OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Algebraic and Modulus Inequalities

Worked solutions 15 questions · 107 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · AQA 7367 · CIE 9231

Questions

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1. The function f is defined by

$$f(x) = |2\cos x - 1| \qquad (0 \leq x \leq 2\pi)$$

Find the set of values of x for which

$$f(x) < 1$$

Give your answer in set notation. **[5]**

Solution

Since $|u| < 1$ exactly when $-1 < u < 1$, the inequality $f(x) < 1$ becomes

$$-1 < 2\cos x - 1 < 1$$

$$0 < 2\cos x < 2$$

$$0 < \cos x < 1$$

The condition $\cos x > 0$ holds on the given domain for

$$0 \leq x < \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} < x \leq 2\pi$$

since $\cos x = 0$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, and $\cos x < 0$ between them.

The condition $\cos x < 1$ excludes only the points where $\cos x = 1$, which on $0 \leq x \leq 2\pi$ are $x = 0$ and $x = 2\pi$. At these points $f(x) = |2 - 1| = 1$, which does not satisfy the strict inequality.

Removing these two endpoints, the set of values is

$$\left\{x \in \mathbb{R} \mid 0 < x < \frac{\pi}{2}\right\} \cup \left\{x \in \mathbb{R} \mid \frac{3\pi}{2} < x < 2\pi\right\}$$

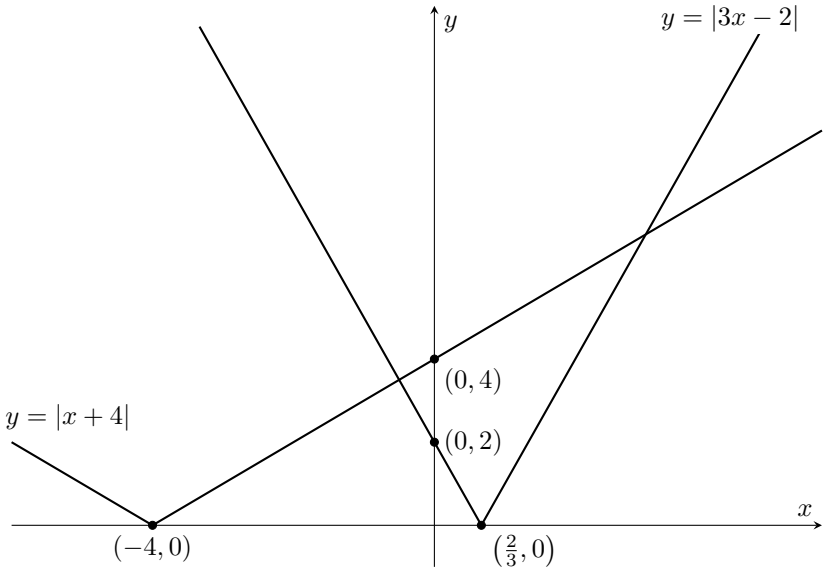
2. **(a)** On the same axes, sketch the graphs of $y = |3x - 2|$ and $y = |x + 4|$, stating the coordinates of the points where each graph meets the coordinate axes. **[3]**

(b) Hence, or otherwise, solve the inequality

$$|3x - 2| > |x + 4|$$ **[3]**

Solution

(a) The sketch should look like:



The graph of $y = |3x - 2|$ is a V shape with its vertex on the x-axis where $3x - 2 = 0$, so it meets the x-axis at $\left(\frac{2}{3}, 0\right)$. Setting $x = 0$ gives $y = |-2| = 2$, so it meets the y-axis at $(0, 2)$.

The graph of $y = |x + 4|$ is a V shape with its vertex on the x-axis where $x + 4 = 0$, so it meets the x-axis at $(-4, 0)$. Setting $x = 0$ gives $y = |4| = 4$, so it meets the y-axis at $(0, 4)$.

(b) The sketch in part (a) shows that the graphs cross twice: once to the right of $x = \frac{2}{3}$, where both expressions inside the modulus signs are positive, and once between $x = -4$ and $x = \frac{2}{3}$, where $3x - 2$ is negative and $x + 4$ is positive.

For the crossing to the right of $x = \frac{2}{3}$:

$$3x - 2 = x + 4 \quad \Longrightarrow \quad 2x = 6 \quad \Longrightarrow \quad x = 3$$

For the crossing between $x = -4$ and $x = \frac{2}{3}$:

$$-(3x - 2) = x + 4 \quad \Longrightarrow \quad -4x = 2 \quad \Longrightarrow \quad x = -\frac{1}{2}$$

From the sketch, the graph of $y = |3x - 2|$ lies above the graph of $y = |x + 4|$ to the left of the crossing at $x = -\frac{1}{2}$ and to the right of the crossing at $x = 3$. Hence

$$x < -\frac{1}{2} \quad \text{or} \quad x > 3$$

3. **(a)** Use algebra to determine the values of x for which

$$\frac{4x+1}{2x+3} \leq -3 \qquad \textbf{[4]}$$

(b) Hence, by using a suitable substitution, find the values of y for which

$$\frac{4+y}{2+3y} \leq -3 \qquad \textbf{[2]}$$

Solution

(a) The inequality is undefined at $x = -\frac{3}{2}$. Multiplying both sides by $(2x+3)^2$, which is positive for $x \neq -\frac{3}{2}$, keeps the direction of the inequality:

$$\begin{aligned}(4x+1)(2x+3) &\leq -3(2x+3)^2\\ (4x+1)(2x+3) + 3(2x+3)^2 &\leq 0\\ (2x+3)[(4x+1) + 3(2x+3)] &\leq 0\\ (2x+3)(10x+10) &\leq 0\\ 10(2x+3)(x+1) &\leq 0\end{aligned}$$

