OPTIONAL PURE / FURTHER CALCULUS

The Weierstrass Substitution

Worked solutions 10 questions · 82 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

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1. Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int_0^{\frac{2}{3}\pi} \frac{1}{5+4\cos x}\,\mathrm{d}x = \frac{\pi}{9} \qquad \textbf{[6]}$$

Solution

With $t = \tan\left(\frac{x}{2}\right)$, differentiating gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}\left(1+t^2\right) \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formula for $\cos x$, then dividing the numerator and denominator by $\cos^2\left(\frac{x}{2}\right)$:

$$\cos x = \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)} = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$$

For the limits, when $x = 0$, $t = \tan 0 = 0$, and when $x = \frac{2}{3}\pi$, $t = \tan\frac{\pi}{3} = \sqrt{3}$.

The denominator becomes

$$5+4\cos x = 5 + \frac{4\left(1-t^2\right)}{1+t^2} = \frac{5+5t^2+4-4t^2}{1+t^2} = \frac{9+t^2}{1+t^2}$$

so the integral becomes

$$\begin{aligned}\int_0^{\frac{2}{3}\pi} \frac{1}{5+4\cos x}\,\mathrm{d}x &= \int_0^{\sqrt{3}} \frac{1+t^2}{9+t^2} \times \frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_0^{\sqrt{3}} \frac{2}{t^2+9}\,\mathrm{d}t\end{aligned}$$

The standard result used is

$$\int \frac{1}{t^2+a^2}\,\mathrm{d}t = \frac{1}{a}\arctan\left(\frac{t}{a}\right) + c$$

and with $a = 3$ this gives

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{2}{t^2+9}\,\mathrm{d}t &= \left[\frac{2}{3}\arctan\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}} \\ &= \frac{2}{3}\arctan\left(\frac{\sqrt{3}}{3}\right) - \frac{2}{3}\arctan 0 \\ &= \frac{2}{3}\arctan\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{2}{3} \times \frac{\pi}{6} \\ &= \frac{\pi}{9}\end{aligned}$$

as required.

2. The integral I is defined by

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x + 2\cos x}\,\mathrm{d}x$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of I, giving your answer in the form $p\ln q$, where p and q are rational numbers. **[6]**

Solution

With $t = \tan\left(\frac{x}{2}\right)$, differentiating gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1+t^2}{2} \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formulae and $\cos^2\left(\frac{x}{2}\right) = \frac{1}{\sec^2\left(\frac{x}{2}\right)} = \frac{1}{1+t^2}$:

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right)(1-t^2) = \frac{1-t^2}{1+t^2}$$

The limits become: when $x = 0$, $t = \tan 0 = 0$; when $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$.

The denominator becomes

$$1 + \frac{2t}{1+t^2} + \frac{2(1-t^2)}{1+t^2} = \frac{1+t^2+2t+2-2t^2}{1+t^2} = \frac{3+2t-t^2}{1+t^2}$$

so

$$I = \int_0^1 \frac{1+t^2}{3+2t-t^2} \times \frac{2}{1+t^2}\,\mathrm{d}t = \int_0^1 \frac{2}{3+2t-t^2}\,\mathrm{d}t = \int_0^1 \frac{2}{(3-t)(1+t)}\,\mathrm{d}t$$

Now use partial fractions:

$$\frac{2}{(3-t)(1+t)} = \frac{A}{3-t} + \frac{B}{1+t} \quad \Longrightarrow \quad 2 = A(1+t) + B(3-t)$$

Putting $t = 3$ gives $2 = 4A$, so $A = \frac{1}{2}$. Putting $t = -1$ gives $2 = 4B$, so $B = \frac{1}{2}$. Hence

$$I = \frac{1}{2}\int_0^1 \left(\frac{1}{3-t} + \frac{1}{1+t}\right)\mathrm{d}t$$

On $0 \le t \le 1$, both $3-t$ and $1+t$ are positive, so the logarithms need no modulus signs:

$$\begin{aligned} I &= \frac{1}{2}[\ln(1+t) - \ln(3-t)]_0^1 \\ &= \frac{1}{2}\Big((\ln 2 - \ln 2) - (\ln 1 - \ln 3)\Big) \\ &= \frac{1}{2}\ln 3 \end{aligned}$$

so the answer is in the form $p\ln q$ with $p = \frac{1}{2}$, $q = 3$.

