frac{8a}{3}y - \frac{5a^2}{3}$$

Completing the square:

$$(x + a)^2 - a^2 + \left(y + \frac{4a}{3}\right)^2 - \frac{16a^2}{9} - \frac{5a^2}{3} = 0$$

$$(x + a)^2 + \left(y + \frac{4a}{3}\right)^2 = \frac{40a^2}{9}$$

So the circle has centre $\left(-a, -\frac{4a}{3}\right)$ and radius r, where $r^2 = \frac{40a^2}{9}$. The enclosed area is 40π, so

$$\pi \times \frac{40a^2}{9} = 40\pi \quad \Longrightarrow \quad a^2 = 9 \quad \Longrightarrow \quad a = \pm 3$$

The centre lies in the first quadrant, so $-a > 0$, that is $a < 0$. Hence

$$a = -3$$

(b) With $a = -3$, the centre is $C(3, 4)$ and the radius is

$$r = \sqrt{\frac{40 \times 9}{9}} = \sqrt{40} = 2\sqrt{10} \approx 6.32$$

The distance of C from the origin O is

$$|OC| = \sqrt{3^2 + 4^2} = 5$$

Since $5 < 2\sqrt{10}$, the origin lies inside the circle. The point of the circle nearest to O therefore lies on the diameter through O, on the far side of O from C, at distance r from C. Its distance from O is $r - |OC|$, so the least value of $|z|$ is

$$2\sqrt{10} - 5$$

8. The locus of points $z = x + iy$ that satisfy

$$\arg\left(\frac{z-3+2i}{z-3-6i}\right) = \frac{3\pi}{4}$$

is an arc of a circle C.

(a) On an Argand diagram, sketch the locus of z. **[3]**

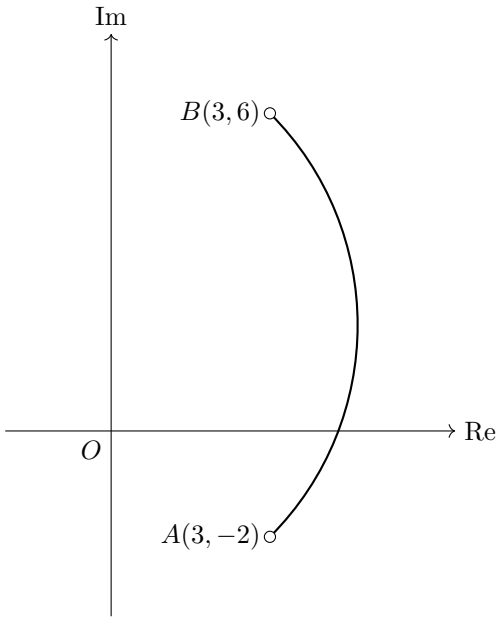
(b) Explain why the centre of C has y-coordinate 2. **[1]**

(c) Find the exact radius of C. **[2]**

(d) Find the x-coordinate of the centre of C and hence find the greatest value of $\text{Re}(z)$ for points on the locus. **[3]**

Solution

(a) The sketch should look like:



Let A and B be the points representing $3-2i$ and $3+6i$. The equation is

$$\arg\big(z-(3-2i)\big) - \arg\big(z-(3+6i)\big) = \frac{3\pi}{4}$$

so A and B are the ends of the arc. They are excluded, since the argument is undefined at $z = 3-2i$ and the quotient is undefined at $z = 3+6i$. The angle $\frac{3\pi}{4}$ is obtuse, so the locus is the minor arc of C.

To decide which side of the line $x = 3$ the arc lies on, take $z = 4+2i$:

$$\frac{1+4i}{1-4i} = \frac{(1+4i)^2}{17} = \frac{-15+8i}{17}$$

which lies in the second quadrant, so its argument is between $\frac{\pi}{2}$ and π. Taking $z = 2+2i$ instead gives

$$\frac{-1+4i}{-1-4i} = \frac{(-1+4i)^2}{17} = \frac{-15-8i}{17}$$

which lies in the third quadrant, with negative argument. Hence the arc lies to the right of the line $x = 3$.