The critical values are $x = -\frac{3}{2}$ and $x = -1$. The left-hand side is a quadratic in x with positive leading coefficient, so it is negative between its roots. It equals zero at $x = -1$, which is included, but $x = -\frac{3}{2}$ is excluded because the original fraction is undefined there. Hence

$$-\frac{3}{2} < x \leq -1$$

(b) Substitute $x = \frac{1}{y}$, with $y \neq 0$, into the left-hand side of the inequality in part (a), and multiply the numerator and denominator by y:

$$\frac{4 \times \frac{1}{y} + 1}{2 \times \frac{1}{y} + 3} = \frac{4+y}{2+3y}$$

So, for $y \neq 0$, the inequality $\frac{4+y}{2+3y} \leq -3$ holds exactly when $x = \frac{1}{y}$ satisfies the inequality in part (a). When $y = 0$ the left-hand side is 2, so $y = 0$ is not a solution. Using part (a):

$$-\frac{3}{2} < \frac{1}{y} \leq -1$$

Here $\frac{1}{y} < 0$, so $y < 0$, and multiplying each inequality by y reverses its direction. From $\frac{1}{y} \leq -1$:

$$1 \geq -y \quad \Longrightarrow \quad y \geq -1$$

From $-\frac{3}{2} < \frac{1}{y}$:

$$-\frac{3}{2}y > 1 \quad \Longrightarrow \quad y < -\frac{2}{3}$$

Combining these:

$$-1 \leq y < -\frac{2}{3}$$

4. The function f is defined by

$$f(x) = 9x^3 + 24x^2 - 44x + 16 \qquad (x \in \mathbb{R})$$

(a) (i) Fully factorise $f(x)$. **[2]**

(ii) Hence solve the inequality $f(x) > 0$. **[2]**

(b) The graph of $y = f(x)$ is reflected in the y-axis. The new graph is then translated by the vector

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

to give the graph of $y = g(x)$.
Write $g(x)$ as a product of linear factors, and hence solve the inequality $g(x) \leq 0$. **[4]**

Solution

(a) (i) Substituting $x = -4$:

$$f(-4) = 9 \times (-64) + 24 \times 16 - 44 \times (-4) + 16 = -576 + 384 + 176 + 16 = 0$$

By the factor theorem, $(x + 4)$ is a factor of $f(x)$. Dividing gives

$$f(x) = (x + 4)\big(9x^2 - 12x + 4\big)$$

The quadratic factor is a perfect square, $9x^2 - 12x + 4 = (3x - 2)^2$, so

$$f(x) = (3x - 2)^2(x + 4)$$

(ii) Using part (a)(i), $f(x) = (3x - 2)^2(x + 4)$.

Since $(3x - 2)^2 \geq 0$ for all x, with equality only at $x = \frac{2}{3}$, the sign of $f(x)$ is the sign of $x + 4$, except at $x = \frac{2}{3}$, where $f(x) = 0$. So $f(x) > 0$ exactly when

$$x + 4 > 0 \quad \text{and} \quad x \neq \frac{2}{3}$$

Hence $-4 < x < \frac{2}{3}$ or $x > \frac{2}{3}$.

(b) Reflecting $y = f(x)$ in the y-axis replaces x with $-x$:

$$y = f(-x) = (-3x - 2)^2(-x + 4) = (3x + 2)^2(4 - x)$$

Translating by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ then replaces x with $x - 2$:

$$\begin{aligned} g(x) &= (3(x - 2) + 2)^2(4 - (x - 2)) \\ &= (3x - 4)^2(6 - x) \end{aligned}$$

So $g(x) = (3x - 4)(3x - 4)(6 - x)$.

Now $(3x - 4)^2 \geq 0$ for all x, with equality only at $x = \frac{4}{3}$.

At $x = \frac{4}{3}$, $g(x) = 0$, so the inequality $g(x) \leq 0$ holds.

For $x \neq \frac{4}{3}$, $(3x - 4)^2 > 0$, so $g(x) \leq 0$ exactly when

$$6 - x \leq 0 \quad \Longrightarrow \quad x \geq 6$$

Hence $g(x) \leq 0$ for $x = \frac{4}{3}$ or $x \geq 6$.

5. The function f is defined by

$$f(x) = \frac{ax + b}{x - 2} \qquad x \in \mathbb{R},\ x \neq 2$$

where a and b are integers.
The graph of $y = f(x)$ has a horizontal asymptote $y = 4$ and passes through the point $(3, 7)$.

(a) Find the value of a and the value of b. **[2]**

(b) Solve the inequality

$$f(x) \geq 2 - x$$

Fully justify your answer. **[6]**

Solution

(a) As $x \to \pm\infty$, the x terms dominate, so $f(x) \to a$. The horizontal asymptote is $y = 4$, so $a = 4$.

The graph passes through $(3, 7)$, so

$$f(3) = \frac{3 \times 4 + b}{3 - 2} = 12 + b = 7$$

Hence $a = 4$ and $b = -5$.

(b) Using part (a):

$$f(x) = \frac{4x - 5}{x - 2}$$

The sign of $x - 2$ is not known, but $(x - 2)^2 > 0$ for $x \neq 2$, so multiplying both sides by $(x - 2)^2$ keeps the direction of the inequality:

$$(4x - 5)(x - 2) \geq (2 - x)(x - 2)^2$$

$$(4x - 5)(x - 2) \geq -(x - 2)^3$$

$$(4x - 5)(x - 2) + (x - 2)^3 \geq 0$$

$$(x - 2)\left[(4x - 5) + (x - 2)^2\right] \geq 0$$

$$(x - 2)\left(4x - 5 + x^2 - 4x + 4\right) \geq 0$$

$$(x - 2)\left(x^2 - 1\right) \geq 0$$

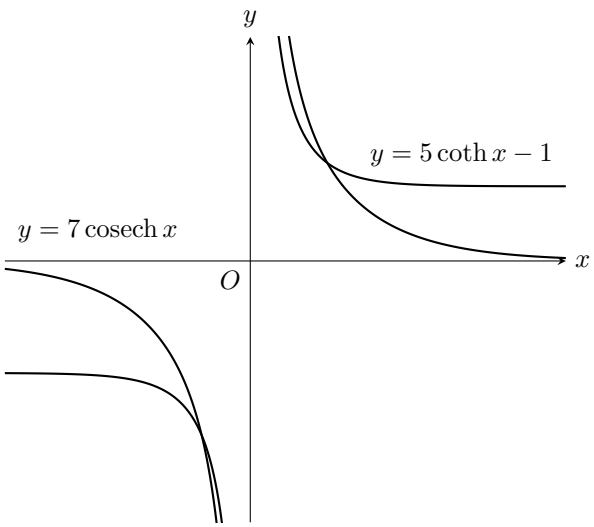
$$(x + 1)(x - 1)(x - 2) \geq 0$$

The critical values are $x = -1$, $x = 1$ and $x = 2$. The cubic $(x + 1)(x - 1)(x - 2)$ has a positive x^3 coefficient and three distinct roots, so it changes sign at each root:

- for $x < -1$, all three factors are negative, so the product is negative;
- for $-1 < x < 1$, only $x - 1$ and $x - 2$ are negative, so the product is positive;
- for $1 < x < 2$, only $x - 2$ is negative, so the product is negative;
- for $x > 2$, all three factors are positive, so the product is positive.

