OPTIONAL PURE / FURTHER CALCULUS

Taylor Series

Worked solutions 15 questions · 132 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

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1. The function f is defined by $f(x) = \sqrt{x}$, $x > 0$.

(a) Find $f'(x)$, $f''(x)$ and $f'''(x)$. **[2]**

(b) Hence find the Taylor series for $f(x)$ about $x = 4$, in ascending powers of $(x-4)$, up to and including the term in $(x-4)^3$. **[3]**

(c) Use your answer to part (b), with a suitable value of x, to find an estimate for $\sqrt{5}$, giving your answer as a fraction in its simplest form. **[2]**

Solution

(a) Write $f(x) = x^{\frac{1}{2}}$ and differentiate repeatedly:

$$f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}}$$
$$f'''(x) = \tfrac{3}{8}x^{-\frac{5}{2}}$$

(b) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Here $a = 4$. Using part (a), with $4^{\frac{1}{2}} = 2$:

$$f(4) = \sqrt{4} = 2$$
$$f'(4) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$
$$f''(4) = -\tfrac{1}{4} \times \tfrac{1}{8} = -\tfrac{1}{32}$$
$$f'''(4) = \tfrac{3}{8} \times \tfrac{1}{32} = \tfrac{3}{256}$$

Substituting these values:

$$f(x) \approx 2 + (x-4) \times \frac{1}{4} + \frac{(x-4)^2}{2!} \times \left(-\frac{1}{32}\right) + \frac{(x-4)^3}{3!} \times \frac{3}{256}$$
$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

(c) Take $x = 5$, which is close to 4, so that $x - 4 = 1$:

$$\sqrt{5} \approx 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512}$$
$$= \frac{1024 + 128 - 8 + 1}{512}$$
$$= \frac{1145}{512}$$

Since $1145 = 5 \times 229$ is odd and $512 = 2^9$, this fraction is in its simplest form.

2. The function f is defined by $f(x) = \cosh 2x$ for $x \in \mathbb{R}$.

(a) Show that $\cosh(2\ln 2) = \frac{17}{8}$, and find the exact value of $\sinh(2\ln 2)$. **[2]**

(b) Hence, by differentiation, find the Taylor series expansion of $f(x)$ about $x = \ln 2$, up to and including the term in $(x - \ln 2)^3$, giving each coefficient as an exact fraction. **[4]**

Solution

(a) Using $\cosh x = \frac{e^x + e^{-x}}{2}$, with $e^{2\ln 2} = e^{\ln 4} = 4$ and $e^{-2\ln 2} = \frac{1}{4}$:

$$\cosh(2\ln 2) = \frac{4 + \frac{1}{4}}{2} = \frac{\frac{17}{4}}{2} = \frac{17}{8}$$

as required.

Similarly, using $\sinh x = \frac{e^x - e^{-x}}{2}$:

$$\sinh(2\ln 2) = \frac{4 - \frac{1}{4}}{2} = \frac{\frac{15}{4}}{2} = \frac{15}{8}$$

(b) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Differentiating $f(x) = \cosh 2x$ repeatedly:

$$f'(x) = 2\sinh 2x$$
$$f''(x) = 4\cosh 2x$$
$$f'''(x) = 8\sinh 2x$$

At $x = \ln 2$, $2x = 2\ln 2$, so using part (a):

$$f(\ln 2) = \frac{17}{8}$$
$$f'(\ln 2) = 2 \times \frac{15}{8} = \frac{15}{4}$$
$$f''(\ln 2) = 4 \times \frac{17}{8} = \frac{17}{2}$$
$$f'''(\ln 2) = 8 \times \frac{15}{8} = 15$$

Substituting into the Taylor series with $a = \ln 2$:

$$f(x) \approx \frac{17}{8} + (x - \ln 2) \times \frac{15}{4} + \frac{(x-\ln 2)^2}{2!} \times \frac{17}{2} + \frac{(x-\ln 2)^3}{3!} \times 15$$
$$= \frac{17}{8} + \frac{15}{4}(x - \ln 2) + \frac{17}{4}(x - \ln 2)^2 + \frac{5}{2}(x - \ln 2)^3$$

