OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Counting, Permutations and Combinations

Worked solutions 10 questions · 79 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

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1. A security code consists of four characters, each chosen from the six digits 1, 2, 3, 4, 5, 6 and the three letters P, Q, R.

(a) Find the number of different codes in which no character is repeated and the first character is a letter. **[2]**

(b) Characters may now be repeated. Find the number of different codes that contain at least one letter. **[3]**

Solution

(a) There are 3 choices for the first character, since it must be one of the letters P, Q, R.

No character is repeated, so 8 characters remain for the other three positions. The order of these matters, so they can be filled in

$$^8P_3 = 8 \times 7 \times 6 = 336$$

ways. The number of codes is therefore

$$3 \times 336 = 1008$$

(b) Each of the four positions can be any of the 9 characters, so the total number of codes is

$$9^4 = 6561$$

A code with no letters uses only the six digits in every position, so there are

$$6^4 = 1296$$

such codes. Every other code contains at least one letter, so the number of codes containing at least one letter is

$$6561 - 1296 = 5265$$

2. Each of the 8 letters in the word TATTOOED is printed on a separate card. The cards are arranged in a row.

(a) Find the number of different arrangements of the letters. **[3]**

(b) Find the number of these arrangements in which the three Ts are next to each other and the first letter is not a T. **[4]**

Solution

(a) The word TATTOOED has 8 letters: three Ts, two Os, and one each of A, E and D. Arranging 8 letters, then dividing by the arrangements of the identical Ts and of the identical Os, gives

$$\frac{8!}{3! \times 2!} = \frac{40\,320}{12} = 3360$$

(b) Treat the three Ts as a single block, TTT. There are then 6 items to arrange: the block and the letters A, O, O, E, D.

The first letter is not a T, so the block cannot be in the first of the 6 places. The block can therefore go in any of the other 5 places.

For each position of the block, the remaining letters A, O, O, E, D fill the other 5 places, which can be done in

$$\frac{5!}{2!} = 60$$

ways, dividing by 2! for the two identical Os. The number of arrangements is

$$5 \times 60 = 300$$

3. A badminton club has 9 adult members and 11 junior members. One of the adults is the club captain, and two of the juniors are twins.
The club must choose a team of 7 players from its members for a tournament.
Find the number of different teams that can be chosen in each of the following cases.

(a) The captain must be in the team. **[1]**

(b) The team must contain exactly 3 adults, and the twins must either both be in the team or both be left out. **[3]**

(c) The team must contain at least 2 adults and at least 2 juniors. **[3]**

Solution

(a) The captain takes one of the 7 places, so the other 6 players are chosen from the remaining $20-1=19$ members:

$$\binom{19}{6} = 27\,132$$

(b) The number of ways to choose the 3 adults from the 9 adults is

$$\binom{9}{3} = 84$$

The other 4 players are juniors. If both twins are in the team, the remaining 2 juniors are chosen from the 9 juniors who are not twins. If both twins are left out, all 4 juniors are chosen from those 9. So the number of ways to choose the juniors is

$$\binom{9}{2} + \binom{9}{4} = 36 + 126 = 162$$

Each choice of adults can go with each choice of juniors, so the number of teams is

$$84 \times 162 = 13\,608$$

(c) If the team has a adults, it has $7-a$ juniors. The conditions $a \geq 2$ and $7-a \geq 2$ give $2 \leq a \leq 5$. Adding the cases:

$$\begin{aligned}
&\binom{9}{2} \times \binom{11}{5} + \binom{9}{3} \times \binom{11}{4} \\
&\quad + \binom{9}{4} \times \binom{11}{3} + \binom{9}{5} \times \binom{11}{2} \\
&= 36 \times 462 + 84 \times 330 + 126 \times 165 + 126 \times 55 \\
&= 16\,632 + 27\,720 + 20\,790 + 6930 \\
&= 72\,072
\end{aligned}$$

4. Three couples and two children are going to the cinema and will sit in a particular row of 8 seats. Find the number of ways in which the eight people can be seated if

