OPTIONAL PURE / GROUPS

Binary Operations and Group Axioms

Mark scheme 11 questions · 123 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · OCR MEI Extra Pure Y435

Questions

Question	Marks by part	Marks	Page
Question 1	(a) 2 (b) 2 (c) 2 (d) 3	9	4
Question 2		5	5
Question 3	(a) 2 (b) 3	5	6
Question 4	(a) 2 (b) 2 (c) 6 (d) 3	13	7
Question 5	(a) 2 (b) 2 (c) 3 (d) 1 (e) 4	12	9
Question 6	(a) 2 (b) 3 (c) 3	8	11
Question 7	(a) 5 (b)(i) 2 (b)(ii) 2 (c)(i) 2 (c)(ii) 3 (d) 2	16	12
Question 8	(a) 3 (b) 6	9	14
Question 9	(a) 3 (b) 2 (c) 3 (d) 2 (e) 3 (f) 1	14	15
Question 10	(a) 3 (b) 7 (c) 2 (d) 2	14	17
Question 11	(a) 6 (b) 1 (c) 1 (d) 7 (e)(i) 1 (e)(ii) 1 (e)(iii) 1	18	19
	Total	**123**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- **[*n*]** ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$(a \circ b) \circ c = (a + b - 4) + c - 4$ $a \circ (b \circ c) = a + (b + c - 4) - 4$	**M1**	Uses the definition twice in each of $(a \circ b) \circ c$ and $a \circ (b \circ c)$. Condone one slip.
	Both $= a + b + c - 8$, so $\circ$ is associative	**A1 AG**	Both bracketings reach $a + b + c - 8$ without error, with a conclusion that $\circ$ is associative.
		[2]	
1(b)	$a + e - 4 = a \Rightarrow e = \ldots$	**M1**	Uses $a \circ e = a$ or $e \circ a = a$ and solves for e.
	$e = 4$, and $4 \geq 4$ so $e \in S$	**A1**	Needs $4 \geq 4$, or $4 \in S$, stated.
		[2]	
1(c)	$a \geq 4,\ b \geq 4 \Rightarrow a + b - 4 \geq 4 + 4 - 4$	**M1**	Uses $a \geq 4$ and $b \geq 4$ to bound $a + b - 4$ below. Not for a numerical example alone.
	$a \circ b \geq 4$, so $a \circ b \in S$ and S is closed	**A1 AG**	Correct inequality reaching $a \circ b \geq 4$ without error, with a conclusion that S is closed.
		[2]	
1(d)	$a + x - 4 =$ "4" $\Rightarrow x = \ldots$	**M1**	Forms $a \circ x =$ their e from (b) and solves for x. Condone sign slips.
	$x = 8 - a$	**A1**	
	$8 - a \geq 4 \Rightarrow a \leq 4$ Only $a = 4$ has an inverse in S Inverse axiom fails, so not a group	**E1**	States that only $a = 4$ has an inverse in S and that S is not a group because the inverse axiom fails.
		[3]	
	Total for Question 1	**9**	

Question	Scheme	Marks	Notes
2	$a, b > 0 \Rightarrow ab > 0$ and $a + b > 0$	**M1**	Uses positivity of both the numerator and the denominator for positive a and b.
	$a \circ b = \frac{ab}{a+b} > 0$, so closed	**A1**	$a \circ b$ positive and real, with a conclusion that the set is closed.
	$(a \circ b) \circ c = \frac{\frac{ab}{a+b} c}{\frac{ab}{a+b} + c}$ $= \frac{abc}{ab+c(a+b)}$	**M1**	Substitutes $a \circ b$ or $b \circ c$ into the operation and clears the inner fraction to a single fraction. Condone slips.
	$= \frac{abc}{ab+bc+ca}$	**A1**	oe, for either bracketing.
	$a \circ (b \circ c) = \frac{a \frac{bc}{b+c}}{a + \frac{bc}{b+c}}$ $= \frac{abc}{ab+bc+ca}$ $\Rightarrow (a \circ b) \circ c = a \circ (b \circ c)$, so associative	**A1 AG**	Both bracketings reduced without error to the same expression, with a conclusion that $\circ$ is associative.
	Total for Question 2	**5**	