3. The function f is defined by $f(x) = \cos 2x$.

(a) By differentiating, find the Taylor series expansion of $f(x)$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$, giving each coefficient in exact form. **[4]**

(b) Use your answer to part (a) to obtain an estimate of $\cos 1$, giving your answer to 6 decimal places. **[2]**

Solution

(a) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Take $a = \frac{\pi}{6}$, so that $2a = \frac{\pi}{3}$. Differentiating and evaluating at $x = \frac{\pi}{6}$:

$$\begin{aligned} f(x) &= \cos 2x, & f\left(\tfrac{\pi}{6}\right) &= \cos \tfrac{\pi}{3} = \tfrac{1}{2} \\ f'(x) &= -2\sin 2x, & f'\left(\tfrac{\pi}{6}\right) &= -2 \times \tfrac{\sqrt{3}}{2} = -\sqrt{3} \\ f''(x) &= -4\cos 2x, & f''\left(\tfrac{\pi}{6}\right) &= -4 \times \tfrac{1}{2} = -2 \\ f'''(x) &= 8\sin 2x, & f'''\left(\tfrac{\pi}{6}\right) &= 8 \times \tfrac{\sqrt{3}}{2} = 4\sqrt{3} \end{aligned}$$

Substituting into the series:

$$\begin{aligned} \cos 2x &\approx \frac{1}{2} + \left(x - \frac{\pi}{6}\right) \times \left(-\sqrt{3}\right) + \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 \times (-2) + \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 \times 4\sqrt{3} \\ &= \frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{6}\right) - \left(x - \frac{\pi}{6}\right)^2 + \frac{2\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^3 \end{aligned}$$

(b) Since $\cos 1 = \cos\left(2 \times \frac{1}{2}\right) = f\left(\frac{1}{2}\right)$, substitute $x = \frac{1}{2}$ into the result of part (a), for which

$$x - \frac{\pi}{6} = \frac{1}{2} - \frac{\pi}{6} \approx -0.0235988$$

This gives

$$\begin{aligned} \cos 1 &\approx \frac{1}{2} - \sqrt{3}\left(\frac{1}{2} - \frac{\pi}{6}\right) - \left(\frac{1}{2} - \frac{\pi}{6}\right)^2 + \frac{2\sqrt{3}}{3}\left(\frac{1}{2} - \frac{\pi}{6}\right)^3 \\ &\approx 0.5 + 0.0408743 - 0.0005569 - 0.0000152 \\ &\approx 0.540302 \qquad (6 \text{ d.p.}) \end{aligned}$$

4. Given that $y = \ln(\sin x)$, where $0 < x < \pi$,

(a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\operatorname{cosec}^2 x \cot x$. **[5]**

(b) Hence find the Taylor series expansion of $\ln(\sin x)$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in exact form. **[3]**

Solution

(a) Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sin x} \times \cos x = \cot x$$

The standard results for differentiating $\cot x$ and $\operatorname{cosec} x$ are

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

so

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\operatorname{cosec}^2 x$$

Writing $-\operatorname{cosec}^2 x = -(\operatorname{cosec} x)^2$ and using the chain rule:

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -2\operatorname{cosec} x \times (-\operatorname{cosec} x \cot x) \\ &= 2\operatorname{cosec}^2 x \cot x\end{aligned}$$

as required.

(b) The Taylor series about $x = a$ is

$$y = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \cdots$$

with $a = \frac{\pi}{4}$. At $x = \frac{\pi}{4}$, $\sin x = \frac{1}{\sqrt{2}}$, $\cot x = 1$ and $\operatorname{cosec}^2 x = 2$. Using part (a):

$$\begin{aligned}y\left(\tfrac{\pi}{4}\right) &= \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2 \\ y'\left(\tfrac{\pi}{4}\right) &= 1 \\ y''\left(\tfrac{\pi}{4}\right) &= -2 \\ y'''\left(\tfrac{\pi}{4}\right) &= 2 \times 2 \times 1 = 4\end{aligned}$$