Equality holds at $x = -1$ and $x = 1$, where $f(-1) = 3 = 2 - (-1)$ and $f(1) = 1 = 2 - 1$, so these values are included. The value $x = 2$ is excluded because f is not defined there. Therefore

$$-1 \leq x \leq 1 \quad \text{or} \quad x > 2$$



6. The diagram above shows part of the graph of $y = 7\,\text{cosech}\,x$ and part of the graph of $y = 5\coth x - 1$. Solve the inequality

$$7\,\text{cosech}\,x < 5\coth x - 1$$

giving your answer in logarithmic form. **[5]**

Solution

Both curves are undefined at $x = 0$, where $\sinh x = 0$, so $x = 0$ is excluded. At a point of intersection, $x \neq 0$ and

$$\frac{7}{\sinh x} = \frac{5\cosh x}{\sinh x} - 1$$

Multiplying through by $\sinh x$ and writing in exponentials gives

$$\begin{aligned} 7 &= 5\cosh x - \sinh x \\ &= \frac{5}{2}\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right) \\ &= 2e^x + 3e^{-x} \end{aligned}$$

Multiplying by e^x and rearranging:

$$\begin{aligned} 2e^{2x} - 7e^x + 3 &= 0 \\ (2e^x - 1)(e^x - 3) &= 0 \end{aligned}$$

so $e^x = \frac{1}{2}$ or $e^x = 3$. The critical values are

$$x = \ln\frac{1}{2} = -\ln 2, \qquad x = \ln 3$$

These are the x-coordinates of the points where the curves cross, in the third and first quadrants respectively. The diagram shows that the graph of $y = 7\,\text{cosech}\,x$ lies below the graph of $y = 5\coth x - 1$ between $x = -\ln 2$ and the asymptote $x = 0$, and to the right of $x = \ln 3$, and above it everywhere else. Hence the solution is

$$-\ln 2 < x < 0 \quad \text{or} \quad x > \ln 3$$

7. The integers a and b are such that the inequality

$$(x^2 + ax - 12)(x^2 - 2x + b) < 0$$

has the solution set

$$\{x \in \mathbb{R} \mid -2 < x < -1\} \cup \{x \in \mathbb{R} \mid 3 < x < c\}$$

Find the values of a, b and c. **[5]**

Solution

The left-hand side is a quartic in x with positive leading coefficient. It is negative exactly when $-2 < x < -1$ or $3 < x < c$, so it changes sign at $x = -2$, $x = -1$, $x = 3$ and $x = c$. Hence its four roots are -2, -1, 3 and c, where $c > 3$, and each quadratic factor has two of them as its roots.

The product of the roots of $x^2 + ax - 12$ is -12. The products of the pairs of roots are

$$(-2) \times (-1) = 2, \qquad (-2) \times 3 = -6, \qquad (-1) \times 3 = -3$$

$$\begin{aligned} (-2) \times c = -12 &\implies c = 6 \\ (-1) \times c = -12 &\implies c = 12 \\ 3 \times c = -12 &\implies c = -4 \end{aligned}$$

Since $c > 3$, the value $c = -4$ is impossible, so the roots of $x^2 + ax - 12$ are either -2 and 6, or -1 and 12.

The sum of the roots of $x^2 - 2x + b$ is 2, and its roots are the other two. If the roots of $x^2 + ax - 12$ are -1 and 12, the other two roots are -2 and 3, whose sum is $1 \neq 2$, so this case is rejected. If they are -2 and 6, the other two roots are -1 and 3, whose sum is 2, as needed. Therefore

$$x^2 + ax - 12 = (x + 2)(x - 6) = x^2 - 4x - 12$$

$$x^2 - 2x + b = (x + 1)(x - 3) = x^2 - 2x - 3$$

Equating coefficients gives $a = -4$ and $b = -3$, and $c = 6$.

8. An engineering student is testing two small robots, A and B, which move along a straight track. The distances, in centimetres, of robots A and B from a fixed marker O on the track are modelled by

$$d_A = |3t - 25|$$
$$d_B = \left|2t^2 - 15t + 3\right|$$

respectively, where t is the time in seconds after the robots are switched on. The model is used for $0 \leq t \leq 10$.

(a) Use algebra to find the set of values of t, with $0 \leq t \leq 10$, for which robot B is further from O than robot A is. **[8]**

In a trial, robot B was recorded as being further from O than robot A for a total of approximately 6 seconds during the first 10 seconds.

(b) Find, in exact form, the total length of time during the first 10 seconds for which the model predicts that robot B is further from O than robot A, and use your answer to assess the validity of the model. **[3]**

Solution

(a) Robot B is further from O than robot A when $d_B > d_A$, that is, when

$$\left|2t^2 - 15t + 3\right| > |3t - 25|$$

Both sides are non-negative, so squaring both sides keeps the inequality the same way round:

$$\left(2t^2 - 15t + 3\right)^2 > (3t - 25)^2$$

$$\left(2t^2 - 15t + 3\right)^2 - (3t - 25)^2 > 0$$

Factorising the left-hand side as a difference of two squares:

$$\left[\left(2t^2 - 15t + 3\right) - (3t - 25)\right]\left[\left(2t^2 - 15t + 3\right) + (3t - 25)\right] > 0$$

$$\left(2t^2 - 18t + 28\right)\left(2t^2 - 12t - 22\right) > 0$$

$$4\left(t^2 - 9t + 14\right)\left(t^2 - 6t - 11\right) > 0$$

$$(t - 2)(t - 7)\left(t^2 - 6t - 11\right) > 0$$

Completing the square, $t^2 - 6t - 11 = 0$ gives

$$(t - 3)^2 = 20 \quad \Longrightarrow \quad t = 3 \pm 2\sqrt{5}$$

so the inequality is

$$(t - 2)(t - 7)\left(t - 3 - 2\sqrt{5}\right)\left(t - 3 + 2\sqrt{5}\right) > 0$$

Since $3 - 2\sqrt{5} \approx -1.47 < 0$, the factor $t - 3 + 2\sqrt{5}$ is positive for $0 \leq t \leq 10$. The inequality therefore reduces to

$$(t - 2)(t - 7)\left(t - 3 - 2\sqrt{5}\right) > 0$$

The critical values in the interval $0 \le t \le 10$ are $t = 2$, $t = 7$ and $t = 3 + 2\sqrt{5} \approx 7.47$. The signs of the factors are

	$0 \le t < 2$	$2 < t < 7$	$7 < t < 3 + 2\sqrt{5}$	$3 + 2\sqrt{5} < t \le 10$
$t - 2$	$-$	$+$	$+$	$+$
$t - 7$	$-$	$-$	$+$	$+$
$t - 3 - 2\sqrt{5}$	$-$	$-$	$-$	$+$
product	$-$	$+$	$-$	$+$

So robot B is further from O than robot A when

$$2 < t < 7 \quad \text{or} \quad 3 + 2\sqrt{5} < t \le 10$$

(b) Using part (a), the total time predicted by the model is

$$\begin{aligned}(7 - 2) + \left(10 - \left(3 + 2\sqrt{5}\right)\right) &= 5 + 7 - 2\sqrt{5} \\ &= \left(12 - 2\sqrt{5}\right) \text{s} \\ &\approx 7.53 \text{ s}\end{aligned}$$

This is about 1.5 seconds more than the recorded total of approximately 6 seconds, an overestimate of about 25%. The model overestimates the time for which robot B is further from O, so it is not very accurate and should be refined.

