OPTIONAL PURE / FURTHER CALCULUS

The t-Formulae

Worked solutions 10 questions · 93 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

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1. **(a)** Write down the t-formula for $\tan x$, where $t = \tan\left(\frac{x}{2}\right)$. **[1]**

(b) (i) Given that $\cos x = \frac{1-t^2}{1+t^2}$, use your answer to part (a) to show that

$$\sin x = \frac{2t}{1+t^2}$$ **[2]**

(ii) Given that $\tan\left(\frac{x}{2}\right) = \frac{1}{\sqrt{5}}$, find the exact values of $\tan x$ and $\sin x$. **[2]**

(c) Use the t-formulae to solve the equation

$$4\sin\theta + 3\cos\theta + 3 = 0$$

for $-180° < \theta \leq 180°$, giving your answers to 1 decimal place where appropriate. Explain why the substitution $t = \tan\left(\frac{\theta}{2}\right)$ does not give every solution of this equation. **[5]**

Solution

(a) Using the double angle formula for $\tan x$ with $t = \tan\left(\frac{x}{2}\right)$:

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1-t^2}$$

(b) (i) Since $\tan x = \frac{\sin x}{\cos x}$, we have $\sin x = \tan x \cos x$. Using part (a):

$$\begin{aligned}\sin x &= \frac{2t}{1-t^2} \times \frac{1-t^2}{1+t^2} \\ &= \frac{2t}{1+t^2}\end{aligned}$$

as required.

(ii) Here $t = \frac{1}{\sqrt{5}}$, so $t^2 = \frac{1}{5}$, giving $1 - t^2 = \frac{4}{5}$ and $1 + t^2 = \frac{6}{5}$. Then

$$\tan x = \frac{\frac{2}{\sqrt{5}}}{\frac{4}{5}} = \frac{2}{\sqrt{5}} \times \frac{5}{4} = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$$

$$\sin x = \frac{\frac{2}{\sqrt{5}}}{\frac{6}{5}} = \frac{2}{\sqrt{5}} \times \frac{5}{6} = \frac{5}{3\sqrt{5}} = \frac{\sqrt{5}}{3}$$

(c) With $t = \tan\left(\frac{\theta}{2}\right)$, the t-formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

Substituting these into the equation:

$$\frac{8t}{1+t^2} + \frac{3\left(1-t^2\right)}{1+t^2} + 3 = 0$$

Multiplying through by $1 + t^2$, which is never zero:

$$\begin{aligned} 8t + 3 - 3t^2 + 3 + 3t^2 &= 0 \\ 8t + 6 &= 0 \\ t &= -\frac{3}{4} \end{aligned}$$

Since $-180° < \theta \leq 180°$, we have $-90° < \frac{\theta}{2} \leq 90°$, and on $-90° < \frac{\theta}{2} < 90°$ each value of $\tan\left(\frac{\theta}{2}\right)$ occurs exactly once. So

$$\begin{aligned} \tan\left(\frac{\theta}{2}\right) = -\frac{3}{4} \quad \Longrightarrow \quad \frac{\theta}{2} &\approx -36.87° \\ \theta &\approx -73.7° \qquad (1 \text{ d.p.}) \end{aligned}$$

The substitution only covers values of θ for which $t = \tan\left(\frac{\theta}{2}\right)$ exists. At $\theta = 180°$, $\frac{\theta}{2} = 90°$ and $\tan 90°$ is undefined, so this value of θ cannot come from any value of t and must be checked separately. Substituting $\theta = 180°$ into the original equation:

$$4\sin 180° + 3\cos 180° + 3 = 4 \times 0 + 3 \times (-1) + 3 = 0$$

so $\theta = 180°$ is also a solution. The t^2 terms cancelling, leaving a linear rather than a quadratic equation in t, reflects this lost solution. The solutions are therefore $\theta \approx -73.7°$ and $\theta = 180°$, and the substitution misses $\theta = 180°$ because $t = \tan 90°$ is undefined there.

