OPTIONAL PURE / GROUPS

Binary Operations and Group Axioms

Questions by topic 11 questions · 123 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

Question 1	*9 marks*	2
Question 2	*5 marks*	3
Question 3	*5 marks*	4
Question 4	*13 marks*	5
Question 5	*12 marks*	6
Question 6	*8 marks*	7
Question 7	*16 marks*	8
Question 8	*9 marks*	9
Question 9	*14 marks*	10
Question 10	*14 marks*	11
Question 11	*18 marks*	12

1. A binary operation $\circ$ is defined on the set $S = \{x \in \mathbb{R} \mid x \geq 4\}$ by

$$a \circ b = a + b - 4 \quad \text{for all } a, b \in S$$

(a) Prove that $\circ$ is associative on S. **[2]**

(b) Find the identity element e for $\circ$, and verify that $e \in S$. **[2]**

(c) Show that S is closed under $\circ$. **[2]**

(d) Find, in terms of a, the real number x such that $a \circ x = e$. Determine which elements of S have an inverse in S, and hence explain why S does **not** form a group under $\circ$. **[3]**

2. A binary operation $\circ$ is defined on the set of positive real numbers by

$$a \circ b = \frac{ab}{a+b}$$

Show that the set of positive real numbers is closed under $\circ$, and prove that $\circ$ is associative. **[5]**

3. Let G be a group under the binary operation $*$, with identity e. For $x \in G$, write x^{-1} for the inverse of x and x^2 for $x * x$.

(a) Prove that if $a * b = a * c$ for some $a, b, c \in G$, then $b = c$, stating which group axiom you use at each step. **[2]**

(b) Hence, or otherwise, prove that if $(a*b)^2 = a^2 * b^2$ for all $a, b \in G$, then G is abelian (commutative). **[3]**

4. The set S consists of the functions f_n, defined for all real x by $f_n(x) = 2^n x + 2^n - 1$, one for each integer n. The binary operation on S is composition of functions, where $f_m f_n$ denotes the composite function $f_m \circ f_n$, so that $f_m f_n(x) = f_m(f_n(x))$.

(a) Show that $f_m f_n = f_{m+n}$ for all integers m and n. **[2]**

(b) Hence show that the binary operation is associative on S. **[2]**

(c) Prove that S is a group under composition of functions. **[6]**

(d) The subset T of S consists of the functions f_n for which $n \geq 0$. Show that T is closed under composition of functions, but that T is not a subgroup of S. **[3]**

5. The set C consists of all complex numbers excluding 0 and 1.
The binary operation $\oplus$ is defined on C by

$$a \oplus b = \frac{ab}{1 - a - b + 2ab}$$

where $a, b \in C$.

(a) Find the identity element of C under $\oplus$. **[2]**

(b) Show that every element x of C has an inverse in C under $\oplus$, giving this inverse in terms of x. **[2]**

(c) Show that $\oplus$ is associative on C. **[3]**

(d) Explain why $(C, \oplus)$ is not a group. **[1]**

(e) The set D is defined by $D = \{x \in \mathbb{R} \mid 0 < x < 1\}$.
Show that $(D, \oplus)$ is a group. **[4]**

6. The elements x and y belong to a group G, whose binary operation is written as multiplication and whose identity is e.
The inverse of an element g of G is written g^{-1}.

(a) Prove that $(xy)^{-1} = y^{-1}x^{-1}$. **[2]**

(b) Given that $g^2 = e$ for every element g of G, use part (a) to prove that G is abelian (commutative). **[3]**

(c) Suppose instead only that $x^2 = e$ and $y^2 = e$. Does it follow that $xy = yx$? Justify your answer by considering the group of non-singular 2×2 matrices with real entries under matrix multiplication. **[3]**

7. The set G is given by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R},\ b \in \mathbb{R},\ a \neq 0 \right\}$$

(a) Show that G forms a group under matrix multiplication, $\times$. You may assume that matrix multiplication is associative. **[5]**