Substituting into the series:

$$\begin{aligned}\ln(\sin x) &\approx -\frac{1}{2}\ln 2 + \left(x - \frac{\pi}{4}\right) \times 1 + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 \times (-2) + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 \times 4 \\ &= -\frac{1}{2}\ln 2 + \left(x - \frac{\pi}{4}\right) - \left(x - \frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3\end{aligned}$$

5. A function is defined by

$$y = \ln(2 + \ln x) \qquad x > \frac{1}{e}$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{3 + \ln x}{x^2(2 + \ln x)^2} \quad \textbf{[4]}$$

(b) Hence, or otherwise, find the Taylor series expansion of y about $x = e$, in ascending powers of $(x - e)$, up to and including the term in $(x - e)^2$, giving each coefficient in exact simplest form. **[3]**

Solution

(a) Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2 + \ln x} \times \frac{1}{x} = \frac{1}{x(2 + \ln x)} = \big(x(2 + \ln x)\big)^{-1}$$

By the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x(2 + \ln x)\big) = (2 + \ln x) + x \times \frac{1}{x} = 3 + \ln x$$

Using the chain rule again:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\big(x(2 + \ln x)\big)^{-2} \times (3 + \ln x) \\ &= -\frac{3 + \ln x}{x^2(2 + \ln x)^2}\end{aligned}$$

as required.

(b) The Taylor series about $x = a$ is

$$y = y(a) + (x - a)y'(a) + \frac{(x - a)^2}{2!}y''(a) + \cdots$$

Take $a = e$, where $\ln e = 1$, so $2 + \ln e = 3$ and $3 + \ln e = 4$. Using part (a) and the first derivative found there:

$$\begin{aligned}y(e) &= \ln 3 \\ y'(e) &= \frac{1}{e \times 3} = \frac{1}{3e} \\ y''(e) &= -\frac{4}{e^2 \times 3^2} = -\frac{4}{9e^2}\end{aligned}$$

Substituting into the series:

$$\begin{aligned}y &\approx \ln 3 + (x - e) \times \frac{1}{3e} + \frac{(x - e)^2}{2!} \times \left(-\frac{4}{9e^2}\right) \\ &= \ln 3 + \frac{1}{3e}(x - e) - \frac{2}{9e^2}(x - e)^2\end{aligned}$$

6. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

(a) Use differentiation to determine the Taylor series expansion of $x \ln x$, in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$. **[4]**

(b) Hence find the exact value of

$$\lim_{x \to 1} \left(\frac{2x \ln x + 1 - x^2}{(x-1)^3} \right)$$ **[2]**

Solution

(a) Let $f(x) = x \ln x$. Differentiating, using the product rule for $f'(x)$:

$$f'(x) = \ln x + x \times \frac{1}{x} = \ln x + 1$$

$$f''(x) = \frac{1}{x}$$

$$f'''(x) = -\frac{1}{x^2}$$

Evaluating at $x = 1$, using $\ln 1 = 0$:

$$f(1) = 0, \qquad f'(1) = 1, \qquad f''(1) = 1, \qquad f'''(1) = -1$$

Substituting into the Taylor series with $a = 1$:

$$x \ln x \approx 0 + (x-1) \times 1 + \frac{(x-1)^2}{2!} \times 1 + \frac{(x-1)^3}{3!} \times (-1)$$

$$= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$$

(b) Multiplying the result of part (a) by 2:

$$2x \ln x = 2(x-1) + (x-1)^2 - \frac{1}{3}(x-1)^3 + \cdots$$

Writing $x = 1 + (x-1)$, the remaining terms of the numerator are

$$1 - x^2 = 1 - (1 + (x-1))^2$$

$$= 1 - 1 - 2(x-1) - (x-1)^2$$

$$= -2(x-1) - (x-1)^2$$

Adding, the terms in $(x-1)$ and $(x-1)^2$ cancel, so

$$2x \ln x + 1 - x^2 = -\frac{1}{3}(x-1)^3 + \cdots$$

where the omitted terms involve $(x-1)^4$ and higher powers. Dividing by $(x-1)^3$ gives

$$\frac{2x \ln x + 1 - x^2}{(x-1)^3} = -\frac{1}{3} + (\text{terms in positive powers of } (x-1))$$