2. Consider the equation

$$7\sin x + \cos x = 5$$

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that the equation can be written as

$$3t^2 - 7t + 2 = 0 \qquad \textbf{[3]}$$

(b) Hence solve the equation for $0° \leq x \leq 720°$, giving your answers to 1 decimal place. Explain why no solutions of the equation in this interval are lost by using the substitution $t = \tan\left(\frac{x}{2}\right)$. **[5]**

Solution

(a) With $t = \tan\left(\frac{x}{2}\right)$, the t-formulae are

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Substituting into $7\sin x + \cos x = 5$:

$$7\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = 5$$

Multiplying both sides by $1 + t^2$:

$$14t + 1 - t^2 = 5 + 5t^2$$

$$6t^2 - 14t + 4 = 0$$

$$3t^2 - 7t + 2 = 0$$

as required.

(b) Using part (a), the quadratic factorises:

$$(3t-1)(t-2) = 0 \quad \Longrightarrow \quad t = \tfrac{1}{3} \text{ or } t = 2$$

Since $0° \leq x \leq 720°$, the half-angle satisfies $0° \leq \frac{x}{2} \leq 360°$, and tan repeats every $180°$.

When $\tan\left(\frac{x}{2}\right) = \frac{1}{3}$:

$$\frac{x}{2} = \arctan\tfrac{1}{3} \approx 18.43° \text{ or } 198.43°$$

$$x \approx 36.87° \text{ or } 396.87°$$

When $\tan\left(\frac{x}{2}\right) = 2$:

$$\frac{x}{2} = \arctan 2 \approx 63.43° \text{ or } 243.43°$$

$$x \approx 126.87° \text{ or } 486.87°$$

So the solutions are

$$x \approx 36.9°,\ 126.9°,\ 396.9°,\ 486.9° \qquad (1 \text{ d.p.})$$

The substitution can only lose values of x where $t = \tan\left(\frac{x}{2}\right)$ is undefined, that is where $\frac{x}{2} = 90°$ or $270°$, so $x = 180°$ or $540°$. At both of these, $\sin x = 0$ and $\cos x = -1$, so

$$7\sin x + \cos x = 7 \times 0 + (-1) = -1 \neq 5$$

Neither value satisfies the original equation, so no solutions in the interval are lost by using the substitution.

3. **(a)** Show that $\sin 2\theta + 3\cos 2\theta + 2$ can be written as

$$\frac{5 + 2t - t^2}{1 + t^2}$$

where $t = \tan\theta$. **[3]**

(b) Hence find the general solution of the equation

$$\sin 2\theta + 3\cos 2\theta + 2 = 3$$

giving your answers in exact form. **[6]**

Solution

(a) With $t = \tan\theta$, the t-formulae are

$$\sin 2\theta = \frac{2t}{1 + t^2}, \qquad \cos 2\theta = \frac{1 - t^2}{1 + t^2}$$

Substituting these and writing 2 over the same denominator:

$$\begin{aligned}\sin 2\theta + 3\cos 2\theta + 2 &= \frac{2t}{1 + t^2} + \frac{3(1 - t^2)}{1 + t^2} + \frac{2(1 + t^2)}{1 + t^2} \\ &= \frac{2t + 3 - 3t^2 + 2 + 2t^2}{1 + t^2} \\ &= \frac{5 + 2t - t^2}{1 + t^2}\end{aligned}$$

as required.

(b) The substitution $t = \tan\theta$ is not defined when $\theta = \frac{\pi}{2} + n\pi$, so these values are checked separately. For them, $2\theta = \pi + 2n\pi$, so $\sin 2\theta = 0$ and $\cos 2\theta = -1$, giving

$$\sin 2\theta + 3\cos 2\theta + 2 = 0 - 3 + 2 = -1 \neq 3$$

so none of these values is a solution.

For all other values of θ, part (a) turns the equation into

$$\begin{aligned}\frac{5 + 2t - t^2}{1 + t^2} &= 3 \\ 5 + 2t - t^2 &= 3 + 3t^2 \\ 4t^2 - 2t - 2 &= 0 \\ 2t^2 - t - 1 &= 0 \\ (2t + 1)(t - 1) &= 0\end{aligned}$$

so $t = 1$ or $t = -\frac{1}{2}$.

The tangent function has period π. When $\tan\theta = 1$, the principal value is $\frac{\pi}{4}$, so $\theta = n\pi + \frac{\pi}{4}$. When $\tan\theta = -\frac{1}{2}$, the principal value is $\arctan\left(-\frac{1}{2}\right) = -\arctan\frac{1}{2}$, so $\theta = n\pi - \arctan\frac{1}{2}$.

