OPTIONAL PURE / MISCELLANEOUS OPTIONAL PURE

Sets of Numbers and Irrationality Proofs

Mark scheme 10 questions · 64 marks

EXAM BOARDS OCR MEI Extra Pure Y435

Questions

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	Total	**64**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$A = \{-3, -2, -1, 0, 1, 2, 3\}$	**B1**	All seven integers and no others, in any order.
		[1]	
1(b)	$A \cap B = \{1, 2, 3\}$	**B1**	
	$A \cup B = \{-3, -2, -1, 0, 1, 2, 3, 4, 6, 12\}$	**B1**	All ten elements and no others, in any order.
		[2]	
1(c)	$-4 \in \mathbb{Z}$, $\frac{3}{7} \in \mathbb{Q}$ $\sqrt{11} \in \mathbb{R}$, $2 - 5i \in \mathbb{C}$	**B1 B1**	**B1** Any two correct. **B1** All four correct.
		[2]	
1(d)	$\sqrt{2} \times \sqrt{2} = 2$, which is rational	**B1**	Any two irrational numbers, which may be equal, with their product shown to be rational. Not if either number is rational.
		[1]	
	Total for Question 1	**6**	

Question	Scheme	Marks	Notes
2(a)	$\log_3 12 = \frac{m}{n},\ n \neq 0$ $3^{\frac{m}{n}} = 12$	**M1**	Assumes $\log_3 12$ is a quotient of integers, any letters, and writes it in exponential form.
	$3^m = 12^n = 2^{2n} \times 3^n$ $\Rightarrow 3^{m-n} = 2^{2n}$	**dM1**	Raises to the power n and writes 12 as $2^2 \times 3$, reaching an equation in powers of 2 and 3 only. Or reaches $3^m = 12^n$ with $m, n > 0$ and compares parity. Condone index slips.
	$2n = 0$ and $m - n = 0 \Rightarrow n = 0$ Contradicts $n \neq 0$, so $\log_3 12 \notin \mathbb{Q}$	**A1 AG**	Correct $3^{m-n} = 2^{2n}$ oe, the given result applied to reach $n = 0$, contradicting a stated $n \neq 0$, and a conclusion. Or 3^m odd and 12^n even with $m, n > 0$, a contradiction, and a conclusion.
		[3]	
2(b)	$\log_3 2 = \frac{r}{s},\ s \neq 0$ $\log_3 12 = 1 + 2\log_3 2 = 1 + \frac{2r}{s}$	**M1**	Assumes $\log_3 2$ is a quotient of integers and expresses $\log_3 12$ in terms of it, by laws of logarithms or powers.
	$\log_3 12 = \frac{s+2r}{s} \in \mathbb{Q}$ Contradicts (a), so $\log_3 2 \notin \mathbb{Q}$	**A1 AG**	Correct $\log_3 12$ as a quotient of integers oe, the contradiction with part (a) stated, and a conclusion.
		[2]	
	Total for Question 2	**5**	

Question	Scheme	Marks	Notes
3	Assumes $b \neq d$: $(d-b)\sqrt{3} = a - c$	**M1**	Assumes $b \neq d$, or considers $d - b \neq 0$, and collects the $\sqrt{3}$ terms on one side. Condone sign slips.
	$\sqrt{3} = \frac{a-c}{d-b}$, rational Contradiction, so $b = d$	**A1**	Division by non-zero $d - b$, the quotient stated rational, contradiction with $\sqrt{3}$ irrational, and $b = d$ concluded.
	$a + b\sqrt{3} = c + b\sqrt{3} \Rightarrow a = c$	**A1 AG**	Substitutes $b = d$ to reach $a = c$ without error. A0 without a closing statement of both $a = c$ and $b = d$.
	Total for Question 3	**3**	

Question	Scheme	Marks	Notes
4	Assumes $\frac{2x-1}{x+3} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$ $\Rightarrow b(2x-1) = a(x+3)$	**M1**	Sets the fraction equal to a quotient of integers and clears the denominators. Condone sign slips.
	$(2b-a)x = 3a+b \Rightarrow x = \ldots$	**dM1**	Collects the terms in x and divides to make x the subject. Condone sign slips.
	$2b-a=0 \Rightarrow 7b=0$, contradicting $b \neq 0$ $x = \frac{3a+b}{2b-a}$, with $2b-a \neq 0$	**A1**	Correct quotient, oe, with a valid reason the denominator is non-zero. A0 if the division is unjustified.
	$3a+b,\ 2b-a \in \mathbb{Z} \Rightarrow x$ rational, contradicting x irrational, so $\frac{2x-1}{x+3}$ is irrational	**A1 AG**	cso. Needs numerator and denominator stated as integers, the contradiction with x irrational, and a conclusion in words. A0 if the conclusion is missing.
	Total for Question 4	**4**	