(b) The points $A(3, -2)$ and $B(3, 6)$ both lie on C, so AB is a chord of C, and the perpendicular bisector of a chord passes through the centre. AB is vertical with midpoint $(3, 2)$, so its perpendicular bisector is the line $y = 2$. Hence the centre of C has y-coordinate 2.

(c) Let P be the centre of C and r its radius. A point on the minor arc and a point on the major arc, with A and B, form a cyclic quadrilateral, whose opposite angles sum to π. So the angle subtended by AB at a point on the major arc is

$$\pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

The angle at the centre is twice the angle at the circumference, so angle $APB = \frac{\pi}{2}$. Triangle APB is right-angled at P with $PA = PB = r$ and $AB = 6 - (-2) = 8$, so by Pythagoras:

$$r^2 + r^2 = 8^2$$

$$r^2 = 32$$

$$r = 4\sqrt{2}$$

(d) Let the centre be $P(h, 2)$, using part (b). Since $PA = 4\sqrt{2}$:

$$(3 - h)^2 + (-2 - 2)^2 = 32$$

$$(3 - h)^2 = 16$$

$$h = -1 \quad \text{or} \quad h = 7$$

The locus is the minor arc, which lies to the right of the chord AB, so the centre lies on the other side of the chord, to the left of $x = 3$. Hence the centre is $(-1, 2)$, with x-coordinate -1.

The point of C with greatest real part is level with the centre, at

$$\left(-1 + 4\sqrt{2},\ 2\right)$$

Since $-1 + 4\sqrt{2} > 3$ and $-2 < 2 < 6$, this point lies on the arc to the right of $x = 3$, so it is on the locus. The greatest value of $\mathrm{Re}(z)$ is

$$4\sqrt{2} - 1 \approx 4.66$$

9. In an Argand diagram, the curve C is given by

$$\arg\left(\frac{z-6}{z+2i}\right) = \frac{3\pi}{4}$$

and represents an arc of a circle.
Given that $z = x + iy$, where x and y are real numbers,

(a) show that this circle has equation

$$x^2 + y^2 + ax + by + c = 0$$

where a, b and c are integers to be determined, and hence find the centre and the exact radius of the circle. **[5]**

(b) determine the exact maximum value of $|z - 8 + 8i|$ for points z on C. **[4]**

Solution

(a) Substituting $z = x + iy$:

$$\frac{z-6}{z+2i} = \frac{(x-6)+iy}{x+(y+2)i}$$

Multiplying the numerator and denominator by $x - (y+2)i$:

$$\begin{aligned}\frac{z-6}{z+2i} &= \frac{[(x-6)+iy][x-(y+2)i]}{x^2+(y+2)^2} \\ &= \frac{x(x-6)+y(y+2)+[xy-(x-6)(y+2)]i}{x^2+(y+2)^2} \\ &= \frac{\left(x^2+y^2-6x+2y\right)+(-2x+6y+12)i}{x^2+(y+2)^2}\end{aligned}$$

The denominator is positive, so the real and imaginary parts have the same signs as $x^2 + y^2 - 6x + 2y$ and $-2x + 6y + 12$. An argument of $\frac{3\pi}{4}$ means the number lies in the second quadrant on the half-line where the imaginary part is minus the real part, so

$$-2x + 6y + 12 = -\left(x^2 + y^2 - 6x + 2y\right)$$

with $-2x + 6y + 12 > 0$. Rearranging:

$$x^2 + y^2 - 8x + 8y + 12 = 0$$

This is of the required form with $a = -8$, $b = 8$, $c = 12$.

Completing the square:

$$\begin{aligned}(x-4)^2 - 16 + (y+4)^2 - 16 + 12 &= 0 \\ (x-4)^2 + (y+4)^2 &= 20\end{aligned}$$

The circle has centre $(4, -4)$, representing $4 - 4i$, and radius $\sqrt{20} = 2\sqrt{5}$.