The general solution is therefore

$$\theta = n\pi + \frac{\pi}{4} \quad \text{or} \quad \theta = n\pi - \arctan\tfrac{1}{2}, \qquad n \in \mathbb{Z}$$

4. Throughout this question, $t = \tan\left(\frac{x}{2}\right)$, where $x \neq n\pi$, $n \in \mathbb{Z}$.

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\operatorname{cosec} x + \cot x \equiv \frac{1}{t} \qquad \textbf{[2]}$$

(b) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ and your answer to part (a) to prove that

$$\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x - \cot x} \equiv \frac{1 + \cos x}{1 - \cos x} \qquad \textbf{[3]}$$

(c) Hence solve, for $0 < x < 2\pi$, the equation

$$\operatorname{cosec} x + \cot x = 3(\operatorname{cosec} x - \cot x) \qquad \textbf{[2]}$$

Solution

(a) The t-formulae are

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

so

$$\begin{aligned}\operatorname{cosec} x + \cot x &= \frac{1}{\sin x} + \frac{\cos x}{\sin x} \\ &= \frac{1+t^2}{2t} + \frac{1-t^2}{2t} \\ &= \frac{2}{2t} \\ &= \frac{1}{t}\end{aligned}$$

as required.

(b) Using the same t-formulae:

$$\begin{aligned}\operatorname{cosec} x - \cot x &= \frac{1+t^2}{2t} - \frac{1-t^2}{2t} \\ &= \frac{2t^2}{2t} \\ &= t\end{aligned}$$

Using part (a), the left-hand side is

$$\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x - \cot x} = \frac{\frac{1}{t}}{t} = \frac{1}{t^2}$$

For the right-hand side:

$$1+\cos x = \frac{(1+t^2)+(1-t^2)}{1+t^2} = \frac{2}{1+t^2}$$

$$1-\cos x = \frac{(1+t^2)-(1-t^2)}{1+t^2} = \frac{2t^2}{1+t^2}$$

so

$$\frac{1+\cos x}{1-\cos x} = \frac{2}{1+t^2} \times \frac{1+t^2}{2t^2} = \frac{1}{t^2}$$

Both sides equal $\frac{1}{t^2}$, so

$$\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x - \cot x} \equiv \frac{1+\cos x}{1-\cos x}$$

as required.

(c) Since $x \neq n\pi$, $t = \tan\left(\frac{x}{2}\right) \neq 0$, so from part (b) $\operatorname{cosec} x - \cot x = t \neq 0$. Dividing both sides of the equation by $\operatorname{cosec} x - \cot x$:

$$\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x - \cot x} = 3$$

Using part (b):

$$\frac{1+\cos x}{1-\cos x} = 3$$

$$1+\cos x = 3 - 3\cos x$$

$$4\cos x = 2$$

$$\cos x = \frac{1}{2}$$

For $0 < x < 2\pi$, neither solution being a multiple of π:

$$x = \frac{\pi}{3}, \qquad x = \frac{5\pi}{3}$$

5. During 2031, the mean daily temperature in a city, $T\,^\circ\text{C}$, is modelled by the equation

$$T = 1.2\sin\left(\frac{x}{60}\right) - 6\cos\left(\frac{x}{60}\right) + 11 \qquad 0 \leq x < 365$$

where x is the number of days after 1st January 2031 and the angle is in radians.

(a) Find, according to the model, the mean temperature on 1st March 2031, giving your answer to 3 significant figures. **[1]**

(b) Use the substitution $t = \tan\left(\frac{x}{120}\right)$ to show that, for $x \neq 60\pi$, T can be written as

$$T = \frac{at^2 + bt + c}{1 + t^2}$$

where a, b and c are constants to be determined. **[3]**

(c) Hence find, according to the model, the date of the first day and the date of the last day in 2031 on which the mean temperature is at least $15\,^\circ\text{C}$. **[5]**

Solution

(a) 2031 is not a leap year, so 1st March 2031 is $31 + 28 = 59$ days after 1st January 2031, giving $x = 59$:

$$T = 1.2\sin\left(\frac{59}{60}\right) - 6\cos\left(\frac{59}{60}\right) + 11 \approx 8.67 \qquad (3\text{ s.f.})$$

The mean temperature on 1st March 2031 is $8.67\,^\circ\text{C}$.