(a) there are no restrictions on where they sit in the row, **[1]**

(b) each couple sits together, **[2]**

(c) neither child sits in a seat at either end of the row, **[2]**

(d) each couple sits together and the two children do not sit next to each other. **[3]**

Solution

(a) The eight people are arranged in the eight seats with no restrictions:

$$8! = 40\,320$$

(b) Treat each couple as a single block. The three blocks and the two children form 5 units, which can be arranged in 5! ways. Within each block the two partners can sit in 2 orders, giving 2^3 further arrangements:

$$5! \times 2^3 = 120 \times 8 = 960$$

(c) There are 6 seats that are not at an end of the row. The first child can sit in any of these 6 seats and the second child in any of the remaining 5, giving $6 \times 5 = 30$ ways. The six adults then fill the remaining 6 seats in $6! = 720$ ways:

$$30 \times 720 = 21\,600$$

(d) First arrange the three couple blocks. The blocks can be ordered in 3! ways, and each couple can sit in 2 orders:

$$3! \times 2^3 = 6 \times 8 = 48$$

The three blocks create 4 gaps: one at each end of the row and one between each pair of neighbouring blocks. A child cannot sit inside a block, since each couple sits together, so each child goes into a gap. The children do not sit next to each other, so they go into different gaps. The first child can go into any of the 4 gaps and the second child into any of the remaining 3, giving $4 \times 3 = 12$ ways.

The total number of arrangements is

$$48 \times 12 = 576$$

5. A security code consists of 7 characters: 4 **distinct** digits taken from the set $\{1, 2, 3, 4, 5, 6\}$ and 3 letters taken from the set $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$.
Letters may be repeated, and the digits and letters may appear in any order within the code.

(a) Find the total number of possible codes. **[2]**

(b) Find the number of codes in which at least two of the letters are next to each other. **[3]**

Solution

(a) Choose which 3 of the 7 positions hold the letters. The remaining 4 positions hold the digits, which are distinct and whose order matters, so they can be filled in ${}^6P_4$ ways. Each letter position can be filled in 5 ways:

$$\binom{7}{3} \times {}^6P_4 \times 5^3 = 35 \times (6 \times 5 \times 4 \times 3) \times 125 = 35 \times 360 \times 125$$

The total number of possible codes is 1 575 000.

(b) Count the complement: the codes in which no two letters are next to each other.

Place the 4 digits in a row. This leaves 5 gaps (one at each end and three between the digits), and no two letters are adjacent exactly when the 3 letters go into 3 different gaps. The number of ways to choose the letter positions is

$$\binom{5}{3} = 10$$

so the number of arrangements of letter positions with at least two letters adjacent is $35 - 10 = 25$. For each of these, the digits and letters are filled in as in part (a):

$$25 \times 360 \times 125 = 1\,125\,000$$

6. A bag contains 8 discs numbered $1, 2, 3, 4, 5, 6, 7, 8$.
Ben chooses 5 of the discs, without replacement, and places them in a row to form a 5-digit number.

(a) How many of these 5-digit numbers have an even first digit and an even last digit? **[2]**

(b) How many of these 5-digit numbers are odd and less than 35 000? **[5]**

Solution

(a) The even digits are 2, 4, 6 and 8. Fill the restricted positions first: the first digit can be any of the 4 even digits, and the last digit any of the 3 even digits that remain. The middle three positions are then filled in order from the 6 discs left, so the number of such 5-digit numbers is

$$4 \times 3 \times (6 \times 5 \times 4) = 4 \times 3 \times 120 = 1440$$

(b) The number is odd, so its last digit is 1, 3, 5 or 7. It is less than 35 000, so its first digit is 1, 2 or 3, and if the first digit is 3 then the second digit must be less than 5, so it is 1, 2 or 4 (the 3 is already used).