3. **(a)** Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to find the integers a, b and c, with $b > 0$, such that

$$\int \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta = \int \frac{a}{(bt + c)^2 + 1}\,\mathrm{d}t \qquad \textbf{[3]}$$

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta \qquad \textbf{[4]}$$

Solution

(a) With $t = \tan\left(\frac{\theta}{2}\right)$, the t-formulae are

$$\sin\theta = \frac{2t}{1 + t^2}, \qquad \cos\theta = \frac{1 - t^2}{1 + t^2}$$

Differentiating $t = \tan\left(\frac{\theta}{2}\right)$ gives

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 + t^2\right) \quad \Longrightarrow \quad \mathrm{d}\theta = \frac{2}{1 + t^2}\,\mathrm{d}t$$

The denominator becomes

$$\begin{aligned} 3 - 2\sin\theta - \cos\theta &= \frac{3\left(1 + t^2\right) - 4t - \left(1 - t^2\right)}{1 + t^2} \\ &= \frac{4t^2 - 4t + 2}{1 + t^2} \end{aligned}$$

So the integral becomes

$$\begin{aligned} \int \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta &= \int \frac{1 + t^2}{4t^2 - 4t + 2} \times \frac{2}{1 + t^2}\,\mathrm{d}t \\ &= \int \frac{2}{4t^2 - 4t + 2}\,\mathrm{d}t \end{aligned}$$

Completing the square, $4t^2 - 4t + 2 = (2t - 1)^2 + 1$, so

$$\int \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta = \int \frac{2}{(2t - 1)^2 + 1}\,\mathrm{d}t$$

This is the required form with $a = 2$, $b = 2$, $c = -1$.

(b) When $\theta = 0$, $t = \tan 0 = 0$, and when $\theta = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$. Using part (a):

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta = \int_0^1 \frac{2}{(2t - 1)^2 + 1}\,\mathrm{d}t$$

Factoring out the scale factor:

$$\frac{2}{(2t - 1)^2 + 1} = \frac{2}{4\left(\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)} = \frac{1}{2} \times \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

The standard result used is

$$\int \frac{1}{x^2+k^2}\,\mathrm{d}x = \frac{1}{k}\arctan\left(\frac{x}{k}\right) + K$$

With $k = \frac{1}{2}$ and x replaced by $t - \frac{1}{2}$:

$$\begin{aligned}
\int_0^1 \frac{2}{(2t-1)^2+1}\,\mathrm{d}t &= \frac{1}{2}\left[2\arctan\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1 \\
&= \left[\arctan(2t-1)\right]_0^1 \\
&= \arctan 1 - \arctan(-1) \\
&= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

4. By using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, find the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{1}{3+5\cos\theta}\,\mathrm{d}\theta \qquad \textbf{[6]}$$

Solution

With $t = \tan\left(\frac{\theta}{2}\right)$, differentiating gives

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}(1+t^2) \quad \Longrightarrow \quad \mathrm{d}\theta = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formula for $\cos\theta$, and dividing top and bottom by $\cos^2\left(\frac{\theta}{2}\right)$:

$$\cos\theta = \frac{\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)+\sin^2\left(\frac{\theta}{2}\right)} = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$$

For the limits, when $\theta = 0$, $t = \tan 0 = 0$, and when $\theta = \frac{1}{2}\pi$, $t = \tan\frac{1}{4}\pi = 1$.

The denominator becomes

$$3+5\cos\theta = 3+\frac{5(1-t^2)}{1+t^2} = \frac{3+3t^2+5-5t^2}{1+t^2} = \frac{8-2t^2}{1+t^2}$$

so the integral becomes

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \frac{1}{3+5\cos\theta}\,\mathrm{d}\theta &= \int_0^1 \frac{1+t^2}{8-2t^2}\times\frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_0^1 \frac{2}{8-2t^2}\,\mathrm{d}t \\ &= \int_0^1 \frac{1}{4-t^2}\,\mathrm{d}t\end{aligned}$$

Now use partial fractions:

$$\frac{1}{(2-t)(2+t)} = \frac{A}{2-t}+\frac{B}{2+t} \quad \Longrightarrow \quad 1 = A(2+t)+B(2-t)$$

Putting $t = 2$ gives $A = \frac{1}{4}$, and putting $t = -2$ gives $B = \frac{1}{4}$.