(b) With $t = \tan\left(\frac{\theta}{2}\right)$, the t-formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

Take $\theta = \frac{x}{60}$, so that $\frac{\theta}{2} = \frac{x}{120}$ and $t = \tan\left(\frac{x}{120}\right)$. This is defined for $x \neq 60\pi$, since then $\frac{x}{120} \neq \frac{\pi}{2}$. Substituting:

$$\begin{aligned} T &= 1.2\left(\frac{2t}{1+t^2}\right) - 6\left(\frac{1-t^2}{1+t^2}\right) + 11 \\ &= \frac{2.4t - 6 + 6t^2 + 11 + 11t^2}{1+t^2} \\ &= \frac{17t^2 + 2.4t + 5}{1+t^2} \end{aligned}$$

So $a = 17$, $b = 2.4$ and $c = 5$.

(c) Using part (b), and since $1 + t^2 > 0$, the condition $T \geq 15$ for $x \neq 60\pi$ becomes

$$\begin{aligned} \frac{17t^2 + 2.4t + 5}{1+t^2} &\geq 15 \\ 17t^2 + 2.4t + 5 &\geq 15 + 15t^2 \\ 2t^2 + 2.4t - 10 &\geq 0 \\ t^2 + 1.2t - 5 &\geq 0 \end{aligned}$$

The roots of $t^2 + 1.2t - 5 = 0$ are

$$t = \frac{-1.2 \pm \sqrt{1.2^2 + 20}}{2} = \frac{-1.2 \pm \sqrt{21.44}}{2}$$

so $t \approx 1.7152$ or $t \approx -2.9152$. The coefficient of t^2 is positive, so the inequality holds for $t \leq -2.9152$ or $t \geq 1.7152$.

For $0 \leq x < 365$, $0 \leq \frac{x}{120} < 3.05 < \pi$.

On $0 \leq \frac{x}{120} < \frac{\pi}{2}$, $t = \tan\left(\frac{x}{120}\right)$ increases from 0, so $t \geq 1.7152$ gives

$$\arctan\left(\frac{-1.2 + \sqrt{21.44}}{2}\right) \leq \frac{x}{120} < \frac{\pi}{2} \quad \Longrightarrow \quad 125.15 \leq x < 60\pi$$

On $\frac{\pi}{2} < \frac{x}{120} < \pi$, t increases from large negative values to 0, and $\tan\left(\frac{x}{120}\right) = -\tan\left(\pi - \frac{x}{120}\right)$, so $t \leq -2.9152$ gives

$$\frac{\pi}{2} < \frac{x}{120} \leq \pi - \arctan\left(\frac{1.2 + \sqrt{21.44}}{2}\right) \quad \Longrightarrow \quad 60\pi < x \leq 228.15$$

At $x = 60\pi$, where t is undefined, the original equation gives

$$T = 1.2 \sin \pi - 6 \cos \pi + 11 = 17 \geq 15$$

So $T \geq 15$ for $125.15 \leq x \leq 228.15$, approximately. Since x counts whole days after 1st January, the mean temperature is at least 15 °C on the days $126 \leq x \leq 228$.

1st May is $x = 31+28+31+30 = 120$, so $x = 126$ is 7th May. 1st August is $x = 120+31+30+31 = 212$, so $x = 228$ is 17th August.

The first day is 7th May 2031 and the last day is 17th August 2031.

6. The depth of water, D metres, at the entrance to a harbour on a particular day is modelled by the equation

$$D = 6 + \frac{12}{5}\sin(30x)^\circ + \frac{16}{5}\cos(30x)^\circ$$

where x is the number of hours after midnight and $0 \leq x < 12$.

(a) Use the model to find the depth of water at the harbour entrance at 06:00. **[1]**

(b) Using the substitution $t = \tan(15x)^\circ$, show that the model can be written as

$$D = \frac{At^2 + Bt + C}{5\,(1 + t^2)}$$

where A, B and C are integers to be determined. **[3]**

(c) Hence find, according to the model, the two times between midnight and noon at which the depth of water at the harbour entrance is 4 m. Give your answers to the nearest minute. **[5]**

(d) Explain why the expression for D found in part (b) cannot be used to find the depth of water at 06:00. **[1]**

Solution

(a) At 06:00, $x = 6$, so $30x = 180$. Since $\sin 180^\circ = 0$ and $\cos 180^\circ = -1$:

$$D = 6 + 0 - \frac{16}{5} = \frac{14}{5} = 2.8$$

The depth of water at 06:00 is 2.8 m.