9. The function f is defined by

$$f(x) = \frac{x^2 + ax + 2}{x + 3}, \qquad x \neq -3$$

where a is a constant and $a > \frac{11}{3}$.

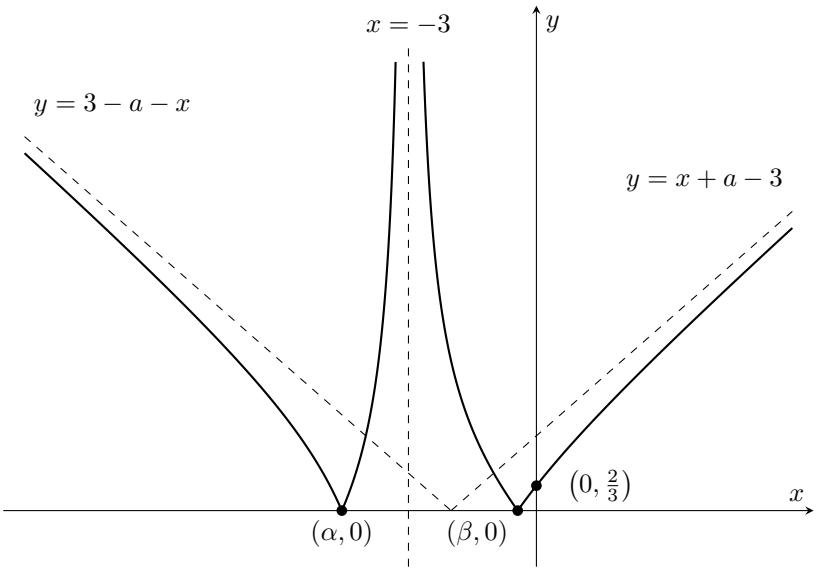
(a) Sketch the graph of $y = |f(x)|$. **[2]**

(b) On your sketch in part (a), draw the line $y = a$. **[1]**

(c) It is given that the line $y = a$ meets the graph of $y = |f(x)|$ at the point where $x = 4$.
Find the value of a and hence find the set of values of x for which $|f(x)| < a$. **[5]**

Solution

(a) The sketch should look like:



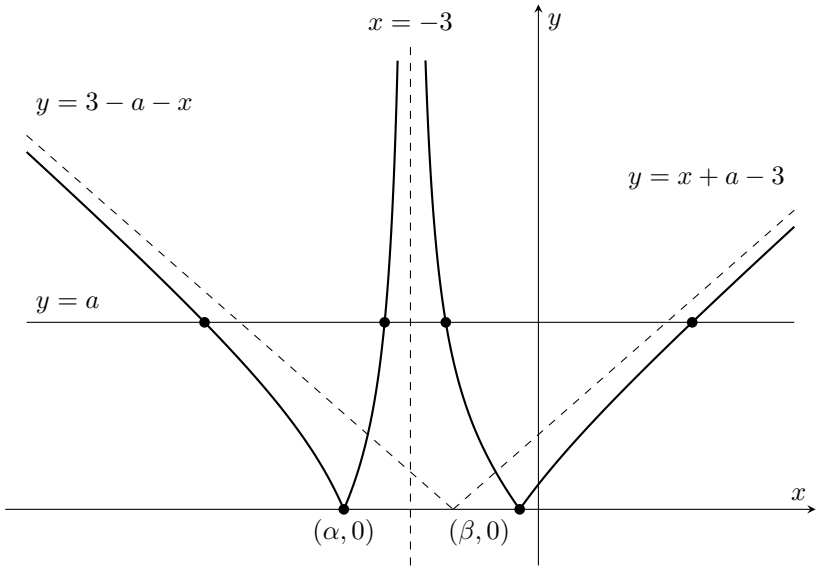
Dividing the numerator by $x + 3$:

$$x^2 + ax + 2 = (x + 3)(x + a - 3) + 11 - 3a \quad \Longrightarrow \quad f(x) = x + a - 3 + \frac{11 - 3a}{x + 3}$$

The line $x = -3$ is a vertical asymptote, with $|f(x)| \to \infty$ on both sides. As $x \to \pm\infty$, $\frac{11-3a}{x+3} \to 0$, so $f(x)$ approaches $x + a - 3$. Hence $y = |f(x)|$ has asymptote $y = x + a - 3$ as $x \to \infty$ and $y = 3 - a - x$ as $x \to -\infty$.

The roots of $x^2 + ax + 2 = 0$ are α and β: the discriminant is $a^2 - 4 \times 1 \times 2 = a^2 - 8 > 0$, since $a^2 > \frac{121}{9} > 8$. Their product is $2 > 0$ and their sum is $-a < 0$, so both are negative, and at $x = -3$ the numerator is $11 - 3a < 0$, so $\alpha < -3 < \beta < 0$. The graph touches the x-axis at $(\alpha, 0)$ and $(\beta, 0)$, where the part below the axis is reflected, giving sharp points. When $x = 0$, $y = \frac{2}{3}$.

(b) The sketch with the line $y = a$ added should look like:



The line $y = a$ meets the graph of $y = |f(x)|$ at four points, two on each side of $x = -3$.