Question	Scheme	Marks	Notes
3(a)	$a^{-1} * (a * b) = a^{-1} * (a * c)$ (inverse)	**M1**	Combines both sides of $a * b = a * c$ on the left with a^{-1}.
	$(a^{-1} * a) * b = (a^{-1} * a) * c$ (associativity) $e * b = e * c$ (inverse) $b = c$ (identity)	**A1 AG**	Reaches $b = c$ without error, naming inverse, associativity and identity where each is used. Combined steps accepted if each axiom is named.
		[2]	
3(b)	$(a * b) * (a * b) = (a * a) * (b * b)$ $\Rightarrow a * ((b * a) * b) = a * ((a * b) * b)$	**M1**	Writes both squares as products of four elements and uses associativity to regroup or drop brackets.
	$(b * a) * b = (a * b) * b$ $\Rightarrow ((b * a) * b) * b^{-1} = ((a * b) * b) * b^{-1}$	**dM1**	Cancels a on the left, by part (a) or with a^{-1}, and b on the right, with b^{-1} or by right cancellation.
	$(b * a) * e = (a * b) * e \Rightarrow b * a = a * b$ $b * a = a * b$ for all $a, b \in G$, so G is abelian	**A1 AG**	Correct expanded equation leading without error to $b * a = a * b$, concluding that it holds for all $a, b \in G$ so G is abelian.
		[3]	
	Total for Question 3	**5**	

Question	Scheme	Marks	Notes
4(a)	$f_m f_n(x) = 2^m(2^n x + 2^n - 1) + 2^m - 1$ $= 2^{m+n}x + 2^{m+n} - 2^m + 2^m - 1$	**M1**	Substitutes $f_n(x)$ for x in f_m and expands the bracket. Condone a sign slip.
	$f_m f_n(x) = 2^{m+n}x + 2^{m+n} - 1 = f_{m+n}(x)$	**A1 AG**	Correct substituted expression simplified without error to $2^{m+n}x + 2^{m+n} - 1$, identified as $f_{m+n}(x)$ or f_{m+n}.
		[2]	
4(b)	$(f_p f_q)f_r = f_{p+q}f_r = f_{(p+q)+r}$ $f_p(f_q f_r) = f_p f_{q+r} = f_{p+(q+r)}$	**M1**	Applies part (a) twice to each bracketing of three general elements of S. Not for particular suffixes.
	$(p+q)+r = p+(q+r) \Rightarrow (f_p f_q)f_r = f_p(f_q f_r)$	**A1 AG**	Both bracketings shown equal, as f_{p+q+r} oe, and a conclusion in words. Associativity of integer addition need not be named.
		[2]	
4(c)	Closure: $f_m f_n = f_{m+n}$	**M1**	Uses part (a) to write the composite of two general elements as a single function with combined suffix.
	$m+n \in \mathbb{Z} \Rightarrow f_{m+n} \in S$	**A1**	Needs $m+n$ stated to be an integer.
	Identity: $f_0(x) = 2^0 x + 2^0 - 1 = x$	**M1**	Identifies f_0 as the identity, by showing $f_0(x) = x$ or by using part (a) to show $f_0 f_n = f_n$.
	$0 \in \mathbb{Z}$, $f_0 f_n = f_n f_0 = f_n$	**A1**	Needs $f_0 \in S$ and f_0 acting as the identity on both sides, shown with part (a) or stated for the identity function.
	Inverses: $f_n f_{-n} = f_{-n} f_n = f_0$	**M1**	Finds an element of S composing with a general f_n to give f_0, using part (a) or by inverting $2^n x + 2^n - 1$. Condone slips.
	$-n \in \mathbb{Z} \Rightarrow f_{-n} \in S$ Associativity from part (b), so S is a group	**A1 AG**	Needs $-n$ stated to be an integer, associativity cited from part (b), and a conclusion that S is a group. Axioms in any order.
		[6]	
4(d)	$m, n \geq 0 \Rightarrow m+n \geq 0 \Rightarrow f_m f_n = f_{m+n} \in T$	**B1**	Needs $m+n \geq 0$, with part (a), and a conclusion that T is closed.

