OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Transformations of the Complex Plane

Questions by topic 12 questions · 115 marks

EXAM BOARDS Edexcel FP2 9FM0/4A

Questions

Question 1	*5 marks*	2
Question 2	*5 marks*	3
Question 3	*8 marks*	4
Question 4	*9 marks*	5
Question 5	*4 marks*	6
Question 6	*12 marks*	7
Question 7	*12 marks*	8
Question 8	*12 marks*	9
Question 9	*12 marks*	10
Question 10	*11 marks*	11
Question 11	*11 marks*	12
Question 12	*14 marks*	13

1. The transformation T from the z-plane to the w-plane is given by

$$w = iz^2$$

(a) Describe fully the geometric effect of T. **[3]**

The region R in the z-plane is given by

$$\left\{ z \in \mathbb{C} \;\middle|\; |z| < 2,\ -\frac{\pi}{6} < \arg z < \frac{\pi}{6} \right\}$$

(b) On an Argand diagram, sketch the image of R under T. Mark the radius of its curved boundary and the arguments of its straight boundaries. **[2]**

2. A transformation from the z-plane to the w-plane is given by

$$w = \frac{2z + i}{z - 1}, \qquad z \neq 1$$

The transformation maps points on the line $\mathrm{Re}(z) = 1$ in the z-plane, other than the point $z = 1$, onto points on the line ℓ in the w-plane.
Given that $w = u + iv$, where u and v are real, find an equation of ℓ in the form $au + bv = c$, where a, b and c are integers. **[5]**

3. The complex numbers $z = x + iy$ and $w = u + iv$, where x, y, u and v are real, are represented by the points P and Q respectively in Argand diagrams, where

$$w = z^2 + 2z$$

(a) Show that $u = x^2 - y^2 + 2x$ and obtain an expression for v in terms of x and y. **[4]**

(b) The point P moves along the line $x = 1$. Find a Cartesian equation of the locus of Q, giving your answer in the form $v^2 = a + bu$, where a and b are constants. **[4]**

4. A transformation from the z-plane to the w-plane is given by

$$w = \frac{(1+i)z + 2 - i}{z - 1}, \qquad z \neq 1$$

The circle C in the z-plane has equation $|z| = 2$.

(a) Show that C is mapped to a circle D in the w-plane, and find a Cartesian equation for D, stating its centre and radius. **[7]**

(b) Sketch C and D on separate Argand diagrams. **[2]**

5. The transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{5z - p}{z + i}, \qquad z \neq -i, \quad p \in \mathbb{R}$$

The point representing $2 + 2i$ is invariant under T.
Find the value of p, and find the complex number represented by the other point that is invariant under T. **[4]**

6. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+1}{z-i}, \qquad z \neq i$$

The circle with equation $|z-1| = 2$ is mapped by T onto the curve C.

(a) Show that every point w on C satisfies

$$|w-(1+i)| = \sqrt{2}\,|w-1| \qquad \textbf{[4]}$$

(b) Hence show that C is a circle, and find its centre and radius. **[6]**

The region $|z-1| < 2$ in the z-plane is mapped by T onto the region R in the w-plane.

(c) Shade the region R on an Argand diagram. **[2]**

7. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + 2i}{z - 2}, \qquad z \neq 2$$

The transformation T maps the circle $|z| = 2$ in the z-plane onto the line ℓ in the w-plane.

(a) Find a Cartesian equation of the line ℓ. **[5]**

The circle $|z - a - bi| = c$ in the z-plane, where a, b and c are real constants with $c > 0$, is mapped by T onto the circle $|w - i| = 2$ in the w-plane.

(b) Find the exact values of a, b and c. **[7]**

8. The complex numbers z and w are represented, respectively, by the points $P(x, y)$ and $Q(u, v)$ in Argand diagrams, where

$$w = \frac{z}{z - 2i}, \qquad z \neq 2i$$

(a) Show that

$$u = \frac{x^2 + y^2 - 2y}{x^2 + (y - 2)^2}$$

and obtain an expression for v in terms of x and y. **[5]**

(b) The point P moves along the line $y = x$. Show that the locus of Q is a circle, and find its centre and its radius. **[7]**

9. The transformation T maps points from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$. The transformation T is given by

$$w = \frac{z-1}{z+i}, \qquad z \neq -i$$

The transformation T maps the line ℓ in the z-plane onto the line with equation $u = 1$ in the w-plane.

(a) Find a Cartesian equation of ℓ in terms of x and y. **[5]**

The transformation T maps the line with equation $y = x$ in the z-plane onto the curve C in the w-plane.

(b) (i) Show that C is a circle, and find its centre and its radius. **[5]**

(ii) Find the exact greatest value of $|w|$ for points w on C. **[2]**

10. A transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{z + 2i}{z - 1}, \qquad z \neq 1$$

The locus of points in the z-plane satisfying $|z - 3 - i| = |z - 1 + i|$ is mapped by T onto a circle C in the w-plane.

(a) Find the coordinates of the centre of C and the exact radius of C. **[9]**

The region $|z - 3 - i| < |z - 1 + i|$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) On a single Argand diagram, sketch the circle C and shade and label the region R. **[2]**

11. A complex number z is represented by the point P in an Argand diagram, where

$$|z-1| = |z+1-2i|$$

(a) Find a Cartesian equation for the locus of P, and sketch this locus on an Argand diagram, marking the coordinates of the points where it meets the axes. **[3]**

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2z}{z+i}, \qquad z \neq -i$$

Given that T maps the locus of P onto a circle C in the w-plane,

(b) find the equation of C, giving your answer in the form

$$|w-(p+qi)| = r$$

where p, q and r are real constants to be found. **[8]**

12. A transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{\left(-1 + \sqrt{3}\, i\right)(z - 3i)}{z + 3i}, \qquad z \neq -3i$$

(a) Show that the imaginary axis in the z-plane is mapped by T onto the line with equation $v = -\sqrt{3}\, u$ in the w-plane. **[3]**

(b) Show that the circle in the z-plane with equation $|z| = 3$ is mapped by T onto a line in the w-plane, stating clearly an equation for this line. **[5]**

The region R in the z-plane is defined by

$$\{z \in \mathbb{C} \mid |z| > 3\} \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$$

(c) Determine the image of R under T, giving your answer in the form $\{w \in \mathbb{C} \mid \alpha < \arg w < \beta\}$, where α and β are rational multiples of π, and sketch this image on an Argand diagram. **[6]**