(c) When $x = 4$:

$$f(4) = \frac{16 + 4a + 2}{7} = \frac{18 + 4a}{7}$$

This is positive since $a > 0$, so $|f(4)| = f(4)$ and

$$\frac{18 + 4a}{7} = a$$

$$18 + 4a = 7a$$

$$a = 6$$

Then $f(x) = \frac{x^2+6x+2}{x+3}$. The boundary values are where $|f(x)| = 6$. Where $f(x) = 6$:

$$x^2 + 6x + 2 = 6x + 18$$

$$x^2 = 16$$

$$x = \pm 4$$

Where $f(x) = -6$:

$$x^2 + 6x + 2 = -6x - 18$$

$$x^2 + 12x + 20 = 0$$

$$(x + 2)(x + 10) = 0$$

$$x = -2 \text{ or } x = -10$$

From the sketch in part (b), on each side of $x = -3$ the graph of $y = |f(x)|$ lies below the line $y = 6$ between the two points of intersection on that side, so $|f(x)| < 6$ for

$$-10 < x < -4 \quad \text{or} \quad -2 < x < 4$$

10. Given that k is a positive constant and that

$$\frac{kx}{x+3} \leq \frac{x-1}{x-2}$$

(a) show that

$$(x-2)(x+3)(px^2+qx+r) \leq 0$$

where p, q and r are expressions in terms of k to be determined. **[3]**

(b) Given also that $px^2+qx+r=0$ when $x=3$, find the value of k.
Hence, or otherwise, find the values of x for which the given inequality holds for this value of k. **[5]**

Solution

(a) The inequality is undefined at $x=-3$ and $x=2$, so $x \neq -3$ and $x \neq 2$. Then $(x+3)^2(x-2)^2 > 0$, so multiplying both sides by it keeps the direction of the inequality:

$$kx(x+3)(x-2)^2 \leq (x-1)(x-2)(x+3)^2$$

Taking everything to the right-hand side and factorising out $(x-2)(x+3)$:

$$0 \leq (x-2)(x+3)[(x-1)(x+3) - kx(x-2)]$$

$$0 \leq (x-2)(x+3)\left[x^2+2x-3-kx^2+2kx\right]$$

$$0 \leq (x-2)(x+3)\left[(1-k)x^2+(2k+2)x-3\right]$$

Multiplying by -1 reverses the inequality:

$$(x-2)(x+3)\left((k-1)x^2-(2k+2)x+3\right) \leq 0$$

This is the required form with $p=k-1$, $q=-2k-2$ and $r=3$.

(b) With $x=3$, $px^2+qx+r=0$ gives

$$\begin{aligned} 9(k-1)+3(-2k-2)+3 &= 0 \\ 3k-12 &= 0 \\ k &= 4 \end{aligned}$$

With $k=4$, $p=3$, $q=-10$ and $r=3$, so the quadratic factor is

$$3x^2-10x+3=(3x-1)(x-3)$$

Using part (a), the inequality becomes

$$(x-2)(x+3)(3x-1)(x-3) \leq 0, \qquad x \neq -3, \quad x \neq 2$$

The critical values are $x=-3$, $x=\frac{1}{3}$, $x=2$ and $x=3$. The left-hand side is a quartic with positive leading coefficient 3 and these four distinct roots, so it is positive for $x>3$ and changes sign at each critical value:

- positive for $x<-3$;
- negative for $-3<x<\frac{1}{3}$;
- positive for $\frac{1}{3}<x<2$;

- negative for $2 < x < 3$;
- positive for $x > 3$.

The quartic is zero at all four critical values, but $x = -3$ and $x = 2$ are excluded because the original inequality is undefined there. The inequality holds for

$$-3 < x \leq \frac{1}{3} \quad \text{or} \quad 2 < x \leq 3$$

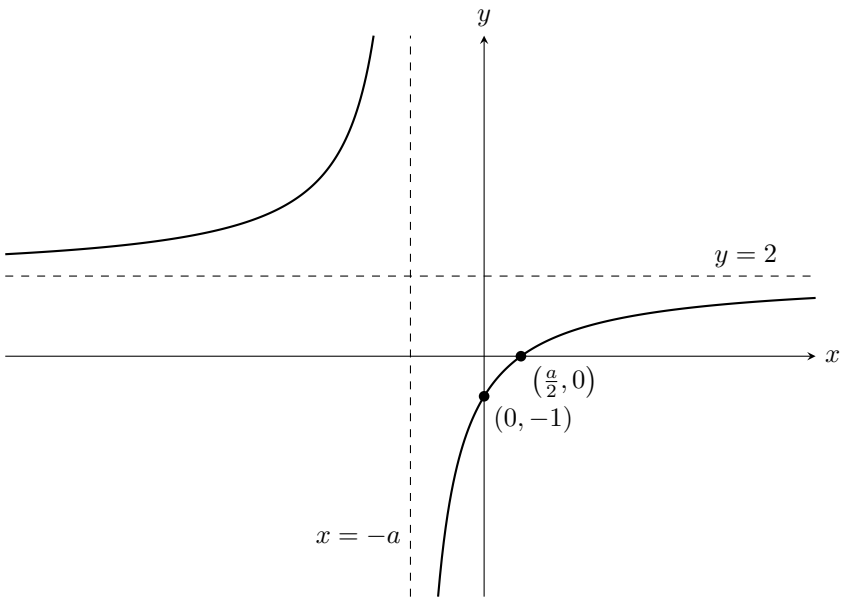
11. Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{2x-a}{x+a}$, stating the equations of its asymptotes and the coordinates of the points where it meets the coordinate axes. **[3]**

(b) Sketch the curve with equation $y = \left|\frac{2x-a}{x+a}\right|$ and find, in terms of a, the set of values of x for which $\left|\frac{2x-a}{x+a}\right| > 3$. **[4]**

Solution

(a) The sketch should look like:



Since $2x - a = 2(x + a) - 3a$, the equation can be written as

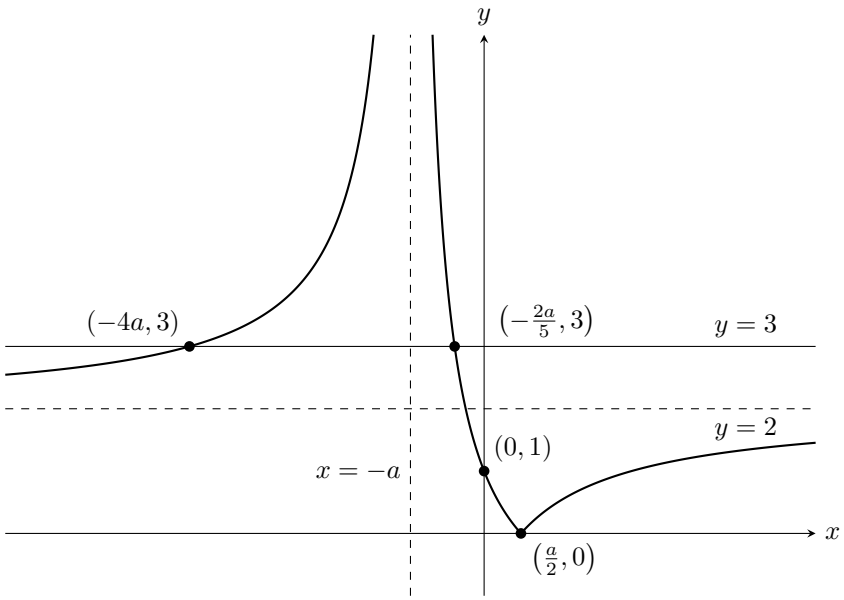
$$y = 2 - \frac{3a}{x + a}$$

As $x \to \pm\infty$, $\frac{3a}{x+a} \to 0$, so $y \to 2$. Hence the curve has horizontal asymptote $y = 2$. The denominator is zero when $x = -a$, so the curve has vertical asymptote $x = -a$.