Question 4 continued

Question	Scheme	Marks	Notes
	$f_1 \in T$, $f_1 f_k = f_{1+k}$ for $k \geq 0$	**M1**	Considers the inverse of a particular or general f_n with $n \geq 1$, by composing with a general element of T or finding its inverse f_{-n} in S.
	$f_{1+k}(0) = 2^{1+k} - 1 \geq 1$, $f_0(0) = 0$ $\Rightarrow f_1$ has no inverse in T, so T is not a subgroup	**A1 AG**	Shows no element of T composes with it to give f_0, or that $f_{-n} \notin T$ as $-n < 0$, and concludes T is not a subgroup.
		[3]	
	Total for Question 4	**13**	

Question	Scheme	Marks	Notes
5(a)	$\frac{ae}{1-a-e+2ae} = a$ $\Rightarrow 2e(1-a) = 1-a \Rightarrow e = \ldots$	**M1**	Forms $a \oplus e = a$, or $e \oplus a = a$, clears the fraction and solves for e. Condone slips in rearranging.
	$e = \frac{1}{2}$	**A1**	
		[2]	
5(b)	$\frac{xy}{1-x-y+2xy} =$ "$\frac{1}{2}$" $\Rightarrow y = \ldots$	**M1**	Forms $x \oplus y$ equal to their identity, clears the fraction and solves for y. Condone slips in rearranging.
	$y = 1 - x$ $x \neq 0, 1 \Rightarrow 1 - x \neq 1, 0$, so $1 - x \in C$	**A1 AG**	Needs $1 - x \neq 0$ and $1 - x \neq 1$ stated. A check of the denominator or of commutativity is not required.
		[2]	
5(c)	$(a \oplus b) \oplus c = \frac{abc}{(1-a-b+2ab)(1-c)-ab+2abc}$	**M1**	Substitutes $a \oplus b$ into one bracketing, either way round, and clears the inner fraction to a single fraction. Condone sign slips.
	$= \frac{abc}{1-a-b-c+ab+bc+ca}$	**A1**	Denominator expanded and simplified, oe.
	$a \oplus (b \oplus c) = \frac{abc}{(1-b-c+2bc)(1-a)-bc+2abc}$ $= \frac{abc}{1-a-b-c+ab+bc+ca}$ Equal, so $\oplus$ is associative	**A1 AG**	Other bracketing reached without error from a correct substituted form, with a conclusion. Accept commutativity with the symmetry of the expression in a, b, c.
		[3]	
5(d)	Not closed: $2, \frac{1}{3} \in C$, but the denominator of $2 \oplus \frac{1}{3}$ is $1 - 2 - \frac{1}{3} + \frac{4}{3} = 0$	**E1**	States C is not closed, or $\oplus$ undefined for some pairs, with a pair in C, specific or general, making the denominator zero.
		[1]	
5(e)	$1 - a - b + 2ab = (1-a)(1-b) + ab > ab > 0$	**M1**	Attempts to bound $a \oplus b$ for $0 < a < 1$ and $0 < b < 1$, such as by writing the denominator as $(1-a)(1-b) + ab$. Not for numerical examples alone.
	$0 < a \oplus b < 1$, so D is closed	**A1**	Needs both bounds and the conclusion of closure.
	Identity $\frac{1}{2} \in D$, since $0 < \frac{1}{2} < 1$	**B1**	

Question 5 continued

Question	Scheme	Marks	Notes
	$0 < x < 1 \Rightarrow 0 < 1 - x < 1$, so $1 - x \in D$ Associative by part (c), as $D \subseteq C$ All four axioms hold, so a group	**A1 AG**	Needs inverse $1 - x$ shown in D, associativity from C, and a conclusion. A0 if any axiom is omitted.
		[4]	
	Total for Question 5	**12**	

Question	Scheme	Marks	Notes
6(a)	$(xy)z = e \Rightarrow x(yz) = e$ $\Rightarrow (x^{-1}x)(yz) = x^{-1} \Rightarrow yz = \ldots$	**M1**	Uses associativity to premultiply $(xy)z = e$ by x^{-1}, or forms $(xy)(y^{-1}x^{-1})$ and regroups it. Condone brackets not shown at one step.
	$z = y^{-1}x^{-1}$, so $(xy)^{-1} = y^{-1}x^{-1}$	**A1 AG**	Printed result reached without error, using associativity and the identity, with no elements commuted. By verification, needs $(xy)(y^{-1}x^{-1}) = e$ shown.
		[2]	
6(b)	$g^2 = e \Rightarrow g^{-1} = g$ for every $g \in G$	**B1**	
	$xy = (xy)^{-1} = y^{-1}x^{-1}$	**M1**	Applies self-inverse to xy and uses part (a). Not for a route that avoids part (a).
	$= yx$, so G is abelian	**A1 AG**	Uses $x^{-1} = x$ and $y^{-1} = y$ without error, and concludes G is abelian for arbitrary x, y.
		[3]	
6(c)	$x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $x^2 = y^2 = \mathbf{I}$	**B1**	Any two real 2×2 matrices, each shown to square to $\mathbf{I}$.
	$xy = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \ldots$ $yx = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \ldots$	**M1**	Multiplies their two matrices in both orders. Condone arithmetic slips.
	$xy = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = yx$ No, it does not follow	**A1**	Both products correct for a valid pair and shown unequal, with a conclusion that it does not follow.
		[3]	
	Total for Question 6	**8**	