First digit 1: the last digit is 3, 5 or 7, giving 3 ways, and the middle three digits are chosen in order from the 6 remaining discs:

$$3 \times (6 \times 5 \times 4) = 360$$

First digit 2: the last digit is 1, 3, 5 or 7, giving 4 ways:

$$4 \times (6 \times 5 \times 4) = 480$$

First digit 3: the second digit is 1, 2 or 4. Once the first, second and last digits are fixed, the two remaining middle digits are chosen in order from the 5 discs left, in $5 \times 4 = 20$ ways.

- Second digit 1: the last digit is 5 or 7, giving $2 \times 20 = 40$.
- Second digit 2: the last digit is 1, 5 or 7, giving $3 \times 20 = 60$.
- Second digit 4: the last digit is 1, 5 or 7, giving $3 \times 20 = 60$.

So there are $40 + 60 + 60 = 160$ numbers beginning with 3.

Adding the three cases, the number of odd 5-digit numbers less than 35 000 is

$$360 + 480 + 160 = 1000$$

7. The first 25 positive integers include the 9 prime numbers $2, 3, 5, 7, 11, 13, 17, 19$ and 23.
Emma chooses 5 distinct numbers at random from the first 25 positive integers. The order in which Emma chooses the numbers does **not** matter.

(a) Calculate the number of possibilities in which Emma's 5 numbers include exactly 3 prime numbers, none of which is even. **[2]**

(b) Calculate the number of possibilities in which Emma's 5 numbers include more non-prime numbers than prime numbers. **[3]**

(c) Calculate the number of possibilities in which Emma's 5 numbers include at least one pair of prime numbers whose difference is exactly 12. **[4]**

Solution

(a) The only even prime is 2, so the 3 primes are chosen from the 8 odd primes. For exactly 3 primes, the other 2 numbers are chosen from the $25 - 9 = 16$ non-prime numbers. The number of possibilities is

$$\binom{8}{3} \times \binom{16}{2} = 56 \times 120 = 6720$$

(b) With 5 numbers, there are more non-primes than primes when there are 3, 4 or 5 non-primes, with 2, 1 or 0 primes respectively. There are 16 non-primes and 9 primes, so the number of possibilities is

$$\begin{aligned}
&\binom{16}{3} \times \binom{9}{2} + \binom{16}{4} \times \binom{9}{1} + \binom{16}{5} \\
&= 560 \times 36 + 1820 \times 9 + 4368 \\
&= 20\,160 + 16\,380 + 4368 \\
&= 40\,908
\end{aligned}$$

(c) For each prime p, check whether $p + 12$ is a prime not exceeding 25:

$$2 + 12 = 14, \quad 3 + 12 = 15, \quad 5 + 12 = 17, \quad 7 + 12 = 19, \quad 11 + 12 = 23, \quad 13 + 12 = 25$$

For larger primes $p + 12 > 25$. So the pairs of primes with difference 12 are

$$\{5, 17\}, \qquad \{7, 19\}, \qquad \{11, 23\}$$

These pairs have no number in common, so 5 numbers can contain at most two of them, since all three would need 6 numbers.

Exactly two pairs: choose 2 of the 3 pairs, then the fifth number from the remaining $25 - 4 = 21$ numbers:

$$\binom{3}{2} \times 21 = 63$$

Exactly one pair: choose the pair in 3 ways, then the other 3 numbers from the remaining 23. Of the $\binom{23}{3} = 1771$ such choices, those that also contain a particular one of the other two pairs number 21 (the third number is any of the remaining 21), and none can contain both other pairs. So the number of possibilities is

$$3 \times (1771 - 2 \times 21) = 3 \times 1729 = 5187$$

Adding the two cases, the number of possibilities is

$$5187 + 63 = 5250$$

8. Charlie has a set of 150 cards. Each of the integers from 1 to 150 is printed on exactly one of the cards.

(a) Determine how many of Charlie's cards show a number that is **not** a multiple of 3, 5 or 7. **[3]**

Charlie then chooses one or more of the cards and places them side by side in a row, so that reading the digits from left to right gives a single number. Two arrangements are different if they use different cards, or the same cards in a different order.