Question	Scheme	Marks	Notes
5(a)	$r = \frac{m}{n}$, m, n integers, $m \neq 0$, $n \neq 0$	**B1**	Both integers, and both stated non-zero; $r \neq 0$ stated with $n \neq 0$ is enough.
	Assumes rx rational: $rx = \frac{p}{q}$, $q \neq 0$ $x = \frac{p}{q} \div \frac{m}{n}$	**M1**	Assumes the product rational as a quotient of integers, and divides by r or multiplies by its reciprocal.
	$x = \frac{pn}{qm}$, pn, qm integers, $qm \neq 0$	**A1**	Denominator shown non-zero from both factors non-zero.
	x rational, contradicting x irrational $\Rightarrow$ the product is irrational	**A1 AG**	States the contradiction with x irrational and concludes the product is irrational. A0 if the conclusion is missing.
		[4]	
5(b)	$\sqrt{75} - \sqrt{12} = 5\sqrt{3} - 2\sqrt{3}$	**M1**	Writes both surds as integer multiples of $\sqrt{3}$. Condone one coefficient slip.
	$= 3\sqrt{3}$ 3 non-zero rational, $\sqrt{3}$ irrational $\Rightarrow$ irrational by (a)	**A1 AG**	Needs $3\sqrt{3}$, with 3 stated non-zero rational, part (a) cited and a conclusion.
		[2]	
	Total for Question 5	**6**	

Question	Scheme	Marks	Notes
6(a)	$\sqrt{3}+\sqrt{7}=\frac{a}{b}$, $a,b\in\mathbb{Z}$, $b\neq 0$ $\Rightarrow 10+2\sqrt{21}=\frac{a^2}{b^2}$	**M1**	Assumes $\sqrt{3}+\sqrt{7}$ rational, as a quotient of integers or a rational r, and squares both sides. Condone slips in the expansion.
	$\sqrt{21}=\frac{a^2-10b^2}{2b^2}$	**dM1**	Rearranges to make $\sqrt{21}$ the subject, as a quotient of integers or in terms of their rational. Condone arithmetic slips.
	a^2-10b^2 and $2b^2\neq 0$ are integers so $\sqrt{21}$ is rational, a contradiction so $\sqrt{3}+\sqrt{7}$ is irrational	**A1 AG**	Needs a^2-10b^2 and $2b^2\neq 0$ stated to be integers, or $\frac{r^2-10}{2}$ stated rational, so $\sqrt{21}$ rational, contradicting the given fact, and a conclusion. A0 if the conclusion is missing.
		[3]	
6(b)	$\sqrt{7}-\sqrt{3}=\frac{p}{q}$, $p,q\in\mathbb{Z}$, $q\neq 0$ $\left(\sqrt{7}-\sqrt{3}\right)\left(\sqrt{7}+\sqrt{3}\right)=4$ $\Rightarrow \sqrt{7}+\sqrt{3}=\frac{4}{\sqrt{7}-\sqrt{3}}=\frac{4q}{p}$	**M1**	Assumes $\sqrt{7}-\sqrt{3}$ rational and uses the difference of two squares to write $\sqrt{7}+\sqrt{3}$ in terms of it.
	so $\sqrt{3}+\sqrt{7}$ is rational, contradicting (a) so $\sqrt{7}-\sqrt{3}$ is irrational	**A1 AG**	Correct working; states $\sqrt{3}+\sqrt{7}$ would be rational, contradicting (a), and concludes. Condone $p\neq 0$ not stated.
		[2]	
	Total for Question 6	**5**	