Question	Scheme	Marks	Notes
7(a)	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \ldots$	**M1**	Multiplies two general elements of G, with different letters for each. Condone one slip in an entry.
	$= \begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix}$ $a, c \neq 0 \Rightarrow ac \neq 0$, so closed	**A1**	Correct product, with $ac \neq 0$ justified from $a \neq 0$ and $c \neq 0$.
	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G \;\; (a = 1,\; b = 0)$	**B1**	
	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$	**M1**	Finds the inverse of a general element, by the 2×2 inverse formula or by solving a product equal to **I**. Condone one slip.
	$= \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \in G$ as $\frac{1}{a} \neq 0$ Associativity given, so G is a group	**A1 AG**	Correct inverse shown in G, associativity cited and G concluded a group, with no errors in the part.
		[5]	
7(b)(i)	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix}$	**M1**	Forms the product of two general elements of S, or uses the product from (a).
	$ac = \pm 1$, $ad + b \in \mathbb{Z}$, so S is closed	**A1**	Both entries justified and closure concluded. Not for numerical examples alone.
		[2]	
7(b)(ii)	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}$	**M1**	Finds the inverse of a general element of S, by the formula or from (a).
	Closed by (b)(i), $\mathbf{I} \in S$ Inverse $\begin{pmatrix} a & -ab \\ 0 & 1 \end{pmatrix} \in S$, as $\frac{1}{a} = a \Rightarrow$ subgroup	**A1**	Closure, identity in S and inverse shown in S, with a conclusion. Associativity need not be mentioned.
		[2]	
7(c)(i)	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\mathbf{AB} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	**M1**	Multiplies any two elements of G in both orders. Condone one slip.
	$\mathbf{AB} \neq \mathbf{BA}$, so not commutative	**A1 AG**	Both products correct for two elements of G, with a statement that they differ, so $\times$ is not commutative.
		[2]	

Question 7 continued

Question	Scheme	Marks	Notes
7(c)(ii)	$\begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & bc+d \\ 0 & 1 \end{pmatrix}$	**M1**	Equates both products of a general element with a general element of G, or with chosen elements. Condone one slip.
	$ad + b = bc + d$	**A1**	oe, including correct conditions from chosen elements, such as $a + b = b + 1$.
	$c = 1,\ d = 1 \Rightarrow a = 1$; $c = 2 \Rightarrow b = 0$ **I** commutes with all, so only **I**	**A1 AG**	Valid choices with $c \neq 0$ leading without error to $a = 1$ and $b = 0$, and **I** stated to commute with every element.
		[3]	
7(d)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \notin G$	**B1**	Any correct sum of two elements of G, shown not in G. Or the zero matrix shown not in G.
	Not closed under addition, so not a group	**E1**	States G is not closed under addition, or has no additive identity, so is not a group, referring to their example.
		[2]	
	Total for Question 7	**16**	