Both $2-t$ and $2+t$ are positive for $0 \le t \le 1$, so the logarithms need no modulus signs:

$$\begin{aligned}\int_0^1 \frac{1}{4-t^2}\,\mathrm{d}t &= \frac{1}{4}\int_0^1 \left(\frac{1}{2-t}+\frac{1}{2+t}\right)\mathrm{d}t \\ &= \frac{1}{4}\left[-\ln(2-t)+\ln(2+t)\right]_0^1 \\ &= \frac{1}{4}\left[\ln\left(\frac{2+t}{2-t}\right)\right]_0^1 \\ &= \frac{1}{4}(\ln 3-\ln 1) \\ &= \frac{1}{4}\ln 3\end{aligned}$$

5. Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5 + 4\cos x + 3\sin x}\,\mathrm{d}x \qquad \textbf{[6]}$$

Solution

With $t = \tan\left(\frac{x}{2}\right)$, the t-formulae are

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Differentiating $t = \tan\left(\frac{x}{2}\right)$ gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right) = \frac{1+t^2}{2} \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

When $x = 0$, $t = \tan 0 = 0$. When $x = \frac{2\pi}{3}$, $t = \tan\frac{\pi}{3} = \sqrt{3}$.

Now rewrite the denominator in terms of t:

$$\begin{aligned} 5 + 4\cos x + 3\sin x &= 5 + \frac{4(1-t^2)}{1+t^2} + \frac{6t}{1+t^2} \\ &= \frac{5 + 5t^2 + 4 - 4t^2 + 6t}{1+t^2} \\ &= \frac{t^2 + 6t + 9}{1+t^2} \\ &= \frac{(t+3)^2}{1+t^2} \end{aligned}$$

So the integral becomes

$$\begin{aligned} \int_0^{\frac{2\pi}{3}} \frac{1}{5 + 4\cos x + 3\sin x}\,\mathrm{d}x &= \int_0^{\sqrt{3}} \frac{1+t^2}{(t+3)^2} \times \frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_0^{\sqrt{3}} \frac{2}{(t+3)^2}\,\mathrm{d}t \\ &= \left[-\frac{2}{t+3}\right]_0^{\sqrt{3}} \\ &= -\frac{2}{3+\sqrt{3}} + \frac{2}{3} \end{aligned}$$

Rationalising the denominator:

$$\frac{2}{3+\sqrt{3}} = \frac{2(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{2(3-\sqrt{3})}{6} = \frac{3-\sqrt{3}}{3}$$

Hence

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5 + 4\cos x + 3\sin x}\,\mathrm{d}x = \frac{2}{3} - \frac{3-\sqrt{3}}{3} = \frac{\sqrt{3}-1}{3}$$

6. Using the substitution $t = \tan\left(\frac{x}{2}\right)$, evaluate the integral

$$\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}x}{1+2\sin x}$$

giving your answer in the form $p\sqrt{3}\ln\left(q+\sqrt{3}\right)$, where p is a rational number and q is an integer, both to be determined. **[9]**

Solution

With $t = \tan\left(\frac{x}{2}\right)$, differentiating gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1+t^2}{2} \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formulae:

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2\tan\left(\frac{x}{2}\right)}{\sec^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$$

When $x = 0$, $t = \tan 0 = 0$, and when $x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

The denominator becomes

$$1+2\sin x = 1 + \frac{4t}{1+t^2} = \frac{t^2+4t+1}{1+t^2}$$

so the integral is

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}x}{1+2\sin x} &= \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{t^2+4t+1} \times \frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{t^2+4t+1}\,\mathrm{d}t \\ &= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(t+2)^2-3}\,\mathrm{d}t\end{aligned}$$

The standard result used is

$$\int \frac{1}{u^2-a^2}\,\mathrm{d}u = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c \qquad (|u| > a)$$