(b) With $t = \tan\left(\frac{\theta}{2}\right)$, the t-formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

With $\theta = (30x)^\circ$, we have $\frac{\theta}{2} = (15x)^\circ$, so $t = \tan(15x)^\circ$ and

$$\begin{aligned} D &= 6 + \frac{12}{5} \times \frac{2t}{1+t^2} + \frac{16}{5} \times \frac{1-t^2}{1+t^2} \\ &= \frac{30\left(1+t^2\right) + 24t + 16\left(1-t^2\right)}{5(1+t^2)} \\ &= \frac{30 + 30t^2 + 24t + 16 - 16t^2}{5(1+t^2)} \\ &= \frac{14t^2 + 24t + 46}{5(1+t^2)} \end{aligned}$$

This is in the required form with $A = 14$, $B = 24$, $C = 46$.

(c) The expression for D in terms of t is not defined at $x = 6$, but part (a) gives $D = 2.8 \neq 4$ there, so no solution is lost by using it. Setting $D = 4$ in the result of part (b):

$$\frac{14t^2 + 24t + 46}{5(1 + t^2)} = 4$$

$$14t^2 + 24t + 46 = 20 + 20t^2$$

$$6t^2 - 24t - 26 = 0$$

$$3t^2 - 12t - 13 = 0$$

Using the quadratic formula:

$$t = \frac{12 \pm \sqrt{144 + 156}}{6} = \frac{12 \pm 10\sqrt{3}}{6} = 2 \pm \frac{5\sqrt{3}}{3}$$

so $t \approx 4.8868$ or $t \approx -0.8868$.

Since $0 \leq x < 12$, the angle $15x$ satisfies $0 \leq 15x < 180$.

For $t = 2 + \frac{5\sqrt{3}}{3}$, $15x$ is acute:

$$15x = \arctan\left(2 + \tfrac{5\sqrt{3}}{3}\right) \approx 78.435 \quad \Longrightarrow \quad x \approx 5.2290$$

and $0.2290 \times 60 \approx 13.7$, so this is 5 hours 14 minutes after midnight.

For $t = 2 - \frac{5\sqrt{3}}{3} < 0$, $15x$ is obtuse:

$$15x = 180 - \arctan\left(\tfrac{5\sqrt{3}}{3} - 2\right) \approx 180 - 41.565 = 138.435 \quad \Longrightarrow \quad x \approx 9.2290$$

and $0.2290 \times 60 \approx 13.7$, so this is 9 hours 14 minutes after midnight.

The depth is 4 m at 05:14 and at 09:14.

(d) At 06:00, $x = 6$, so $t = \tan 90°$, which is undefined. Hence the expression for D in terms of t cannot be evaluated at 06:00.

7. **(a)** Use $t = \tan\left(\frac{\theta}{2}\right)$ to show that, where both sides are defined,

$$\frac{13 + 12\cos\theta + 5\sin\theta}{13 - 12\cos\theta - 5\sin\theta} \equiv \left(\frac{t+5}{5t-1}\right)^2 \qquad \textbf{[4]}$$

(b) The acute angle α is such that $\tan\alpha = \frac{1}{5}$.
Hence show that, where both sides are defined,

$$\frac{13 + 12\cos\theta + 5\sin\theta}{13 - 12\cos\theta - 5\sin\theta} \equiv \cot^2\left(\frac{\theta}{2} - \alpha\right) \qquad \textbf{[3]}$$

Solution

(a) With $t = \tan\left(\frac{\theta}{2}\right)$, the t-formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

