OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

# Sets of Numbers and Irrationality Proofs

Questions by topic 10 questions · 64 marks

EXAM BOARDS OCR MEI Extra Pure Y435

## Questions


| | | |
|---|---|---|
| Question 1 | *6 marks* | 2 |
| Question 2 | *5 marks* | 3 |
| Question 3 | *3 marks* | 4 |
| Question 4 | *4 marks* | 5 |
| Question 5 | *6 marks* | 6 |
| Question 6 | *5 marks* | 7 |
| Question 7 | *8 marks* | 8 |
| Question 8 | *9 marks* | 9 |
| Question 9 | *9 marks* | 10 |
| Question 10 | *9 marks* | 11 |

**1.** The sets $A$ and $B$ are defined by

$$A = \{x \in \mathbb{Z} \mid x^2 < 10\}, \qquad B = \{x \in \mathbb{N} \mid x \text{ is a factor of } 12\}$$

**(a)** List the elements of $A$. **[1]**

**(b)** Find $A \cap B$ and $A \cup B$. **[2]**

**(c)** For each of the numbers

$$-4, \qquad \frac{3}{7}, \qquad \sqrt{11}, \qquad 2 - 5i$$

state the smallest of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to which it belongs. **[2]**

**(d)** Give a counterexample to show that the set of irrational numbers is not closed under multiplication. **[1]**

**2.** In this question you may assume that if $p$ and $q$ are distinct prime numbers and $p^\alpha = q^\beta$, where $\alpha, \beta \in \mathbb{Z}$, then $\alpha = 0$ and $\beta = 0$.

**(a)** Prove by contradiction that $\log_3 12 \notin \mathbb{Q}$. **[3]**

**(b)** Hence prove that $\log_3 2 \notin \mathbb{Q}$. **[2]**

**3.** The numbers $a$, $b$, $c$ and $d$ are rational, and

$$a + b\sqrt{3} = c + d\sqrt{3}$$

Given that $\sqrt{3}$ is irrational, prove that $a = c$ and $b = d$. **[3]**

**4.** The real number $x$ is irrational, so that $x + 3 \neq 0$. Use proof by contradiction to show that

$$\frac{2x - 1}{x + 3}$$

is also irrational. **[4]**

**5.** **(a)** Prove by contradiction that the product of a non-zero rational number and an irrational number is always irrational. **[4]**

**(b)** Hence, given that $\sqrt{3}$ is irrational, prove that $\sqrt{75} - \sqrt{12}$ is irrational. **[2]**

**6.** You may assume that $\sqrt{21}$ is irrational.

**(a)** Use proof by contradiction to prove that $\sqrt{3} + \sqrt{7}$ is irrational. **[3]**

**(b)** Hence prove that $\sqrt{7} - \sqrt{3}$ is irrational. **[2]**

**7.** A student is attempting to prove that $\sqrt[3]{4}$ is irrational.
The first three lines of their proof are shown below.

| | |
|---|---|
| Assume that $\sqrt[3]{4}$ is rational, so it can be written as $\sqrt[3]{4} = \frac{a}{b}$. | **Line 1** |
| Cubing both sides gives $4 = \frac{a^3}{b^3}$. Hence $a^3 = 4b^3$. | **Line 2** |
| Hence $a^3$ is even, so $a$ is even, and $a = 2k$ for some integer $k$. | **Line 3** |

**(a)** State the conditions on $a$ and $b$ that are required in **Line 1**. **[2]**

**(b)** In **Line 3**, the student uses the fact that if $a^3$ is even then $a$ is even.
Prove that, for any integer $n$, if $n^3$ is even then $n$ is even. **[3]**

**(c)** Complete the proof that $\sqrt[3]{4}$ is irrational. **[3]**

**8.** In this question you may assume that every integer greater than 1 can be written as a product of primes in exactly one way, apart from the order of the factors.

**(a)** Let $m$ be an integer with $m > 1$. Prove that every prime appears to an even power in the prime factorisation of $m^2$. **[3]**

**(b)** Using part (a) and by considering the power of 2 in the prime factorisations of both sides of a suitable equation, prove by contradiction that $\sqrt{18}$ is irrational.
Explain why considering the power of 3 instead would not lead to a contradiction. **[6]**

**9.** Consider the cubic equation

$$x^3 - 3x - 1 = 0 \qquad (*)$$

**(a)** Suppose that $x = \frac{p}{q}$ is a root of $(*)$, where $p$ and $q$ are integers with $q > 0$ and $\frac{p}{q}$ is in its lowest terms. Show that $q$ divides $p^3$ and that $p$ divides $q^3$. Hence prove that $(*)$ has no rational roots. **[4]**

**(b)** Using the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, show that $x = 2\cos 20^\circ$ is a root of $(*)$. Hence prove by contradiction that $\cos 20^\circ$ is irrational. **[5]**

**10.** You are given that $q \in \mathbb{Z}$ with $q \geq 1$ and that

$$S = \frac{1}{(2q+1)(2q+2)} + \frac{1}{(2q+1)(2q+2)(2q+3)(2q+4)} + \frac{1}{(2q+1)(2q+2)\cdots(2q+6)} + \cdots$$

**(a)** By considering a suitable geometric series, show that

$$S < \frac{1}{4q(q+1)}$$ **[3]**

**(b)** Deduce that $S \notin \mathbb{Z}$. **[2]**

You are also given that

$$\cosh 1 = \sum_{r=0}^{\infty} \frac{1}{(2r)!}$$

**(c)** Assume that $\cosh 1 = \frac{p}{q}$, where $p$ and $q$ are positive integers. By considering $(2q)! \cosh 1$ and using the result of part (b), prove by contradiction that $\cosh 1$ is irrational. **[4]**