Applying this with $a = \sqrt{3}$ and u replaced by $t+2$: for $0 \le t \le \frac{1}{\sqrt{3}}$, $t+2 > \sqrt{3}$, so the fraction inside the logarithm is positive and

$$\begin{aligned}\int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(t+2)^2-3}\,\mathrm{d}t &= 2\left[\frac{1}{2\sqrt{3}}\ln\left(\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right)\right]_0^{\frac{1}{\sqrt{3}}} \\ &= \frac{1}{\sqrt{3}}\left[\ln\left(\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right)\right]_0^{\frac{1}{\sqrt{3}}}\end{aligned}$$

At the upper limit, multiplying the numerator and denominator by $\sqrt{3}$:

$$\frac{\frac{1}{\sqrt{3}}+2-\sqrt{3}}{\frac{1}{\sqrt{3}}+2+\sqrt{3}} = \frac{1+2\sqrt{3}-3}{1+2\sqrt{3}+3} = \frac{2\sqrt{3}-2}{4+2\sqrt{3}} = \frac{\sqrt{3}-1}{2+\sqrt{3}}$$

$$= \frac{(\sqrt{3}-1)(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2\sqrt{3}-3-2+\sqrt{3}}{4-3} = 3\sqrt{3}-5$$

At the lower limit:

$$\frac{2-\sqrt{3}}{2+\sqrt{3}} = \frac{(2-\sqrt{3})^2}{(2+\sqrt{3})(2-\sqrt{3})} = 7-4\sqrt{3}$$

Since $(7-4\sqrt{3})(7+4\sqrt{3}) = 49-48 = 1$, dividing by $7-4\sqrt{3}$ is the same as multiplying by $7+4\sqrt{3}$. Hence

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}x}{1+2\sin x} &= \frac{1}{\sqrt{3}}\Big(\ln\Big(3\sqrt{3}-5\Big) - \ln\Big(7-4\sqrt{3}\Big)\Big) \\
&= \frac{1}{\sqrt{3}}\ln\left(\frac{3\sqrt{3}-5}{7-4\sqrt{3}}\right) \\
&= \frac{1}{\sqrt{3}}\ln\Big((3\sqrt{3}-5)(7+4\sqrt{3})\Big) \\
&= \frac{1}{\sqrt{3}}\ln\Big(21\sqrt{3}+36-35-20\sqrt{3}\Big) \\
&= \frac{1}{\sqrt{3}}\ln\Big(1+\sqrt{3}\Big) \\
&= \frac{1}{3}\sqrt{3}\,\ln\Big(1+\sqrt{3}\Big)
\end{aligned}$$

so the answer is in the form $p\sqrt{3}\,\ln\big(q+\sqrt{3}\big)$ with $p=\frac{1}{3}$, $q=1$.

7. The integral I is defined by

$$I = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{2 - \cos x}\,\mathrm{d}x$$

(a) By first writing $\frac{\cos x}{2-\cos x}$ in the form $A + \frac{B}{2-\cos x}$, where A and B are constants, and then using the substitution $t = \tan\left(\frac{x}{2}\right)$, show that

$$I = -\frac{\pi}{2} + \int_0^1 \frac{4}{1+3t^2}\,\mathrm{d}t$$ **[5]**

(b) Hence find the exact value of I, giving your answer in the form $p\pi + q\pi\sqrt{3}$, where p and q are rational numbers. **[4]**

Solution

(a) Write the numerator as $\cos x = 2 - (2 - \cos x)$:

$$\frac{\cos x}{2-\cos x} = \frac{2-(2-\cos x)}{2-\cos x} = -1 + \frac{2}{2-\cos x}$$

So $A = -1$ and $B = 2$. Hence

$$I = \int_0^{\frac{1}{2}\pi} -1\,\mathrm{d}x + \int_0^{\frac{1}{2}\pi} \frac{2}{2-\cos x}\,\mathrm{d}x = -\frac{\pi}{2} + \int_0^{\frac{1}{2}\pi} \frac{2}{2-\cos x}\,\mathrm{d}x$$