(b) Since $|z - 8 + 8i| = |z - (8 - 8i)|$, this is the distance from z to the point $P(8, -8)$. Let $M(4, -4)$ be the centre of the circle. Then

$$MP = \sqrt{(8-4)^2 + (-8+4)^2} = 4\sqrt{2}$$

Since $4\sqrt{2} > 2\sqrt{5}$, P lies outside the circle. The point F of the circle furthest from P lies on the line PM, on the far side of M from P, at a distance $MP + 2\sqrt{5}$ from P. The unit vector in the direction from P to M is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$

so

$$\overrightarrow{OF} = \overrightarrow{OM} + \frac{2\sqrt{5}}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}4-\sqrt{10}\\-4+\sqrt{10}\end{pmatrix}$$

and F is the point $\left(4-\sqrt{10},\, -4+\sqrt{10}\right)$.

It remains to show that F lies on the arc C. On the circle, the working in part (a) shows that the real part of $\frac{z-6}{z+2i}$ is minus its imaginary part, so F is on C provided $-2x + 6y + 12 > 0$ there. At F:

$$-2\left(4-\sqrt{10}\right) + 6\left(-4+\sqrt{10}\right) + 12 = 8\sqrt{10} - 20$$

Since $8\sqrt{10} > 20$, this is positive and F lies on C. Hence the maximum value is

$$MP + 2\sqrt{5} = 4\sqrt{2} + 2\sqrt{5}$$

10. A complex number z is represented by the point P on an Argand diagram.
Given that

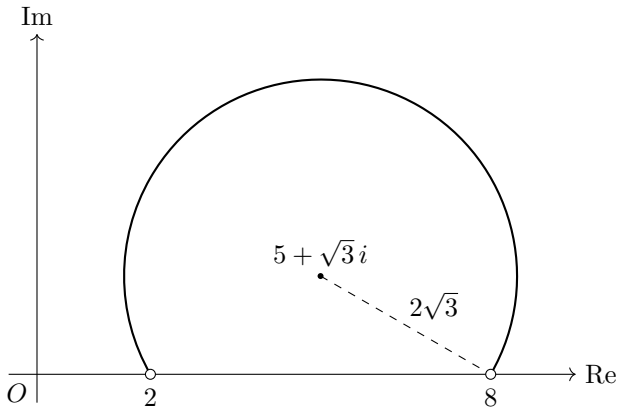
$$\arg\left(\frac{z-8}{z-2}\right) = \frac{\pi}{3}$$

(a) sketch the locus of P as z varies, **[3]**

(b) find the exact maximum value of $\arg z$ for points on this locus, and find, in the form $a+ib$, where a and b are exact real numbers, the value of z at which this maximum occurs. **[6]**

Solution

(a) The sketch should look like:



Let Q and S be the points representing 2 and 8. Then

$$\arg\left(\frac{z-8}{z-2}\right) = \arg(z-8) - \arg(z-2)$$

which is the angle turned anticlockwise from the direction of $\overrightarrow{QP}$ to the direction of $\overrightarrow{SP}$. So angle $QPS = \frac{\pi}{3}$, with P above the real axis: for P below the real axis the turn is clockwise and the argument is negative.

Since the chord QS subtends a constant angle at P, P lies on an arc of a circle through Q and S. Let C be the centre. The angle at the centre is twice the angle at the circumference, so angle $QCS = \frac{2\pi}{3}$. Triangle QCS is isosceles, so with M the midpoint of QS (the point 5), $QM = 3$ and angle $QCM = \frac{\pi}{3}$. Hence

$$\text{radius} = CQ = \frac{3}{\sin\frac{\pi}{3}} = 2\sqrt{3}$$

$$CM = \frac{3}{\tan\frac{\pi}{3}} = \sqrt{3}$$

The angle at P is acute, so P is on the major arc, on the same side of QS as the centre. So C is the point $5+\sqrt{3}\,i$.