(b) Determine the number of different arrangements of cards that produce a 4-digit number. **[4]**

Solution

(a) Let A, B and C be the sets of cards showing a multiple of 3, 5 and 7 respectively. Since $3 \times 50 = 150$, $5 \times 30 = 150$ and $7 \times 21 = 147$, the sizes of these sets are

$$n(A) = 50, \qquad n(B) = 30, \qquad n(C) = 21$$

Since 3, 5 and 7 are prime, a number is a multiple of two or more of them exactly when it is a multiple of their product. Counting the multiples of 15, 21, 35 and 105 up to 150 gives

$$n(A \cap B) = 10, \qquad n(A \cap C) = 7, \qquad n(B \cap C) = 4, \qquad n(A \cap B \cap C) = 1$$

The inclusion-exclusion principle gives

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \\ &= 50 + 30 + 21 - 10 - 7 - 4 + 1 \\ &= 81 \end{aligned}$$

So the number of cards showing a number that is not a multiple of 3, 5 or 7 is

$$150 - 81 = 69$$

(b) The cards 1 to 9 are 9 one-digit cards, 10 to 99 are 90 two-digit cards, and 100 to 150 are 51 three-digit cards. No card begins with the digit 0, so every arrangement whose cards carry 4 digits in total produces a 4-digit number. The cases are split by the numbers of digits on the cards used.

One three-digit card and one one-digit card: there are 51×9 ways to choose the cards, and the three-digit card goes first or last, so the number of arrangements is

$$51 \times 9 \times 2 = 918$$

Two two-digit cards: the first card can be any of the 90 and the second any of the remaining 89, so the number of arrangements is

$$90 \times 89 = 8010$$

One two-digit card and two one-digit cards: the two-digit card is chosen in 90 ways, the one-digit cards in $\binom{9}{2} = 36$ ways, and the three cards are arranged in 3! ways, so the number of arrangements is

$$90 \times 36 \times 3! = 19\,440$$

Four one-digit cards: these are ordered selections of 4 of the 9 one-digit cards, so the number of arrangements is

$${}^{9}P_{4} = 9 \times 8 \times 7 \times 6 = 3024$$

Adding the four cases, the total number of arrangements is

$$918 + 8010 + 19\,440 + 3024 = 31\,392$$

9. Numbers are formed using the digits 0, 1, 2 and 3, and no number may begin with the digit 0.

(a) Find the number of different 3-digit numbers that can be formed

(i) if no digit may be used more than once, **[2]**

(ii) if each digit may be used any number of times, **[2]**

(iii) if each digit may be used any number of times and at least one digit appears more than once. **[2]**

(b) Find the number of different **even** 4-digit numbers that can be formed if each digit may be used at most twice. **[6]**

Solution

(a) (i) The first digit cannot be 0, so it has 3 choices. Since no digit is repeated, there are then 3 digits left for the second place and 2 for the third, giving

$$3 \times 3 \times 2 = 18$$

(ii) The first digit has 3 choices (1, 2 or 3), and the second and third digits have 4 choices each, giving

$$3 \times 4 \times 4 = 48$$

(iii) The numbers counted in part (a)(ii) in which no digit appears more than once are exactly the numbers counted in part (a)(i). Every other number in part (a)(ii) has at least one digit appearing more than once, so the number required is

$$48 - 18 = 30$$

(b) An even number ends in 0 or 2. First count the even 4-digit numbers with no restriction on repetition, then subtract those in which some digit is used three or more times.

With no restriction, the first digit has 3 choices, the second and third digits have 4 choices each, and the last digit has 2 choices, giving

$$3 \times 4 \times 4 \times 2 = 96$$

There are only four places, so at most one digit can be used three or more times in a number. The numbers to be removed can therefore be counted digit by digit, with no number counted twice.

Digit 0: the first digit is not 0, so three 0s must fill the last three places, and the first digit is 1, 2 or 3. This gives 3 numbers.

Digit 1: the last digit is 0 or 2, so three 1s must fill the first three places, giving 1110 and 1112. This gives 2 numbers.

Digit 3: in the same way, the numbers are 3330 and 3332. This gives 2 numbers.