Question	Scheme	Marks	Notes
8(a)	$\det(\mathbf{A}) = 1, \quad \det(\mathbf{B}) = 2$ $\mathbf{B}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$	**M1**	Uses the inverse formula, swapping the leading diagonal, negating the other entries and dividing by the determinant, for at least one matrix. Condone one sign slip. Correct inverses written down imply the mark.
	$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$ $\mathbf{B}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$	**A1**	Both inverses correct, oe.
	$-\frac{1}{2}, \frac{3}{2} \notin \mathbb{Z} \Rightarrow \mathbf{B}^{-1} \notin M$ $\mathbf{B}$ has no inverse in M, so M is not a group	**A1 AG**	Correct $\mathbf{B}^{-1}$ with a non-integer entry identified, concluding $\mathbf{B}$ has no inverse in M, so M is not a group.
		[3]	
8(b)	$G = \{\mathbf{X} \in M \mid \det(\mathbf{X}) = \pm 1\}$	**B1**	Accept in words. Not for $\det(\mathbf{X}) = 1$ alone or a nonzero determinant. No justification of largest needed.
	Closure: $\det(\mathbf{XY}) = \det(\mathbf{X})\det(\mathbf{Y})$	**M1**	Uses the determinant of a product for two general elements of G. Not for numerical examples only.
	$= (\pm 1)(\pm 1) = \pm 1$ Integer entries, so $\mathbf{XY} \in G$	**A1**	Product shown to have integer entries and determinant ± 1, concluding it is in G.
	Identity: $\mathbf{I}$, $\det(\mathbf{I}) = 1 \Rightarrow \mathbf{I} \in G$	**B1**	Identity named as $\mathbf{I}$, with $\mathbf{I} \in G$ stated.
	Inverses: $\mathbf{X} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $ps - qr = \pm 1$ $\Rightarrow \mathbf{X}^{-1} = \frac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$	**M1**	Applies the inverse formula to a general element with determinant ± 1. Not for numerical examples only.
	$\mathbf{X}^{-1} = \pm\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$, $\det(\mathbf{X}^{-1}) = \pm 1$ $\Rightarrow \mathbf{X}^{-1} \in G$ Associative, so G is a group	**A1 AG**	$\mathbf{X}^{-1}$ shown to have integer entries and determinant ± 1, associativity stated, and a conclusion, with no errors.
		[6]	
	Total for Question 8	**9**	

Question	Scheme	Marks	Notes
9(a)	$\big((a,b)*(c,d)\big)*(e,f) = \big(ac,\ ad+bc^3\big)*(e,f)$ $= \big(ace,\ acf+(ad+bc^3)e^3\big)$	**M1**	Applies the definition twice to either bracketing of three general elements. Condone slips.
	$= \big(ace,\ acf+ade^3+bc^3e^3\big)$	**A1**	Correct expanded pair for either bracketing.
	$(a,b)*\big((c,d)*(e,f)\big) = (a,b)*\big(ce,\ cf+de^3\big)$ $= \big(ace,\ acf+ade^3+bc^3e^3\big)$ Same pair $\Rightarrow *$ is associative	**A1 AG**	Both bracketings expanded without error to the same pair, with a conclusion that $*$ is associative.
		[3]	
9(b)	$(x,y)*(u,v) = \big(xu,\ xv+yu^3\big) = (x,y)$ $\Rightarrow u = \ldots,\ v = \ldots$	**M1**	Combines a general element with (u,v) on either side and equates components, or tests a stated candidate on one side. Condone slips.
	$(1,0)$ $(x,y)*(1,0) = (x,\ 0+y) = (x,y)$ $(1,0)*(x,y) = (x,\ y+0) = (x,y)$	**A1**	$(1,0)$, with both products shown equal to a general (x,y). A0 if only one side is checked.
		[2]	
9(c)	$(x,y)*(-1,1) = (-x,\ x-y)$ $(-1,1)*(x,y) = \big(-x,\ x^3-y\big)$ $\Rightarrow x-y = x^3-y$	**M1**	Forms both products of (x,y) with $(-1,1)$ and equates them, or their second components. Condone sign slips.
	$x^3-x=0$	**A1**	Correct equation in x alone, oe.
	$x = 0,\ 1$ or -1, with y any real number	**A1**	All three values and no others, with y stated as unrestricted. Accept the elements listed, such as $(0,y)$, $(1,y)$, $(-1,y)$.
		[3]	
9(d)	$(m,n)*(p,q) = \big(mp,\ mq+np^3\big) =$ "$(1,0)$" $\Rightarrow p = \ldots,\ q = \ldots$	**M1**	Sets (m,n) combined with (p,q), on either side, equal to their identity from (b), and solves for both components. Condone slips.
	$\big(\frac{1}{m},\ -\frac{n}{m^4}\big)$	**A1**	oe, such as $\big(m^{-1},\ -nm^{-4}\big)$. No check on the other side is needed.
		[2]	
9(e)	$(x,y)*(x,y) = \big(x^2,\ xy+yx^3\big) =$ "$(1,0)$"	**M1**	Sets $(x,y)*(x,y)$ equal to their identity, or sets (x,y) equal to their inverse from (d). Condone slips.