As $x \to 1$, every term in a positive power of $(x-1)$ tends to 0, so

$$\lim_{x \to 1} \left(\frac{2x \ln x + 1 - x^2}{(x-1)^3} \right) = -\frac{1}{3}$$

7. **(a)** By writing $\ln x = \ln 3 + \ln\left(1 + \frac{x-3}{3}\right)$ and using the series expansion of $\ln(1+u)$, find the Taylor series of $\ln x$ about $x = 3$ in ascending powers of $(x-3)$, up to and including the term in $(x-3)^3$. State the set of values of x for which the expansion is valid. **[4]**

(b) Use your series to find an approximation to $\ln 2.7$, giving your answer in the form $\ln 3 - p$, where p is a fraction in its lowest terms. **[2]**

Solution

(a) Since $1 + \frac{x-3}{3} = \frac{x}{3}$, the given rearrangement is

$$\ln x = \ln 3 + \ln\left(\frac{x}{3}\right) = \ln 3 + \ln\left(1 + \frac{x-3}{3}\right)$$

The standard series, with its variable written as u, is

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots \qquad (-1 < u \leq 1)$$

Put $u = \frac{x-3}{3}$. Then

$$\frac{u^2}{2} = \frac{(x-3)^2}{2 \times 9} = \frac{(x-3)^2}{18}$$
$$\frac{u^3}{3} = \frac{(x-3)^3}{3 \times 27} = \frac{(x-3)^3}{81}$$

so

$$\ln x \approx \ln 3 + \frac{x-3}{3} - \frac{(x-3)^2}{18} + \frac{(x-3)^3}{81}$$

The expansion is valid when $-1 < u \leq 1$:

$$-1 < \frac{x-3}{3} \leq 1 \quad \Longrightarrow \quad -3 < x - 3 \leq 3 \quad \Longrightarrow \quad 0 < x \leq 6$$

(b) Take $x = 2.7$, which lies in $0 < x \leq 6$, so $x - 3 = -\frac{3}{10}$. The terms of the series are

$$\frac{x-3}{3} = -\frac{1}{10}$$
$$\frac{(x-3)^2}{18} = \frac{9}{100 \times 18} = \frac{1}{200}$$
$$\frac{(x-3)^3}{81} = -\frac{27}{1000 \times 81} = -\frac{1}{3000}$$

Substituting into the series from part (a):

$$\ln 2.7 \approx \ln 3 - \frac{1}{10} - \frac{1}{200} - \frac{1}{3000}$$
$$= \ln 3 - \frac{300 + 15 + 1}{3000}$$
$$= \ln 3 - \frac{316}{3000}$$
$$= \ln 3 - \frac{79}{750}$$

This is in the form $\ln 3 - p$ with $p = \frac{79}{750}$.

8. The function f is defined by $f(x) = \arctan x$.

(a) Using repeated differentiation, find the Taylor series expansion of $f(x)$ about $x = 1$, up to and including the term in $(x-1)^3$. **[6]**

(b) Use your series to find an approximation to $\arctan 1.2$ in the form $\frac{\pi}{4} + \frac{p}{q}$, where p and q are positive integers with no common factor. **[2]**

(c) Given that $\arctan 1.2 = 0.876058$ to 6 decimal places, use your answer to part (b) to find an estimate for π, giving your answer to 4 decimal places. **[2]**

Solution

(a) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Here $a = 1$. Differentiating repeatedly:

$$f(x) = \arctan x$$

$$f'(x) = \frac{1}{1+x^2} = \left(1+x^2\right)^{-1}$$

$$f''(x) = -2x\left(1+x^2\right)^{-2}$$

Using the product rule on $f''(x)$:

$$\begin{aligned} f'''(x) &= -2\left(1+x^2\right)^{-2} + (-2x) \times (-2)(2x)\left(1+x^2\right)^{-3} \\ &= \frac{-2\left(1+x^2\right) + 8x^2}{\left(1+x^2\right)^3} \\ &= \frac{6x^2 - 2}{\left(1+x^2\right)^3} \end{aligned}$$