Digit 2: the number 2222 uses 2 four times. For exactly three 2s, consider where the other digit goes. In the first place it is 1 or 3, giving 2 numbers. In the second or third place it is 0, 1 or 3, giving 3 numbers for each place. In the last place it must be even and not 2, so it is 0, giving 1 number. The total for digit 2 is

$$1 + 2 + 3 + 3 + 1 = 10$$

The number of numbers to be removed is

$$3 + 2 + 2 + 10 = 17$$

so the number of even 4-digit numbers in which each digit is used at most twice is

$$96 - 17 = 79$$

10. A test consists of 5 algebra questions, A, B, C, D and E, and 3 geometry questions, G, H and I. The examiner first plans to arrange all 8 questions in a random order, regardless of topic.

(a) (i) How many different arrangements are possible? **[1]**

(ii) Find the probability that no two geometry questions are next to each other. **[3]**

Later, the examiner decides that the questions should be arranged in two sections, one for each topic. Which section comes first is chosen at random, and the questions within each section are arranged in a random order.

(b) (i) How many different arrangements are now possible? **[2]**

(ii) Find the probability that questions A and B are next to each other. **[2]**

(iii) Find the probability that at least three other questions lie between questions C and H. **[4]**

Solution

(a) (i) The 8 questions are all different, so the number of arrangements is

$$8! = 40\,320$$

(ii) First arrange the 5 algebra questions, in $5! = 120$ ways. This leaves 6 gaps, one before each algebra question and one at the end:

$$_\, A \,_\, B \,_\, C \,_\, D \,_\, E \,_$$

No two geometry questions are next to each other exactly when G, H and I go into three different gaps. The geometry questions are different, so they can be placed in the gaps in ${}^6P_3 = 6 \times 5 \times 4 = 120$ ways. The number of arrangements with no two geometry questions together is

$$5! \times 6 \times 5 \times 4 = 120 \times 120 = 14\,400$$

All 8! arrangements are equally likely, so using part (a)(i):

$$P(\text{no two geometry questions together}) = \frac{14\,400}{40\,320} = \frac{5}{14}$$

(b) (i) There are 2 choices for which section comes first, 5! orders for the algebra section and 3! orders for the geometry section, so the number of arrangements is

$$2 \times 5! \times 3! = 2 \times 120 \times 6 = 1440$$

(ii) A and B are both in the algebra section, so whether they are next to each other depends only on the order within that section. Treat A and B as a single block: the block, C, D and E can be arranged in 4! ways, and the block itself in 2 ways, so $4! \times 2 = 48$ of the $5! = 120$ equally likely orders of the algebra section have A and B together, so

$$P(A \text{ and } B \text{ next to each other}) = \frac{48}{120} = \frac{2}{5}$$

(iii) Let C be the mth question in the algebra section, where $1 \leq m \leq 5$, and let H be the nth question in the geometry section, where $1 \leq n \leq 3$. The two sections are arranged independently at random, so each of the $5 \times 3 = 15$ pairs (m, n) is equally likely, whichever section comes first.

Algebra section first. Between C and H lie the $5-m$ algebra questions after C and the $n-1$ geometry questions before H. At least three questions lie between them when

$$(5-m)+(n-1)\geq 3$$

$$m\leq n+1$$

This gives $m=1,2$ when $n=1$; $m=1,2,3$ when $n=2$; and $m=1,2,3,4$ when $n=3$: $2+3+4=9$ pairs.

Geometry section first. Between H and C lie the $3-n$ geometry questions after H and the $m-1$ algebra questions before C. At least three questions lie between them when

$$(3-n)+(m-1)\geq 3$$

$$m\geq n+1$$

This gives $m=2,3,4,5$ when $n=1$; $m=3,4,5$ when $n=2$; and $m=4,5$ when $n=3$: $4+3+2=9$ pairs.

Each section comes first with probability $\frac{1}{2}$, so

$$P(\text{at least three questions between } C \text{ and } H)=\frac{1}{2}\times\frac{9}{15}+\frac{1}{2}\times\frac{9}{15}=\frac{3}{5}$$