Question 9 continued

Question	Scheme	Marks	Notes
	$x^2 = 1 \Rightarrow x = \pm 1$ $y\left(x + x^3\right) = 0$	**A1**	$x = \pm 1$ and a correct second-component equation, oe.
	$(1, 0)$ and $(-1, 0)$	**A1**	Both elements and no others.
		[3]	
9(f)	$(0, y) * (p, q) = \left(0,\ yp^3\right) \neq (1, 0)$ $\Rightarrow (0, y)$ has no inverse	**E1**	States that an element with first component 0, such as $(0, 0)$, has no inverse. Not for "not every element has an inverse" alone.
		[1]	
	Total for Question 9	**14**	

Question	Scheme	Marks	Notes
10(a)	$\det M(a,b) = a(a+2b) - 2b^2 = 0$	**M1**	Forms $a(a+2b) - 2b \times b$ and sets it equal to 0, by contradiction or to solve. Condone sign slips.
	$(a+b)^2 = 3b^2$	**A1**	oe, such as $a = \left(-1 \pm \sqrt{3}\right) b$ from solving as a quadratic in a.
	$b = 0 \Rightarrow a = 0$, not allowed $b \neq 0 \Rightarrow \sqrt{3} = \pm\frac{a+b}{b} \in \mathbb{Q}$, contradiction	**A1 AG**	cso. Needs $b = 0$ ruled out, $\sqrt{3}$ shown rational otherwise, and the contradiction or conclusion stated.
		[3]	
10(b)	$M(a,b)M(c,d)$ $= \begin{pmatrix} ac+2bd & 2ad+2bc+4bd \\ ad+bc+2bd & ac+2ad+2bc+6bd \end{pmatrix}$	**M1**	Multiplies two general elements of G with different parameters, all four entries attempted. Condone slips.
	$= M(p,q)$ $p = ac+2bd, \quad q = ad+bc+2bd$	**A1**	Correct p and q, with the top-right entry $2q$ and bottom-right entry $p+2q$ checked.
	$p, q \in \mathbb{Q}$ $\det M(p,q) = \det M(a,b) \det M(c,d) \neq 0$ $\Rightarrow p, q$ not both 0, so closed	**A1**	Both p, q rational and not both 0, using part (a) or otherwise, and closure concluded.
	$M(1,0) = \mathbf{I} \in G$	**B1**	
	$M(a,b)^{-1} = \frac{1}{a^2+2ab-2b^2} \begin{pmatrix} a+2b & -2b \\ -b & a \end{pmatrix}$	**M1**	Uses the 2×2 inverse formula on a general element, dividing by their determinant. Condone sign slips.
	$= M(x,y)$ $x = \frac{a+2b}{a^2+2ab-2b^2}, \quad y = -\frac{b}{a^2+2ab-2b^2}$	**A1**	Correct x and y oe, with the top-right entry $2y$ and bottom-right entry $x+2y$ checked.
	$x, y \in \mathbb{Q}$, $\det M(x,y) = \frac{1}{a^2+2ab-2b^2} \neq 0$ Associativity given, so G is a group	**A1 AG**	cso. Needs x, y rational and not both 0 with a reason, associativity stated, and G concluded a group.
		[7]	
10(c)	$M(2,0) = 2\mathbf{I} \in H$ $M(2,0)^{-1} = \frac{1}{2}\mathbf{I} = M\left(\frac{1}{2}, 0\right)$	**M1**	Finds the inverse of a specific or general element of H and tests whether its entries are integers. Condone slips.
	$\frac{1}{2} \notin \mathbb{Z} \Rightarrow M(2,0)^{-1} \notin H$, not a subgroup	**A1**	Correct inverse, a non-integer entry identified, and H concluded not a subgroup.
		[2]	

Question 10 continued

Question	Scheme	Marks	Notes
10(d)	$\{\mathbf{I}, -\mathbf{I}\}$, $-\mathbf{I} = M(-1, 0) \in G$	**B1**	
	$(-\mathbf{I})^2 = \mathbf{I}$: closed, identity, self-inverse G infinite, subgroup of order 2, so false	**E1**	Shows closure, identity and inverses for their subgroup, notes G is infinite, and concludes the statement is false.
		[2]	
	Total for Question 10	**14**	