Question	Scheme	Marks	Notes
7(a)	a and b integers, $b \neq 0$	**B1**	Accept positive integers, or b a positive integer, in place of $b \neq 0$.
	a and b have no common factor other than 1	**B1**	Accept highest common factor 1, coprime, or $\frac{a}{b}$ in its lowest terms.
		[2]	
7(b)	Considers n odd: $n = 2m + 1$ $n^3 = 8m^3 + 12m^2 + 6m + 1$	**M1**	Considers odd n, as $2m + 1$ oe, for a contrapositive or contradiction, and expands n^3 to four terms. Condone one coefficient slip.
	$n^3 = 2\left(4m^3 + 6m^2 + 3m\right) + 1$, odd	**A1**	Correct expansion written as $2\times$ an integer $+1$, with n^3 stated odd.
	Contrapositive: n^3 even $\Rightarrow n$ even	**A1 AG**	cso. Concludes the printed statement by the contrapositive, or by contradiction with n^3 assumed even. A0 if that link is missing.
		[3]	
7(c)	$(2k)^3 = 4b^3 \Rightarrow b^3 = \ldots$	**M1**	Substitutes $a = 2k$ into $a^3 = 4b^3$ and simplifies. Condone one arithmetic slip.
	$b^3 = 2k^3 \Rightarrow b^3$ even $\Rightarrow b$ even	**A1**	Needs b even concluded from b^3 even, citing part (b) or the result.
	a, b both even: contradicts no common factor $\Rightarrow \sqrt[3]{4}$ irrational	**A1 AG**	cso. Needs the common factor 2 contradicting the assumption and the conclusion that $\sqrt[3]{4}$ is irrational.
		[3]	
	Total for Question 7	**8**	

Question	Scheme	Marks	Notes
8(a)	$m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ $m^2 = (p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k})^2 = \ldots$	**M1**	Writes m as a product of powers of distinct primes, with general exponents, and squares it.
	$m^2 = p_1^{2n_1} p_2^{2n_2} \cdots p_k^{2n_k}$	**A1**	
	Unique, so this is the factorisation of m^2: each $2n_i$ is even, and any other prime has power 0, also even	**A1 AG**	Cites uniqueness and that every exponent $2n_i$ is even, with a conclusion. Condone primes not dividing m^2 omitted. A0 for an example in place of general exponents.
		[3]	
8(b)	$\sqrt{18} = \frac{a}{b} \Rightarrow a^2 = 18b^2$	**M1**	Assumes $\sqrt{18}$ is a quotient of integers, with any letters, and squares to an equation in integers.
	Power of 2 in a^2 is even, by (a)	**B1**	Applies part (a) to a^2. Condone $a > 1$ not stated.
	$18b^2 = 2 \times 3^2 \times b^2$ $\Rightarrow$ power of 2 is $1 + 2t$	**M1**	Obtains the power of 2 in $18b^2$ as 1 plus the power of 2 in b^2. Condone no letter for that power.
	Odd, including $b = 1$ (power 1)	**A1**	Odd, with the power in b^2 even by part (a) and $b = 1$ covered, separately or as power 0.
	Even = odd for $a^2 = 18b^2$: contradicts uniqueness, so $\sqrt{18}$ is irrational	**A1 AG**	Cites uniqueness of prime factorisation for the contradiction and concludes $\sqrt{18}$ is irrational, with no error in the working.
	Power of 3: even in a^2, 2+ even in $18b^2$, so no contradiction; $18 = 2 \times 3^2$	**E1**	States the power of 3 is even on both sides, so no contradiction, because 3 appears to an even power in 18.
		[6]	
	Total for Question 8	**9**	