When $x = 0$, $y = -\frac{a}{a} = -1$, so the curve meets the y-axis at $(0, -1)$.

When $y = 0$, $2x - a = 0$, so $x = \frac{a}{2}$ and the curve meets the x-axis at $\left(\frac{a}{2}, 0\right)$.

(b) The sketch should look like:



The part of the curve in part (a) that lies below the x-axis, for $-a < x < \frac{a}{2}$, is reflected in the x-axis. The curve now meets the y-axis at $(0, 1)$ and touches the x-axis at $\left(\frac{a}{2}, 0\right)$. The asymptotes are still $x = -a$ and $y = 2$, and $y \to +\infty$ on both sides of $x = -a$.

From the sketch, the curve lies above the line $y = 3$ only on either side of the asymptote $x = -a$.

For $x < -a$, the curve is the original branch, so solve

$$\frac{2x - a}{x + a} = 3$$

$$2x - a = 3x + 3a$$

$$x = -4a$$

For $x > -a$, the curve is the reflected part, so solve

$$\frac{2x - a}{x + a} = -3$$

$$2x - a = -3x - 3a$$

$$5x = -2a$$

$$x = -\frac{2a}{5}$$

Reading from the sketch, $\left|\frac{2x-a}{x+a}\right| > 3$ when

$$-4a < x < -a \quad \text{or} \quad -a < x < -\frac{2a}{5}$$

12. The curve C has equation

$$y = \frac{x-3}{2-x-x^2}$$

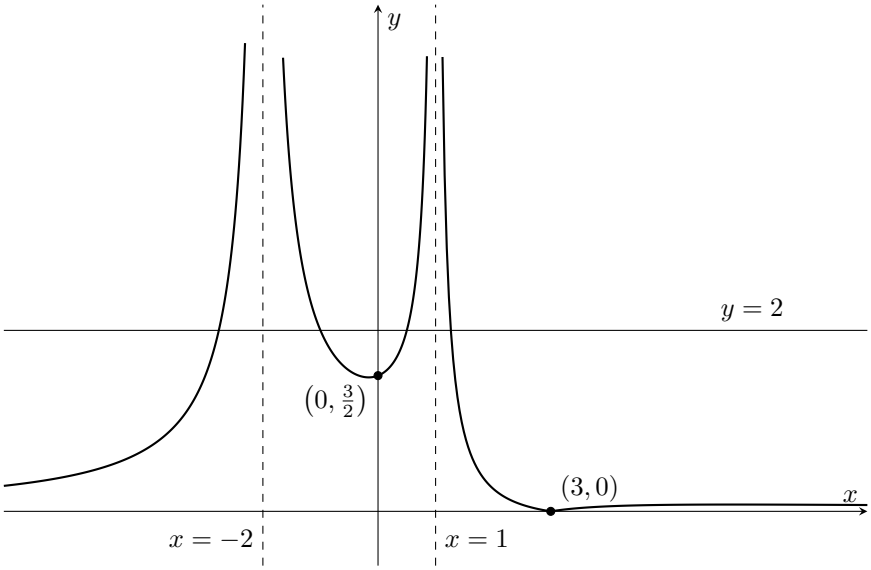
Sketch the curve with equation

$$y = \left|\frac{x-3}{2-x-x^2}\right|$$

and find, in exact form, the set of values of x for which $|x-3| > 2\left|2-x-x^2\right|$. **[6]**

Solution

The sketch should look like:



The denominator factorises as $2-x-x^2 = -(x+2)(x-1)$, so the curve has vertical asymptotes $x=-2$ and $x=1$. As $x \to \pm\infty$, the x^2 term in the denominator dominates, so $y \to 0$. Hence the curve has horizontal asymptote $y=0$.

When $x=0$, $y = \left|-\frac{3}{2}\right| = \frac{3}{2}$. When $y=0$, $x=3$, where the curve touches the x-axis. The branch of C between the asymptotes lies below the x-axis, and it is reflected in the x-axis to give the middle branch above.

When $x=-2$ or $x=1$, the right-hand side of the inequality is 0, while the left-hand side is 5 or 2, so the inequality holds at both values. For every other x, $\left|2-x-x^2\right| > 0$, and dividing by $\left|2-x-x^2\right|$ gives

$$\left|\frac{x-3}{2-x-x^2}\right| > 2$$

so the inequality holds where the sketch lies above the line $y=2$.

The critical values satisfy $x-3 = 2\left(2-x-x^2\right)$ or $x-3 = -2\left(2-x-x^2\right)$. The first gives

$$x-3 = 4-2x-2x^2$$

$$2x^2+3x-7=0$$

$$x = \frac{-3\pm\sqrt{65}}{4}$$

and the second gives

$$x - 3 = -4 + 2x + 2x^2$$

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0 \quad \Longrightarrow \quad x = \frac{1}{2} \text{ or } x = -1$$

In increasing order, the critical values and asymptotes are $\frac{-3-\sqrt{65}}{4} \approx -2.77$, -2, -1, $\frac{1}{2}$, 1 and $\frac{-3+\sqrt{65}}{4} \approx 1.27$. From the sketch, the curve lies above $y = 2$ for $\frac{-3-\sqrt{65}}{4} < x < -2$, $-2 < x < -1$, $\frac{1}{2} < x < 1$ and $1 < x < \frac{-3+\sqrt{65}}{4}$. Including $x = -2$ and $x = 1$, the set of values is

$$\frac{-3 - \sqrt{65}}{4} < x < -1 \quad \text{or} \quad \frac{1}{2} < x < \frac{-3 + \sqrt{65}}{4}$$