Substitute these into the left-hand side, then multiply the numerator and the denominator by $1 + t^2$:

$$\begin{aligned}
\frac{13 + 12\cos\theta + 5\sin\theta}{13 - 12\cos\theta - 5\sin\theta} &= \frac{13 + \frac{12(1-t^2)}{1+t^2} + \frac{10t}{1+t^2}}{13 - \frac{12(1-t^2)}{1+t^2} - \frac{10t}{1+t^2}} \\
&= \frac{13(1+t^2) + 12(1-t^2) + 10t}{13(1+t^2) - 12(1-t^2) - 10t} \\
&= \frac{13 + 13t^2 + 12 - 12t^2 + 10t}{13 + 13t^2 - 12 + 12t^2 - 10t} \\
&= \frac{t^2 + 10t + 25}{25t^2 - 10t + 1} \\
&= \frac{(t+5)^2}{(5t-1)^2} \\
&= \left(\frac{t+5}{5t-1}\right)^2
\end{aligned}$$

as required.

(b) The addition formula for tangent is

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

With $A = \frac{\theta}{2}$ and $B = \alpha$, we have $\tan A = t$ and $\tan B = \frac{1}{5}$, so

$$\begin{aligned}
\tan\left(\frac{\theta}{2} - \alpha\right) &= \frac{t - \frac{1}{5}}{1 + \frac{1}{5}t} \\
&= \frac{5t - 1}{5 + t}
\end{aligned}$$

Taking the reciprocal gives

$$\cot\left(\frac{\theta}{2}-\alpha\right)=\frac{t+5}{5t-1}$$

Squaring and using part (a):

$$\frac{13+12\cos\theta+5\sin\theta}{13-12\cos\theta-5\sin\theta}\equiv\left(\frac{t+5}{5t-1}\right)^2\equiv\cot^2\left(\frac{\theta}{2}-\alpha\right)$$

as required.

8. Throughout this question, $t = \tan\left(\frac{x}{2}\right)$.

(a) Prove that

$$\sin x \equiv \frac{2t}{1+t^2} \qquad \textbf{[3]}$$

(b) You may use without proof the corresponding results for $\cos x$ and $\tan x$ in terms of t.
Show that the equation

$$4\tan x + 3\sin x + 6\cos x + 6 = 0$$

can be written as

$$t^3 - 6t^2 + 7t + 6 = 0 \qquad \textbf{[4]}$$

(c) Hence solve, for $-180° < x < 180°$, the equation

$$4\tan x + 3\sin x + 6\cos x + 6 = 0$$

giving each answer to 1 decimal place. **[6]**

Solution

(a) Using the double angle formula for sin, and $\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 1$:

$$\begin{aligned}\sin x &= 2\sin\left(\tfrac{x}{2}\right)\cos\left(\tfrac{x}{2}\right)\\ &= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)}\end{aligned}$$

Since $t = \tan\left(\frac{x}{2}\right)$ is defined, $\cos\left(\frac{x}{2}\right) \neq 0$. Dividing the numerator and denominator by $\cos^2\left(\frac{x}{2}\right)$:

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$$

as required.

(b) The t-formulae for $\cos x$ and $\tan x$ are

$$\cos x = \frac{1-t^2}{1+t^2}, \qquad \tan x = \frac{2t}{1-t^2}$$

Substituting these and the result of part (a) into the equation:

$$\frac{8t}{1-t^2} + \frac{6t}{1+t^2} + \frac{6(1-t^2)}{1+t^2} + 6 = 0$$

The last three terms combine over $1+t^2$:

$$\frac{6t + 6 - 6t^2 + 6 + 6t^2}{1+t^2} = \frac{6t+12}{1+t^2}$$

so the equation becomes

$$\frac{8t}{1-t^2} + \frac{6t+12}{1+t^2} = 0$$

Multiplying through by $(1-t^2)(1+t^2)$:

$$8t(1+t^2)+(6t+12)(1-t^2)=0$$
$$8t+8t^3+6t-6t^3+12-12t^2=0$$
$$2t^3-12t^2+14t+12=0$$
$$t^3-6t^2+7t+6=0$$

as required.

(c) Let $f(t)=t^3-6t^2+7t+6$. Then

$$f(3)=27-54+21+6=0$$

so $(t-3)$ is a factor, and

$$(t-3)(t^2-3t-2)=0$$

The quadratic factor gives

$$t=\frac{3\pm\sqrt{17}}{2}$$

so the solutions in t are $t=3$, $t=\frac{3+\sqrt{17}}{2}\approx 3.562$ and $t=\frac{3-\sqrt{17}}{2}\approx -0.562$.