Question	Scheme	Marks	Notes
9(a)	$\frac{p^3}{q^3} - \frac{3p}{q} - 1 = 0$ $\Rightarrow p^3 - 3pq^2 - q^3 = 0$	**M1**	Substitutes $x = \frac{p}{q}$ and multiplies through by q^3 to clear the fractions. Condone sign slips.
	$p^3 = q\left(3pq + q^2\right)$ $q^3 = p\left(p^2 - 3q^2\right)$	**A1**	Both, or each other term shown a multiple of q (or p), with both divisibilities stated.
	No common prime factor, so $q \mid p^3 \Rightarrow q = 1$ $p \mid q^3 = 1 \Rightarrow p = \pm 1$	**dM1**	Uses the lowest terms with the divisibilities to deduce $q = 1$ and $p = \pm 1$, in either order. Condone $q = 1$ asserted from no common factor.
	$x = 1 : 1 - 3 - 1 = -3 \neq 0$ $x = -1 : -1 + 3 - 1 = 1 \neq 0$ No rational roots	**A1 AG**	Both $x = 1$ and $x = -1$ shown not to satisfy $(*)$, and the conclusion that $(*)$ has no rational roots.
		[4]	
9(b)	$\cos 60° = 4\cos^3 20° - 3\cos 20°$ $\Rightarrow 4\cos^3 20° - 3\cos 20° = \frac{1}{2}$	**M1**	Puts $\theta = 20°$ in the identity and uses $\cos 60° = \frac{1}{2}$, soi, at any stage of the working.
	$x^3 - 3x - 1 = 8\cos^3 20° - 6\cos 20° - 1$ $= 2 \times$ "$\frac{1}{2}$" $- 1$	**dM1**	Substitutes $x = 2\cos 20°$ into the cubic and replaces $4\cos^3 20° - 3\cos 20°$ by their value. Condone a coefficient slip.
	$= 0$, so $2\cos 20°$ is a root	**A1**	cso, with a statement that $2\cos 20°$ is a root.
	Assume $\cos 20° = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$ $\Rightarrow 2\cos 20° = \frac{2m}{n}$, a rational root Contradicts (a), so $\cos 20°$ is irrational	**M1** **A1 AG**	**M1** Assumes $\cos 20°$ rational and deduces that $2\cos 20°$ is a rational root of $(*)$. Writing $\frac{m}{n}$ is not needed. **A1 AG** The contradiction with (a) stated, and the conclusion that $\cos 20°$ is irrational.
		[5]	
	Total for Question 9	**9**	

Question	Scheme	Marks	Notes
10(a)	$(2q+1)(2q+2)\cdots(2q+2k) > (2q+1)^{2k}$ $\Rightarrow S < \frac{1}{(2q+1)^2} + \frac{1}{(2q+1)^4} + \cdots$	**M1**	Uses a lower bound on the denominators of S to compare S term by term with a geometric series, such as one with ratio $\frac{1}{(2q+1)^2}$.
	$\dfrac{\frac{1}{(2q+1)^2}}{1-\frac{1}{(2q+1)^2}}$	**dM1**	Uses the sum to infinity formula with the first term and common ratio of their geometric series.
	$= \frac{1}{(2q+1)^2-1} = \frac{1}{4q(q+1)}$ $\Rightarrow S < \frac{1}{4q(q+1)}$	**A1 AG**	Sum simplified without error to $\frac{1}{4q(q+1)}$, or to a value shown at most this, and $S < \frac{1}{4q(q+1)}$ concluded.
		[3]	
10(b)	$S > 0$, $q \geq 1 \Rightarrow 4q(q+1) \geq 8$	**M1**	States $S > 0$ and uses $q \geq 1$ to show $\frac{1}{4q(q+1)} \leq 1$, oe such as $\leq \frac{1}{8}$.
	$0 < S < \frac{1}{8} < 1 \Rightarrow S \notin \mathbb{Z}$	**A1 AG**	Needs $0 < S < 1$ and the conclusion that no integer lies strictly between 0 and 1.
		[2]	
10(c)	$(2q)!\cosh 1 = \frac{(2q)!}{q} \times p$ $= 2(2q-1)!\,p \in \mathbb{Z}$	**B1**	Integer, with a reason such as q a factor of $(2q)!$. Not for the assertion alone.
	$(2q)!\cosh 1 = \sum_{r=0}^{q} \frac{(2q)!}{(2r)!}$ $+ \sum_{r=q+1}^{\infty} \frac{(2q)!}{(2r)!}$	**M1**	Multiplies the series by $(2q)!$ and separates the terms with $r \leq q$ from those with $r > q$, in sigma notation or written out.
	$\sum_{r=0}^{q} \frac{(2q)!}{(2r)!} = N \in \mathbb{Z}$	**A1**	First sum an integer, with a reason such as $(2r)!$ dividing $(2q)!$ for $r \leq q$.
	$\sum_{r=q+1}^{\infty} \frac{(2q)!}{(2r)!} = \frac{1}{(2q+1)(2q+2)} + \cdots$ $= S$ $S = (2q)!\cosh 1 - N \in \mathbb{Z}$ Contradicts (b), so $\cosh 1$ is irrational	**A1 AG**	Second sum shown to be S, S an integer as a difference of integers, the contradiction with part (b), and the conclusion that $\cosh 1$ is irrational.
		[4]	
	Total for Question 10	**9**	