13. The curve C has equation $y = f(x)$, where

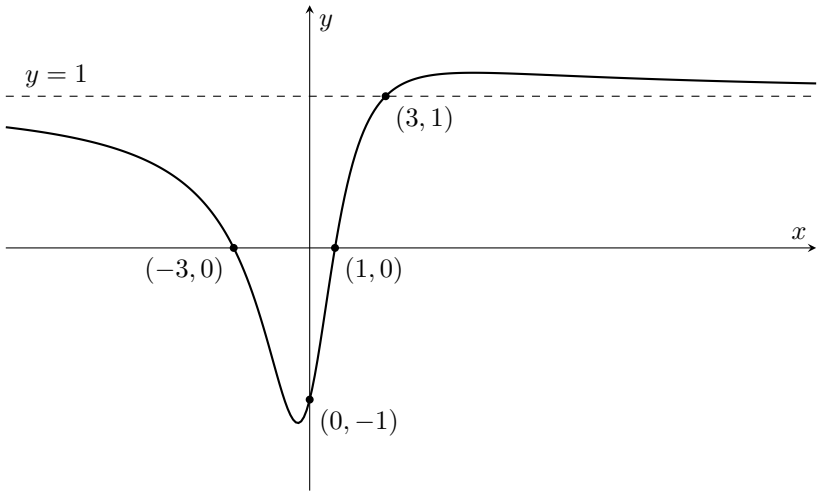
$$f(x) = \frac{x^2 + 3}{x^2 + 2x - 3}$$

(a) Sketch the curve with equation $y = \frac{1}{f(x)}$, stating the coordinates of any points where it meets the coordinate axes and the equation of its asymptote. **[3]**

(b) Find the set of values of x for which $\frac{1}{f(x)} \geq f(x)$. **[4]**

Solution

(a) The sketch of $y = \frac{1}{f(x)}$ should look like:



Here

$$\frac{1}{f(x)} = \frac{x^2 + 2x - 3}{x^2 + 3} = \frac{(x+3)(x-1)}{x^2 + 3}$$

Since $x^2 + 3 > 0$ for all x, the curve has no vertical asymptotes. As $x \to \pm\infty$, the x^2 terms dominate, so $\frac{1}{f(x)} \to 1$. Hence the asymptote is $y = 1$. The curve meets it where $x^2 + 2x - 3 = x^2 + 3$, that is at $x = 3$, so it crosses its asymptote at $(3, 1)$.

When $x = 0$, $y = -\frac{3}{3} = -1$, so the curve meets the y-axis at $(0, -1)$.

When $y = 0$, $(x + 3)(x - 1) = 0$, so the curve meets the x-axis at $(-3, 0)$ and $(1, 0)$.

(b) Write $g(x) = \frac{1}{f(x)}$. The function f is undefined where $x^2 + 2x - 3 = 0$, so $x = -3$ and $x = 1$ are excluded, and for every other x, $g(x) \neq 0$. The inequality is $g(x) \geq \frac{1}{g(x)}$. Multiplying both sides by $g(x)^2 > 0$:

$$g(x)^3 \geq g(x)$$

$$g(x)^3 - g(x) \geq 0$$

$$g(x)(g(x) - 1)(g(x) + 1) \geq 0$$

This cubic in $g(x)$ has positive leading coefficient and roots -1, 0 and 1, so it is non-negative for $-1 \leq g(x) \leq 0$ or $g(x) \geq 1$. Since $g(x) \neq 0$, the inequality holds where

$$-1 \leq g(x) < 0 \qquad \text{or} \qquad g(x) \geq 1$$

From part (a), $g(x) = 1$ at $x = 3$ and $g(x) = 0$ at $x = -3$ and $x = 1$. Solving $g(x) = -1$:

$$x^2 + 2x - 3 = -\left(x^2 + 3\right)$$

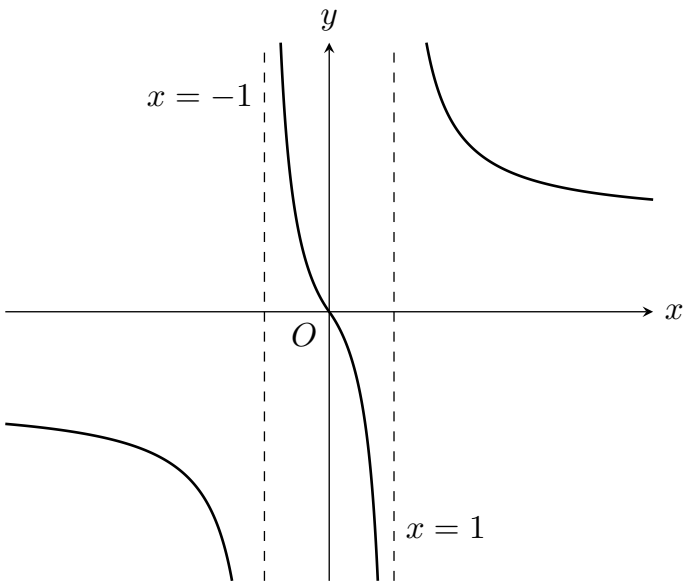
$$2x^2 + 2x = 0$$

$$2x(x + 1) = 0$$

so $x = -1$ or $x = 0$.

From the sketch in part (a), the curve lies on or above $y = 1$ exactly for $x \geq 3$. It lies below the x-axis for $-3 < x < 1$, where it dips below $y = -1$ only between $x = -1$ and $x = 0$. So $-1 \leq g(x) < 0$ for $-3 < x \leq -1$ or $0 \leq x < 1$, and the set of values is

$$-3 < x \leq -1 \quad \text{or} \quad 0 \leq x < 1 \quad \text{or} \quad x \geq 3$$



14. The diagram above shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{2x}{|x| - 1}$$

Use algebra to determine the values of x for which

$$\frac{2x}{|x| - 1} \geq x + 3$$ **[8]**

Solution

The critical values are the x-coordinates where the curve $y = f(x)$ meets the line $y = x + 3$, together with the vertical asymptotes $x = -1$ and $x = 1$.

For $x \geq 0$, $|x| = x$, so with $x \neq 1$ the intersections satisfy

$$\frac{2x}{x - 1} = x + 3$$

$$2x = (x + 3)(x - 1)$$

$$2x = x^2 + 2x - 3$$

$$x^2 = 3$$

Since $x \geq 0$, this gives $x = \sqrt{3}$.