For $-180°<x<180°$, $-90°<\frac{x}{2}<90°$, so each value of t gives exactly one value $\frac{x}{2}=\arctan t$. None of these values of t is ± 1, so $\tan x$ is defined at each.

When $t=3$:

$$\tfrac{x}{2}=\arctan 3\approx 71.565° \quad\Longrightarrow\quad x\approx 143.1° \qquad (1\text{ d.p.})$$

When $t=\frac{3+\sqrt{17}}{2}$:

$$\tfrac{x}{2}=\arctan\left(\frac{3+\sqrt{17}}{2}\right)\approx 74.317° \quad\Longrightarrow\quad x\approx 148.6° \qquad (1\text{ d.p.})$$

When $t=\frac{3-\sqrt{17}}{2}$:

$$\tfrac{x}{2}=\arctan\left(\frac{3-\sqrt{17}}{2}\right)\approx -29.317° \quad\Longrightarrow\quad x\approx -58.6° \qquad (1\text{ d.p.})$$

The solutions are $x\approx -58.6°$, $x\approx 143.1°$ and $x\approx 148.6°$.

9.

$$f(x) = \frac{1+\cos x}{3+2\sin x+\cos x}, \qquad 0 < x < 2\pi$$

Using the substitution $t = \tan\left(\frac{x}{2}\right)$,

(a) show that, for $x \neq \pi$, $f(x)$ can be written in the form

$$\frac{1}{(t+1)^2+1} \quad \textbf{[3]}$$

(b) Hence solve the equation $f(x) = \frac{1}{5}$, giving each answer exactly where possible and otherwise to 2 decimal places. **[4]**

(c) Find the greatest value of $f(x)$ and state the exact value of x at which it occurs. **[2]**

Solution

(a) With $t = \tan\left(\frac{x}{2}\right)$, the t-formulae are

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

The numerator becomes

$$1+\cos x = \frac{(1+t^2)+(1-t^2)}{1+t^2} = \frac{2}{1+t^2}$$

The denominator becomes

$$3+2\sin x+\cos x = \frac{3(1+t^2)+4t+(1-t^2)}{1+t^2}$$
$$= \frac{2t^2+4t+4}{1+t^2}$$

Dividing, the factors of $1+t^2$ cancel:

$$f(x) = \frac{2}{2t^2+4t+4}$$
$$= \frac{1}{t^2+2t+2}$$
$$= \frac{1}{(t+1)^2+1}$$

as required.

(b) First consider $x = \pi$, where the form in part (a) does not apply. Here

$$f(\pi) = \frac{1+(-1)}{3+0+(-1)} = 0 \neq \frac{1}{5}$$

so $x = \pi$ is not a solution. For $x \neq \pi$, using part (a):

$$\frac{1}{(t+1)^2+1} = \frac{1}{5}$$

$$(t+1)^2 = 4$$

$$t+1 = \pm 2 \quad \Longrightarrow \quad t = 1 \text{ or } t = -3$$

Since $0 < x < 2\pi$, the range for $\frac{x}{2}$ is $0 < \frac{x}{2} < \pi$.

When $\tan\left(\frac{x}{2}\right) = 1$:

$$\frac{x}{2} = \frac{\pi}{4} \quad \Longrightarrow \quad x = \frac{\pi}{2}$$

When $\tan\left(\frac{x}{2}\right) = -3$, the angle $\frac{x}{2}$ lies in the second quadrant, so

$$\frac{x}{2} = \pi - \arctan 3$$

$$x = 2\pi - 2\arctan 3 \approx 3.79 \qquad (2 \text{ d.p.})$$

The solutions are $x = \frac{\pi}{2}$ and $x = 2\pi - 2\arctan 3 \approx 3.79$.

(c) For $x \neq \pi$, $(t+1)^2 \geq 0$, so $(t+1)^2 + 1 \geq 1$, with equality when $t = -1$. Hence $f(x) \leq 1$, and since $f(\pi) = 0$, the greatest value of $f(x)$ is 1. It occurs when $\tan\left(\frac{x}{2}\right) = -1$ with $0 < \frac{x}{2} < \pi$, so $\frac{x}{2}$ lies in the second quadrant:

$$\frac{x}{2} = \frac{3\pi}{4} \quad \Longrightarrow \quad x = \frac{3\pi}{2}$$

The greatest value of $f(x)$ is 1, at $x = \frac{3\pi}{2}$.