Now use $t = \tan\left(\frac{x}{2}\right)$. Differentiating:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1+t^2) \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formula and dividing top and bottom by $\cos^2\left(\frac{x}{2}\right)$:

$$\cos x = \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$$

so

$$2 - \cos x = \frac{2(1+t^2) - (1-t^2)}{1+t^2} = \frac{1+3t^2}{1+t^2}$$

When $x = 0$, $t = \tan 0 = 0$, and when $x = \frac{1}{2}\pi$, $t = \tan\frac{1}{4}\pi = 1$. Therefore

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \frac{2}{2-\cos x}\,\mathrm{d}x &= \int_0^1 \frac{2(1+t^2)}{1+3t^2} \times \frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_0^1 \frac{4}{1+3t^2}\,\mathrm{d}t\end{aligned}$$

and so

$$I = -\frac{\pi}{2} + \int_0^1 \frac{4}{1+3t^2}\,\mathrm{d}t$$

as required.

(b) The standard result used is

$$\int \frac{1}{t^2+a^2}\,\mathrm{d}t = \frac{1}{a}\arctan\left(\frac{t}{a}\right) + c$$

Factoring out 3 from the denominator gives $a^2 = \frac{1}{3}$, so $a = \frac{1}{\sqrt{3}}$:

$$\begin{aligned}
\int_0^1 \frac{4}{1+3t^2}\,\mathrm{d}t &= \frac{4}{3}\int_0^1 \frac{1}{t^2+\frac{1}{3}}\,\mathrm{d}t \\
&= \frac{4}{3}\Big[\sqrt{3}\arctan\Big(\sqrt{3}\,t\Big)\Big]_0^1 \\
&= \frac{4\sqrt{3}}{3}\Big(\arctan\sqrt{3} - \arctan 0\Big) \\
&= \frac{4\sqrt{3}}{3}\times\frac{\pi}{3} \\
&= \frac{4}{9}\pi\sqrt{3}
\end{aligned}$$

Using part (a):

$$I = -\frac{1}{2}\pi + \frac{4}{9}\pi\sqrt{3}$$

so the answer is in the form $p\pi + q\pi\sqrt{3}$ with $p = -\frac{1}{2}$ and $q = \frac{4}{9}$.

8. The integral I is defined by

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x + \cos x} \, \mathrm{d}x$$

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \int_0^1 \frac{2t}{(1+t)\,(1+t^2)} \, \mathrm{d}t \qquad \textbf{[4]}$$

(b) Hence find the exact value of I, giving your answer in the form $a\pi + b\ln 2$, where a and b are rational. **[7]**

Solution

(a) With $t = \tan\left(\frac{x}{2}\right)$, differentiating gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}\left(1 + t^2\right) \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formulae, and $\cos^2\left(\frac{x}{2}\right) = \frac{1}{\sec^2\left(\frac{x}{2}\right)} = \frac{1}{1+t^2}$:

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right)\left(1 - t^2\right) = \frac{1-t^2}{1+t^2}$$

The denominator of the integrand becomes

$$1 + \sin x + \cos x = \frac{\left(1+t^2\right) + 2t + \left(1-t^2\right)}{1+t^2} = \frac{2(1+t)}{1+t^2}$$

so

$$\frac{\sin x}{1 + \sin x + \cos x} = \frac{2t}{1+t^2} \times \frac{1+t^2}{2(1+t)} = \frac{t}{1+t}$$

For the limits, when $x = 0$, $t = \tan 0 = 0$, and when $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$. Therefore

$$I = \int_0^1 \frac{t}{1+t} \times \frac{2}{1+t^2}\,\mathrm{d}t = \int_0^1 \frac{2t}{(1+t)(1+t^2)}\,\mathrm{d}t$$

as required.