Question	Scheme	Marks	Notes
11(a)	$0 \circ 0 = k_4 = 10$	**B1**	
	$1 \circ 0 = k_2 + k_4 = 6 \Rightarrow k_2 = -4$	**B1**	
	$e \circ a = (k_1 e + k_3)a + 10 - 4e = a$ $\Rightarrow k_1 e + k_3 = 1,\ 10 - 4e = 0 \Rightarrow e = \ldots$	**M1**	Uses $e \circ a = a$ or $a \circ e = a$ for all a, equating coefficients of a or constant terms. Condone sign slips.
	$e = \frac{5}{2}$	**A1**	
	$a \circ e = (k_1 e - 4)a + k_3 e + 10 = a$ $\Rightarrow$ "$\frac{5}{2}$"$k_1 - 4 = 1 \Rightarrow k_1 = \ldots,\ k_3 = \ldots$	**M1**	Uses the other identity condition with their e, equating coefficients, and solves for k_1 and k_3. Condone sign slips.
	$k_1 = 2,\ k_3 = -4$ $a \circ b = 2ab - 4a - 4b + 10$	**A1 AG**	Correct identity equations leading without error to $k_1 = 2$, $k_3 = -4$ and the printed operation. Not for checking the printed operation satisfies the conditions.
		[6]	
11(b)	$e = \frac{5}{2}$	**B1**	
		[1]	
11(c)	$b \circ a = 2ba - 4b - 4a + 10 = a \circ b$, so commutative	**E1**	States commutative, with $b \circ a$ shown equal to $a \circ b$ for all a, b, or the symmetry in a and b noted. Not for a numerical example alone.
		[1]	
11(d)	$a \circ b = 2(a-2)(b-2) + 2$	**M1**	Uses $a \circ b - 2 = 2(a-2)(b-2)$, or another valid method, to test whether $a \circ b = 2$ can occur. Not for numerical examples alone.
	$a, b \neq 2 \Rightarrow a \circ b - 2 \neq 0$, so closed	**A1**	Correct reason that $a \circ b \neq 2$ for all $a, b \in A$, and conclusion that A is closed.
	$e = \frac{5}{2} \in A$	**B1**	Identity $\frac{5}{2}$ stated to be in A, or $\frac{5}{2} \neq 2$. Part (a) may be quoted.
	$2(a-2)(b-2) + 2 = \frac{5}{2} \Rightarrow b = \ldots$	**M1**	Solves $a \circ b =$ their e for b, from either form of $a \circ b$. Condone sign slips.
	$a^{-1} = 2 + \frac{1}{4(a-2)} \neq 2$, so in A	**A1**	Correct inverse oe, such as $\frac{8a-15}{4(a-2)}$, with a reason it is in A.

Question 11 continued

Question	Scheme	Marks	Notes
	$(a \circ b) \circ c = 2(a \circ b - 2)(c - 2) + 2$ $a \circ (b \circ c) = 2(a - 2)(b \circ c - 2) + 2$	**M1**	Expands both $(a \circ b) \circ c$ and $a \circ (b \circ c)$ in terms of a, b, c, from either form of the operation. Condone slips.
	$(a \circ b) \circ c = 4(a - 2)(b - 2)(c - 2) + 2$ $= a \circ (b \circ c)$ Associative, so $(A, \circ)$ is a group	**A1**	cso. Both bracketings correct in any form and shown equal, with conclusions that $\circ$ is associative and $(A, \circ)$ is a group.
		[7]	
11(e)(i)	$a, b > 2 \Rightarrow a \circ b = 2(a - 2)(b - 2) + 2 > 2$ $e = \frac{5}{2} > 2$, inverse $2 + \frac{1}{4(a-2)} > 2$ Still a group, so no change	**E1**	States it is still a group, with reasons that $a \circ b > 2$, $\frac{5}{2} > 2$ and the inverse $2 + \frac{1}{4(a-2)} > 2$ for $a, b > 2$. Not for the conclusion alone.
		[1]	
11(e)(ii)	$e = \frac{5}{2} \notin \mathbb{Z}$, so no identity in A Not a group, so the answer changes	**E1**	States it is not a group because the identity $\frac{5}{2}$ is not an integer. Not for the conclusion alone.
		[1]	
11(e)(iii)	$0 \circ 0 = 10 \notin A$, so not closed Not a group, so the answer changes	**E1**	States it is not a group, with a failing axiom shown, such as a counterexample to closure, $a \circ b > 2$ for $a, b < 2$, or $\frac{5}{2} \notin A$.
		[1]	
	Total for Question 11	**18**	