10. Using the substitution $t = \tan\left(\frac{x}{2}\right)$, find the general solution, in radians, to the equation

$$4\sin x + \tan x = 3\cot\left(\frac{x}{2}\right)$$

You should determine whether any solutions of the equation are lost by using this substitution, and give your final answer in as simple a form as possible. **[11]**

Solution

With $t = \tan\left(\frac{x}{2}\right)$, the t-formulae are

$$\sin x = \frac{2t}{1+t^2}, \qquad \tan x = \frac{2t}{1-t^2}$$

and $\cot\left(\frac{x}{2}\right) = \frac{1}{t}$. Substituting these into the equation gives

$$\frac{8t}{1+t^2} + \frac{2t}{1-t^2} = \frac{3}{t}$$

Here $t \neq 0$, since $\cot\left(\frac{x}{2}\right)$ is undefined there, and $t \neq \pm 1$, since $\tan x$ is undefined there. Multiply both sides by $t(1+t^2)(1-t^2)$, using $(1+t^2)(1-t^2) = 1-t^4$:

$$8t^2(1-t^2) + 2t^2(1+t^2) = 3(1-t^4)$$

$$8t^2 - 8t^4 + 2t^2 + 2t^4 = 3 - 3t^4$$

$$3t^4 - 10t^2 + 3 = 0$$

$$(3t^2 - 1)(t^2 - 3) = 0$$

So $t^2 = \frac{1}{3}$ or $t^2 = 3$, giving $t = \pm\frac{1}{\sqrt{3}}$ or $t = \pm\sqrt{3}$. None of these is 0 or ± 1, so all are valid.

The tangent function has period π, so solve for $\frac{x}{2}$ and then double:

$$\tan\left(\frac{x}{2}\right) = \frac{1}{\sqrt{3}} \implies \frac{x}{2} = \frac{\pi}{6} + n\pi \implies x = \frac{\pi}{3} + 2n\pi$$

$$\tan\left(\frac{x}{2}\right) = -\frac{1}{\sqrt{3}} \implies \frac{x}{2} = -\frac{\pi}{6} + n\pi \implies x = -\frac{\pi}{3} + 2n\pi$$

$$\tan\left(\frac{x}{2}\right) = \sqrt{3} \implies \frac{x}{2} = \frac{\pi}{3} + n\pi \implies x = \frac{2\pi}{3} + 2n\pi$$

$$\tan\left(\frac{x}{2}\right) = -\sqrt{3} \implies \frac{x}{2} = -\frac{\pi}{3} + n\pi \implies x = -\frac{2\pi}{3} + 2n\pi$$

where $n \in \mathbb{Z}$. Since $-\frac{2\pi}{3} + 2n\pi = \frac{\pi}{3} + (2n-1)\pi$, the first and fourth families together are the odd and even multiples of π added to $\frac{\pi}{3}$, that is $x = \frac{\pi}{3} + n\pi$. Similarly $\frac{2\pi}{3} + 2n\pi = -\frac{\pi}{3} + (2n+1)\pi$, so the second and third families together are $x = -\frac{\pi}{3} + n\pi$. Hence these solutions are $x = n\pi \pm \frac{\pi}{3}$.

Lost solutions: the substitution $t = \tan\left(\frac{x}{2}\right)$ is undefined when $\frac{x}{2} = \frac{\pi}{2} + n\pi$, that is when $x = (2n+1)\pi$, so these values must be checked in the original equation. When $x = (2n+1)\pi$:

$$\sin x = 0, \qquad \tan x = 0, \qquad \cot\left(\frac{x}{2}\right) = \cot\left(\frac{\pi}{2} + n\pi\right) = 0$$

so the left-hand side is $4 \times 0 + 0 = 0$ and the right-hand side is $3 \times 0 = 0$. Therefore $x = (2n+1)\pi$ are solutions of the equation that are lost by the substitution.

The general solution, in radians, is

$$x = n\pi \pm \frac{\pi}{3} \quad \text{or} \quad x = (2n+1)\pi, \qquad n \in \mathbb{Z}$$