(b) Write the integrand in partial fractions:

$$\frac{2t}{(1+t)(1+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2} \quad \Longrightarrow \quad 2t = A\left(1+t^2\right) + (Bt+C)(1+t)$$

Putting $t = -1$ gives $-2 = 2A$, so $A = -1$. Equating coefficients of t^2 gives $0 = A + B$, so $B = 1$. Equating constant terms gives $0 = A + C$, so $C = 1$. Hence

$$I = \int_0^1 \left(-\frac{1}{1+t} + \frac{t}{1+t^2} + \frac{1}{1+t^2}\right)\mathrm{d}t$$

The standard result used is

$$\int \frac{1}{t^2+k^2}\,\mathrm{d}t = \frac{1}{k}\arctan\left(\frac{t}{k}\right) + c$$

with $k = 1$, and $\int \frac{t}{1+t^2}\,\mathrm{d}t = \frac{1}{2}\ln\left(1+t^2\right) + c$. Both $1+t$ and $1+t^2$ are positive for $0 \le t \le 1$, so the logarithms need no modulus signs:

$$\begin{aligned} I &= \left[-\ln(1+t) + \frac{1}{2}\ln\left(1+t^2\right) + \arctan t\right]_0^1 \\ &= \left(-\ln 2 + \frac{1}{2}\ln 2 + \frac{\pi}{4}\right) - \left(-\ln 1 + \frac{1}{2}\ln 1 + \arctan 0\right) \\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2 \end{aligned}$$

so the answer is in the form $a\pi + b\ln 2$ with $a = \frac{1}{4}$, $b = -\frac{1}{2}$.

9. The integral I is defined by

$$I = \int_0^{\frac{1}{2}\pi} \frac{1-\cos\theta}{1+\sin\theta}\,\mathrm{d}\theta$$

(a) By using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, show that

$$I = \int_0^1 \frac{4t^2}{(1+t)^2(1+t^2)}\,\mathrm{d}t \qquad \textbf{[4]}$$

(b) Hence find the exact value of I, giving your answer in the form $a+b\ln 2$, where a and b are integers. **[7]**

Solution

(a) With $t = \tan\left(\frac{\theta}{2}\right)$, the t-formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

Differentiating $t = \tan\left(\frac{\theta}{2}\right)$ with respect to θ:

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}(1+t^2) \quad \Longrightarrow \quad \mathrm{d}\theta = \frac{2}{1+t^2}\,\mathrm{d}t$$

The numerator and denominator of the integrand become

$$1-\cos\theta = \frac{(1+t^2)-(1-t^2)}{1+t^2} = \frac{2t^2}{1+t^2}$$

$$1+\sin\theta = \frac{1+t^2+2t}{1+t^2} = \frac{(1+t)^2}{1+t^2}$$

so

$$\frac{1-\cos\theta}{1+\sin\theta} = \frac{2t^2}{1+t^2} \times \frac{1+t^2}{(1+t)^2} = \frac{2t^2}{(1+t)^2}$$

For the limits: when $\theta = 0$, $t = \tan 0 = 0$, and when $\theta = \frac{1}{2}\pi$, $t = \tan\frac{1}{4}\pi = 1$. Therefore

$$I = \int_0^1 \frac{2t^2}{(1+t)^2} \times \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int_0^1 \frac{4t^2}{(1+t)^2(1+t^2)}\,\mathrm{d}t$$

as required.

(b) Using part (a), write the integrand in partial fractions:

$$\frac{4t^2}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}$$

Multiplying through by $(1+t)^2(1+t^2)$ gives

$$4t^2 = A(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2$$

Substituting values of t and equating coefficients:

$$\begin{aligned} t=-1: &\quad 4=2B \quad\Longrightarrow\quad B=2 \\ t=0: &\quad 0=A+B+D \quad\Longrightarrow\quad A+D=-2 \\ t^3: &\quad 0=A+C \\ t^2: &\quad 4=A+B+2C+D \end{aligned}$$

The t^2 equation gives $4=-2+2+2C$, so $C=2$, and then $A=-2$ and $D=0$. Hence

$$I=\int_0^1\left(-\frac{2}{1+t}+\frac{2}{(1+t)^2}+\frac{2t}{1+t^2}\right)\mathrm{d}t$$

Since $1+t>0$ and $1+t^2>0$ for $0\le t\le 1$, the logarithms need no modulus signs:

$$\begin{aligned} I &= \left[-2\ln(1+t)-\frac{2}{1+t}+\ln\left(1+t^2\right)\right]_0^1 \\ &= (-2\ln 2-1+\ln 2)-(0-2+0) \\ &= 1-\ln 2 \end{aligned}$$

so the answer is in the form $a+b\ln 2$ with $a=1$, $b=-1$.