For $x < 0$, $|x| = -x$, so with $x \neq -1$ the intersections satisfy

$$\frac{2x}{-x - 1} = x + 3$$

$$2x = -(x + 3)(x + 1)$$

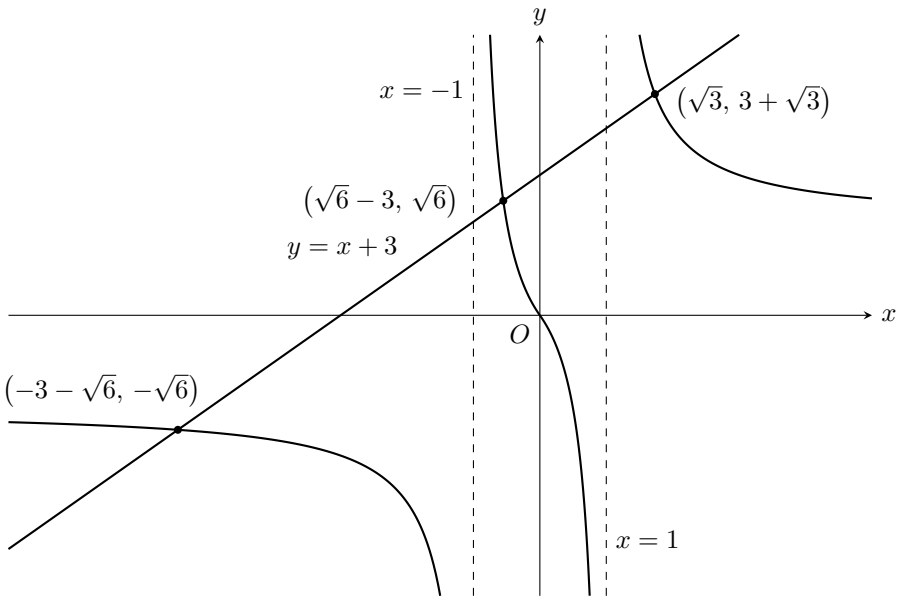
$$2x = -x^2 - 4x - 3$$

$$x^2 + 6x + 3 = 0$$

$$x = -3 \pm \sqrt{6}$$

Since $\sqrt{6} < 3$, both $-3 + \sqrt{6}$ and $-3 - \sqrt{6}$ are negative, so both are valid.

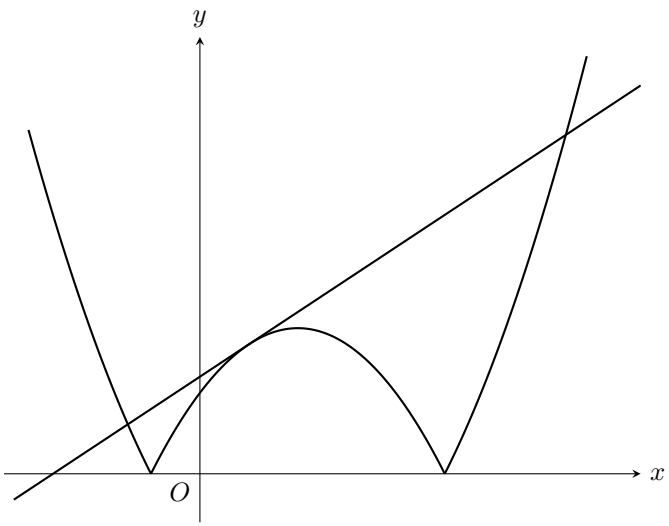
Adding the line $y = x + 3$ to the sketch of $y = f(x)$, the diagram looks like:



The inequality holds where the curve lies on or above the line. From the sketch, this happens on the branch $x < -1$ for $x \leq -3 - \sqrt{6}$, on the branch $-1 < x < 1$ for $-1 < x \leq \sqrt{6} - 3$, and on the branch $x > 1$ for $1 < x \leq \sqrt{3}$. The values $x = \pm 1$ are excluded, since f is undefined there.

The solution is

$$x \leq -3 - \sqrt{6} \quad \text{or} \quad -1 < x \leq \sqrt{6} - 3 \quad \text{or} \quad 1 < x \leq \sqrt{3}$$



15. The diagram above shows a sketch of the curve with equation $y = |x^2 - 4x - 5|$ and a sketch of the straight line with equation $y = mx + c$, where m and c are positive constants.
The equation

$$|x^2 - 4x - 5| = mx + c$$

has exactly 3 roots, as shown in the diagram above.

(a) Show that

$$m^2 - 8m - 4c + 36 = 0 \quad \textbf{[2]}$$

Given that the line passes through the point $(-2, 2)$,

(b) determine the value of m and the value of c, justifying your answer. **[4]**

(c) Hence solve

$$|x^2 - 4x - 5| > mx + c \quad \textbf{[3]}$$

Solution

(a) Since $x^2 - 4x - 5 = (x + 1)(x - 5)$, this is negative for $-1 < x < 5$, so on that interval the curve is

$$y = -(x^2 - 4x - 5) = 5 + 4x - x^2$$

The diagram shows that the line meets each outer branch once and touches this part of the curve, giving the 3 roots. At the point of contact, equating the line and the curve gives

$$5 + 4x - x^2 = mx + c$$

$$x^2 + (m - 4)x + (c - 5) = 0$$

and this quadratic has a repeated root, so its discriminant is zero:

$$(m - 4)^2 - 4 \times 1 \times (c - 5) = 0$$

$$m^2 - 8m + 16 - 4c + 20 = 0$$

$$m^2 - 8m - 4c + 36 = 0$$

as required.

(b) The line passes through $(-2, 2)$, so

$$2 = -2m + c \quad \Longrightarrow \quad c = 2m + 2$$

Substituting into the result of part (a):

$$m^2 - 8m - 4(2m + 2) + 36 = 0$$

$$m^2 - 16m + 28 = 0$$

$$(m - 2)(m - 14) = 0$$

so $m = 2$ or $m = 14$. The repeated root of $x^2 + (m - 4)x + (c - 5) = 0$ is

$$x = \frac{4 - m}{2}$$

When $m = 14$, this gives $x = -5$, which does not satisfy $-1 < x < 5$, so the line would not touch the part of the curve between $x = -1$ and $x = 5$, and $m = 14$ is rejected. When $m = 2$, the point of contact is at $x = 1$, which does lie in $-1 < x < 5$. Hence

$$m = 2, \qquad c = 2 \times 2 + 2 = 6$$

(c) The inequality is $|x^2 - 4x - 5| > 2x + 6$. For $x \leq -1$ or $x \geq 5$, the curve is $y = x^2 - 4x - 5$, and it meets the line where

$$x^2 - 4x - 5 = 2x + 6$$

$$x^2 - 6x - 11 = 0$$

$$x = 3 \pm 2\sqrt{5}$$

Here $3 - 2\sqrt{5} \approx -1.47 < -1$ and $3 + 2\sqrt{5} \approx 7.47 > 5$, so both roots lie on the outer branches. For $-1 < x < 5$, part (b) shows that the line touches the curve only at $x = 1$, where the two sides are equal, so the strict inequality fails there. The diagram shows that the curve lies above the line to the left of the first root and to the right of the last, so

$$x < 3 - 2\sqrt{5} \quad \text{or} \quad x > 3 + 2\sqrt{5}$$