Evaluating at $x = 1$:

$$f(1) = \frac{\pi}{4}, \qquad f'(1) = \frac{1}{2}, \qquad f''(1) = -\frac{2}{4} = -\frac{1}{2}, \qquad f'''(1) = \frac{4}{8} = \frac{1}{2}$$

Substituting into the series:

$$\begin{aligned} \arctan x &\approx \frac{\pi}{4} + (x-1) \times \frac{1}{2} + \frac{(x-1)^2}{2!} \times \left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!} \times \frac{1}{2} \\ &= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 \end{aligned}$$

(b) Put $x = 1.2$, so that $x - 1 = 0.2$:

$$\begin{aligned}\arctan 1.2 &\approx \frac{\pi}{4} + \frac{1}{2}(0.2) - \frac{1}{4}(0.2)^2 + \frac{1}{12}(0.2)^3 \\ &= \frac{\pi}{4} + 0.1 - 0.01 + \frac{0.008}{12} \\ &= \frac{\pi}{4} + \frac{135}{1500} + \frac{1}{1500} \\ &= \frac{\pi}{4} + \frac{136}{1500} \\ &= \frac{\pi}{4} + \frac{34}{375}\end{aligned}$$

Since $34 = 2 \times 17$ and $375 = 3 \times 5^3$ have no common factor, $p = 34$ and $q = 375$.

(c) Using part (b) with $\arctan 1.2 = 0.876058$:

$$\begin{aligned}\frac{\pi}{4} &\approx 0.876058 - \frac{34}{375} \\ &\approx 0.7853913\end{aligned}$$

so

$$\pi \approx 4 \times 0.7853913 \approx 3.1416 \qquad (4 \text{ d.p.})$$

9. It is given that

$$y = \ln\left(3 + e^{2x}\right)$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{12e^{2x}}{(3 + e^{2x})^2} \quad \textbf{[4]}$$

(b) Hence determine the Maclaurin series expansion for y, in ascending powers of x, up to and including the term in x^2, giving each term in simplest form. **[3]**

(c) Use the series expansion for $\cos x$ in ascending powers of x to determine, in simplest form, the first 2 non-zero terms in ascending powers of x of the series for $\cos 2x$. **[1]**

(d) Use your answers to parts (b) and (c) to find the exact value of

$$\lim_{x \to 0} \frac{\ln\left(3 + e^{2x}\right) - 2\ln 2 - \frac{1}{2}x}{1 - \cos 2x} \quad \textbf{[3]}$$

Solution

(a) Differentiating using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2e^{2x}}{3 + e^{2x}}$$

The quotient rule is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

with $u = 2e^{2x}$ and $v = 3 + e^{2x}$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 4e^{2x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 2e^{2x}$. Hence

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{4e^{2x}\left(3 + e^{2x}\right) - 2e^{2x} \times 2e^{2x}}{\left(3 + e^{2x}\right)^2} \\ &= \frac{12e^{2x} + 4e^{4x} - 4e^{4x}}{\left(3 + e^{2x}\right)^2} \\ &= \frac{12e^{2x}}{\left(3 + e^{2x}\right)^2}\end{aligned}$$

as required.

(b) Maclaurin's series is

$$y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \cdots$$

Using part (a) and the first derivative found there:

$$y(0) = \ln 4 = 2\ln 2$$

$$y'(0) = \frac{2}{3+1} = \frac{1}{2}$$

$$y''(0) = \frac{12}{(3+1)^2} = \frac{3}{4}$$

Substituting into the series:

$$y \approx 2\ln 2 + x \times \frac{1}{2} + \frac{x^2}{2!} \times \frac{3}{4}$$

$$= 2\ln 2 + \frac{1}{2}x + \frac{3}{8}x^2$$

(c) The series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \cdots \qquad \text{(for all } x\text{)}$$

Replacing x by $2x$:

$$\cos 2x \approx 1 - \frac{(2x)^2}{2!} = 1 - 2x^2$$

(d) Using part (b), the numerator is

$$\left(2\ln 2 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right) - 2\ln 2 - \frac{1}{2}x = \frac{3}{8}x^2 + (\text{terms in } x^3 \text{ and higher})$$