10. The integral I is defined by

$$I = \int_0^{\frac{1}{3}\pi} \frac{\sin x}{1+\sin x}\,\mathrm{d}x$$

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{4t}{(1+t)^2(1+t^2)}\,\mathrm{d}t \qquad \textbf{[4]}$$

(b) Express $\frac{4t}{(1+t)^2(1+t^2)}$ in partial fractions. **[3]**

(c) Hence show that $I = \frac{1}{3}\pi + 1 - \sqrt{3}$. **[4]**

Solution

(a) With $t = \tan\left(\frac{x}{2}\right)$, differentiating gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1+t^2) \quad \Longrightarrow \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Using the double angle formula for $\sin x$:

$$\begin{aligned}\sin x &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\\ &= 2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)\\ &= \frac{2\tan\left(\frac{x}{2}\right)}{\sec^2\left(\frac{x}{2}\right)}\\ &= \frac{2t}{1+t^2}\end{aligned}$$

Then

$$1+\sin x = \frac{1+t^2+2t}{1+t^2} = \frac{(1+t)^2}{1+t^2}$$

so

$$\frac{\sin x}{1+\sin x} = \frac{2t}{1+t^2} \times \frac{1+t^2}{(1+t)^2} = \frac{2t}{(1+t)^2}$$

When $x = 0$, $t = \tan 0 = 0$, and when $x = \frac{1}{3}\pi$, $t = \tan\frac{1}{6}\pi = \frac{1}{\sqrt{3}}$. Therefore

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{(1+t)^2} \times \frac{2}{1+t^2}\,\mathrm{d}t = \int_0^{\frac{1}{\sqrt{3}}} \frac{4t}{(1+t)^2(1+t^2)}\,\mathrm{d}t$$

as required.

(b) Since $1+t^2$ does not factorise, let

$$\frac{4t}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}$$

Multiplying through by $(1+t)^2(1+t^2)$:

$$4t = A(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2$$

Setting $t = -1$ gives $-4 = 2B$, so $B = -2$. Expanding the right-hand side:

$$4t = (A + C)t^3 + (A + B + 2C + D)t^2 + (A + C + 2D)t + (A + B + D)$$

Equating coefficients:

$$\begin{aligned} t^3: &\quad A + C = 0 \\ t^2: &\quad A + B + 2C + D = 0 \\ t^0: &\quad A + B + D = 0 \end{aligned}$$

Subtracting the third equation from the second gives $2C = 0$, so $C = 0$ and hence $A = 0$. Then $D = -A - B = 2$, so

$$\frac{4t}{(1+t)^2(1+t^2)} = \frac{2}{1+t^2} - \frac{2}{(1+t)^2}$$

(c) Using parts (a) and (b):

$$I = \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{2}{1+t^2} - \frac{2}{(1+t)^2} \right) \mathrm{d}t$$

The standard result used is

$$\int \frac{1}{t^2 + a^2}\, \mathrm{d}t = \frac{1}{a} \arctan\left(\frac{t}{a}\right) + c$$

with $a = 1$. Also $\int -2(1+t)^{-2}\, \mathrm{d}t = 2(1+t)^{-1} + c$, so

$$\begin{aligned} I &= \left[2\arctan t + \frac{2}{1+t} \right]_0^{\frac{1}{\sqrt{3}}} \\ &= 2\arctan\frac{1}{\sqrt{3}} + \frac{2}{1+\frac{1}{\sqrt{3}}} - (0+2) \\ &= 2 \times \frac{\pi}{6} + \frac{2\sqrt{3}}{\sqrt{3}+1} - 2 \\ &= \frac{\pi}{3} + \frac{2\sqrt{3}}{\sqrt{3}+1} - 2 \end{aligned}$$

Rationalising the denominator:

$$\frac{2\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{2\sqrt{3}(\sqrt{3}-1)}{3-1} = 3 - \sqrt{3}$$

so

$$I = \frac{\pi}{3} + 3 - \sqrt{3} - 2 = \frac{1}{3}\pi + 1 - \sqrt{3}$$

as required.