The locus is the major arc of the circle $|z-(5+\sqrt{3}\,i)| = 2\sqrt{3}$ above the real axis, excluding the points 2 and 8 (where the argument is undefined).

(b) Let $z = x+iy$. The circle from part (a) has Cartesian equation

$$(x-5)^2 + \left(y-\sqrt{3}\right)^2 = 12$$

As the half-line from O through P turns anticlockwise, $\arg z$ increases, so the greatest value of $\arg z$ on the arc occurs where the line $y = mx$ through O touches the circle. Substituting $y = mx$:

$$(x-5)^2 + \left(mx - \sqrt{3}\right)^2 = 12$$

$$x^2 - 10x + 25 + m^2x^2 - 2\sqrt{3}\,mx + 3 - 12 = 0$$

$$\left(1+m^2\right)x^2 - \left(10 + 2\sqrt{3}\,m\right)x + 16 = 0$$

For the line to be a tangent, the discriminant is zero:

$$\left(10 + 2\sqrt{3}\,m\right)^2 - 4 \times \left(1+m^2\right) \times 16 = 0$$

$$100 + 40\sqrt{3}\,m + 12m^2 - 64 - 64m^2 = 0$$

$$52m^2 - 40\sqrt{3}\,m - 36 = 0$$

$$13m^2 - 10\sqrt{3}\,m - 9 = 0$$

so

$$m = \frac{10\sqrt{3} \pm \sqrt{300 + 468}}{26} = \frac{10\sqrt{3} \pm 16\sqrt{3}}{26} \quad \Longrightarrow \quad m = \sqrt{3} \ \text{ or } \ m = -\frac{3\sqrt{3}}{13}$$

The greater argument comes from the steeper tangent with positive gradient, $m = \sqrt{3}$. At the repeated root:

$$x = \frac{10 + 2\sqrt{3} \times \sqrt{3}}{2(1+3)} = \frac{16}{8} = 2, \qquad y = \sqrt{3} \times 2 = 2\sqrt{3}$$

This point has $y > 0$, so it lies on the arc. It is in the first quadrant, so

$$\arg z = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$$

The maximum value of $\arg z$ is $\frac{\pi}{3}$, occurring at $z = 2 + 2\sqrt{3}\,i$, so $a = 2$, $b = 2\sqrt{3}$.

11. A circle C in the complex plane has equation

$$|z-(4-2i)| = \alpha\,|z-(4+i)|$$

where α is a real constant with $\alpha > 1$.
Given that the real axis is a tangent to C,

(a) sketch, on an Argand diagram, the circle C, **[2]**

(b) explain why $\alpha = 2$. **[1]**

The circle C is contained in the region

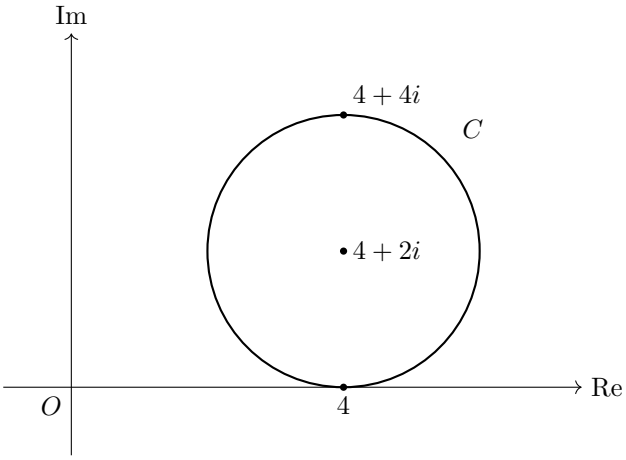
$$R = \{z \in \mathbb{C} \mid 0 \leq \arg z \leq \beta\}$$

(c) Determine the least possible value of β, giving your answer in an exact form. **[4]**

(d) For this least value of β, find, in the form $x+iy$, the complex number z on C for which $\arg z = \beta$. **[3]**