(b) The set S is defined by

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \{-1, 1\},\ b \in \mathbb{Z} \right\}$$

(i) Determine whether S is closed under $\times$. **[2]**

(ii) Determine whether S is a subgroup of $(G, \times)$, justifying your answer. **[2]**

(c) (i) By giving a suitable example, show that $\times$ is not commutative on G. **[2]**

(ii) Show that the identity matrix is the only element of G that commutes with every element of G. **[3]**

(d) With the help of an example, explain why G does **not** form a group under matrix addition. **[2]**

8. Let M be the set of all 2×2 matrices whose entries are integers. Matrix multiplication is a binary operation on M.
The matrices $\mathbf{A}$ and $\mathbf{B}$ in M are given by

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

(a) Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, and hence show that M does not form a group under matrix multiplication. **[3]**

(b) State the largest subset of M which forms a group under matrix multiplication, and show that it is a group. **[6]**

9. The binary operation $*$ is defined on the set of ordered pairs (x, y), where $x, y \in \mathbb{R}$, by

$$(a, b) * (c, d) = \left(ac,\ ad + bc^3\right)$$

(a) Prove that $*$ is associative. **[3]**

(b) Find the identity element for $*$, showing that it is an identity on both sides. **[2]**

(c) Find all the elements which commute with $(-1, 1)$. **[3]**

(d) It is given that the particular element (m, n) has an inverse. Find this inverse in terms of m and n. **[2]**

(e) Find all self-inverse elements. **[3]**

(f) Give a reason why the set of ordered pairs (x, y), under the operation $*$, does not form a group. **[1]**

10. For rational numbers a and b, the matrix $M(a, b)$ is defined by

$$M(a,b) = \begin{pmatrix} a & 2b \\ b & a+2b \end{pmatrix}$$

(a) Given that $\sqrt{3}$ is an irrational number, prove that $\det M(a, b) \neq 0$ for all $a, b \in \mathbb{Q}$ where a and b are not both 0. **[3]**

A set G is defined by $G = \{M(a, b) \mid a, b \in \mathbb{Q},\ a \text{ and } b \text{ not both } 0\}$.

(b) Prove that G is a group under matrix multiplication. You may assume that matrix multiplication is associative. **[7]**

(c) A subset H of G is defined by $H = \{M(a, b) \mid a, b \in \mathbb{Z},\ a \text{ and } b \text{ not both } 0\}$. Determine whether or not H is a subgroup of $(G, \times)$, where $\times$ denotes matrix multiplication. **[2]**

(d) Using $(G, \times)$, prove by counter-example that the statement "Every subgroup of an infinite group, other than the subgroup containing only the identity, is itself infinite" is false. **[2]**

11. A binary operation, $\circ$, is defined on a set of numbers, A, in terms of constants k_1, k_2, k_3 and k_4 (which are not necessarily in A) by

$$a \circ b = k_1ab + k_2a + k_3b + k_4, \qquad \text{for } a, b \in A$$

Addition, subtraction and multiplication of numbers have their usual notation and meaning.
You are initially given the following information about $\circ$ and A.

- $A = \{x \in \mathbb{R} \mid x \neq 2\}$
- $0 \circ 0 = 10$ and $1 \circ 0 = 6$
- An identity element, e, exists for $\circ$ in A

(a) Show that $a \circ b = 2ab - 4a - 4b + 10$. **[6]**

(b) State the value of e. **[1]**

(c) Explain whether $\circ$ is commutative over A. **[1]**

(d) Determine whether or not $(A, \circ)$ is a group. **[7]**

(e) Explain whether your answer to part (d) would change in each of the following cases, giving details of any change.

(i) $A = \{x \in \mathbb{R} \mid x > 2\}$ **[1]**

(ii) $A = \{n \in \mathbb{Z} \mid n \neq 2\}$ **[1]**

(iii) $A = \{x \in \mathbb{R} \mid x < 2\}$ **[1]**