Solution

(a) The sketch should look like:



Writing $z = x+iy$ and squaring both sides, C has Cartesian equation

$$(x-4)^2 + (y+2)^2 = \alpha^2\left[(x-4)^2 + (y-1)^2\right]$$

By part (b), $\alpha = 2$, so

$$\begin{aligned}(x-4)^2 + (y+2)^2 &= 4(x-4)^2 + 4(y-1)^2\\ 0 &= 3(x-4)^2 + 4y^2 - 8y + 4 - y^2 - 4y - 4\\ 0 &= 3(x-4)^2 + 3y^2 - 12y\\ (x-4)^2 + (y-2)^2 &= 4\end{aligned}$$

So C is the circle with centre $4+2i$ and radius 2. It touches the real axis at 4, its highest point is $4+4i$, and it lies entirely in the first quadrant.

(b) Putting $y = 0$ in the equation of C from part (a):

$$(x-4)^2 + 4 = \alpha^2\left[(x-4)^2 + 1\right]$$

$$(\alpha^2 - 1)(x-4)^2 = 4 - \alpha^2$$

Since $\alpha > 1$, $\alpha^2 - 1 > 0$. The real axis is a tangent, so this equation has exactly one solution for x, which happens only when $4 - \alpha^2 = 0$. As $\alpha > 1$, this gives $\alpha = 2$.

(c) Every point of C has $x \geq 2$ and $y \geq 0$, so the least β is the argument of the half-line from O that touches C on its upper side. Let this half-line lie on $y = mx$, where $m = \tan\beta$. Substituting into $(x-4)^2 + (y-2)^2 = 4$:

$$(x-4)^2 + (mx-2)^2 = 4$$

$$x^2 - 8x + 16 + m^2x^2 - 4mx + 4 = 4$$

$$(1+m^2)x^2 - (8+4m)x + 16 = 0$$

For the line to be a tangent, the discriminant $b^2 - 4ac$ is zero:

$$(8+4m)^2 - 4 \times (1+m^2) \times 16 = 0$$

$$64 + 64m + 16m^2 - 64 - 64m^2 = 0$$

$$16m(4-3m) = 0$$

So $m = 0$ or $m = \frac{4}{3}$. The tangent $m = 0$ is the real axis, so the other tangent from O has gradient $\frac{4}{3}$, and C lies between these two tangents. Hence the least value of β is

$$\beta = \arctan\tfrac{4}{3}$$

(d) With $m = \frac{4}{3}$, the quadratic in part (c) has a repeated root:

$$x = \frac{8+4m}{2(1+m^2)} = \frac{8+\frac{16}{3}}{2 \times \frac{25}{9}} = \frac{40}{3} \times \frac{9}{50} = \frac{12}{5}$$

Then

$$y = \frac{4}{3} \times \frac{12}{5} = \frac{16}{5}$$

so

$$z = \frac{12}{5} + \frac{16}{5}i$$

12. The circle C has equation

$$|z + 7 - 2i| = 3\,|z - 1 - 2i|$$

where $z = x + iy$.

(a) Show that an equation for C is

$$x^2 + y^2 - 4x - 4y - 1 = 0 \qquad \textbf{[3]}$$

The half-line with equation

$$\arg(z - ai) = \frac{\pi}{3}$$

where a is a real constant, is a tangent to C.

(b) Determine the exact value of a, and find the complex number representing the point at which the half-line touches C, giving your answer in the form $p + iq$, where p and q are exact real numbers. **[7]**

Solution

(a) With $z = x + iy$, we have $z + 7 - 2i = (x + 7) + (y - 2)i$ and $z - 1 - 2i = (x - 1) + (y - 2)i$. Squaring the moduli, the equation of C becomes

$$(x + 7)^2 + (y - 2)^2 = 9\left[(x - 1)^2 + (y - 2)^2\right]$$

Expanding both sides:

$$\begin{aligned} x^2 + 14x + 49 + y^2 - 4y + 4 &= 9x^2 - 18x + 9 + 9y^2 - 36y + 36 \\ 0 &= 8x^2 + 8y^2 - 32x - 32y - 8 \\ 0 &= x^2 + y^2 - 4x - 4y - 1 \end{aligned}$$

So an equation for C is $x^2 + y^2 - 4x - 4y - 1 = 0$, as required.

(b) The half-line $\arg(z - ai) = \frac{\pi}{3}$ starts at the point $(0, a)$, which is not included, and makes an angle $\frac{\pi}{3}$ with the positive real direction. Its gradient is $\tan\frac{\pi}{3} = \sqrt{3}$, so it is the part of the line

$$y = \sqrt{3}x + a$$

with $x > 0$.

Substituting into the equation of C:

$$x^2 + \left(\sqrt{3}x + a\right)^2 - 4x - 4\left(\sqrt{3}x + a\right) - 1 = 0$$

$$x^2 + 3x^2 + 2\sqrt{3}ax + a^2 - 4x - 4\sqrt{3}x - 4a - 1 = 0$$

$$4x^2 + \left(2\sqrt{3}a - 4 - 4\sqrt{3}\right)x + \left(a^2 - 4a - 1\right) = 0$$

For the line to be a tangent, this quadratic has a repeated root, so its discriminant is zero:

$$\left(2\sqrt{3}a - 4 - 4\sqrt{3}\right)^2 - 4 \times 4 \times \left(a^2 - 4a - 1\right) = 0$$

Dividing by 4 and expanding:

$$\left(\sqrt{3}a - 2 - 2\sqrt{3}\right)^2 - 4\left(a^2 - 4a - 1\right) = 0$$

$$3a^2 - \left(12 + 4\sqrt{3}\right)a + 16 + 8\sqrt{3} - 4a^2 + 16a + 4 = 0$$

$$-a^2 + \left(4 - 4\sqrt{3}\right)a + 20 + 8\sqrt{3} = 0$$

$$a^2 + \left(4\sqrt{3} - 4\right)a - 20 - 8\sqrt{3} = 0$$

The discriminant of this quadratic in a is

$$\left(4\sqrt{3} - 4\right)^2 + 4\left(20 + 8\sqrt{3}\right) = 48 - 32\sqrt{3} + 16 + 80 + 32\sqrt{3} = 144$$

so

$$a = \frac{4 - 4\sqrt{3} \pm 12}{2} \quad \Longrightarrow \quad a = 8 - 2\sqrt{3} \quad \text{or} \quad a = -4 - 2\sqrt{3}$$

The repeated root of the quadratic in x is

$$x = -\frac{2\sqrt{3}a - 4 - 4\sqrt{3}}{8}$$

When $a = 8 - 2\sqrt{3}$, $2\sqrt{3}a = 16\sqrt{3} - 12$, so

$$x = -\frac{12\sqrt{3} - 16}{8} = 2 - \frac{3\sqrt{3}}{2} < 0$$

The line touches C where $x < 0$, which is not on the half-line, so this value is rejected.

When $a = -4 - 2\sqrt{3}$, $2\sqrt{3}a = -8\sqrt{3} - 12$, so

$$x = \frac{12\sqrt{3} + 16}{8} = 2 + \frac{3\sqrt{3}}{2} > 0$$

which is on the half-line, and then

$$y = \sqrt{3}\left(2 + \frac{3\sqrt{3}}{2}\right) - 4 - 2\sqrt{3} = 2\sqrt{3} + \frac{9}{2} - 4 - 2\sqrt{3} = \frac{1}{2}$$

Hence $a = -4 - 2\sqrt{3}$, and the half-line touches C at the point represented by

$$\left(2 + \frac{3\sqrt{3}}{2}\right) + \frac{1}{2}i$$

which is in the form $p + iq$ with $p = 2 + \frac{3\sqrt{3}}{2}$, $q = \frac{1}{2}$.