Using part (c), the denominator is

$$1 - \left(1 - 2x^2 + \cdots\right) = 2x^2 + (\text{terms in } x^4 \text{ and higher})$$

Dividing the numerator and denominator by x^2:

$$\lim_{x\to 0} \frac{\ln\left(3 + e^{2x}\right) - 2\ln 2 - \frac{1}{2}x}{1 - \cos 2x} = \lim_{x\to 0} \frac{\frac{3}{8} + (\text{terms in } x \text{ and higher})}{2 + (\text{terms in } x^2 \text{ and higher})}$$

$$= \frac{\frac{3}{8}}{2}$$

$$= \frac{3}{16}$$

10. It is given that $y = \tan x$.

(a) Show that

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = a\tan^4 x + b\tan^2 x + c$$

where a, b and c are integers to be determined. **[4]**

(b) Hence find the Taylor series expansion of $\tan x$ about $\frac{\pi}{6}$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$, up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$, giving each coefficient in exact simplest form. **[3]**

(c) Use your answer to part (b) to find an approximate value of $\tan 0.5$, giving your answer to four significant figures. Explain why this approximation would be expected to be accurate. **[3]**

Solution

(a) Differentiating $y = \tan x$ and using $\sec^2 x = 1 + \tan^2 x$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x = 1 + \tan^2 x$$

Differentiating again, by the chain rule:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\tan x \sec^2 x \\ &= 2\tan x\left(1 + \tan^2 x\right) \\ &= 2\tan x + 2\tan^3 x\end{aligned}$$

Differentiating once more:

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \left(2 + 6\tan^2 x\right)\sec^2 x \\ &= \left(2 + 6\tan^2 x\right)\left(1 + \tan^2 x\right) \\ &= 2 + 2\tan^2 x + 6\tan^2 x + 6\tan^4 x \\ &= 6\tan^4 x + 8\tan^2 x + 2\end{aligned}$$

This is in the required form with $a = 6$, $b = 8$, $c = 2$.

(b) As the question already uses a, the Taylor series about $x = \alpha$ is

$$y = y(\alpha) + (x - \alpha)y'(\alpha) + \frac{(x-\alpha)^2}{2!}y''(\alpha) + \frac{(x-\alpha)^3}{3!}y'''(\alpha) + \cdots$$

Take $\alpha = \frac{\pi}{6}$. At $x = \frac{\pi}{6}$, $\tan x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ and $\tan^2 x = \frac{1}{3}$. Using the derivatives from part (a):

$$y\left(\tfrac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$y'\left(\tfrac{\pi}{6}\right) = 1 + \frac{1}{3} = \frac{4}{3}$$

$$y''\left(\tfrac{\pi}{6}\right) = 2 \times \frac{\sqrt{3}}{3} \times \frac{4}{3} = \frac{8\sqrt{3}}{9}$$

$$y'''\left(\tfrac{\pi}{6}\right) = 6 \times \frac{1}{9} + 8 \times \frac{1}{3} + 2 = \frac{2}{3} + \frac{8}{3} + 2 = \frac{16}{3}$$

Substituting into the series:

$$\begin{aligned}\tan x &\approx \frac{\sqrt{3}}{3} + \left(x - \frac{\pi}{6}\right) \times \frac{4}{3} + \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 \times \frac{8\sqrt{3}}{9} + \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 \times \frac{16}{3} \\ &= \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3\end{aligned}$$

(c) Put $x = 0.5$ in the series from part (b), so that

$$x - \frac{\pi}{6} = 0.5 - \frac{\pi}{6} \approx -0.0235988$$

The four terms are then

$$\frac{\sqrt{3}}{3} \approx 0.5773503$$

$$\frac{4}{3}\left(0.5 - \frac{\pi}{6}\right) \approx -0.0314650$$

$$\frac{4\sqrt{3}}{9}\left(0.5 - \frac{\pi}{6}\right)^2 \approx 0.0004287$$

$$\frac{8}{9}\left(0.5 - \frac{\pi}{6}\right)^3 \approx -0.0000117$$

Adding them:

$$\tan 0.5 \approx 0.5463023 \approx 0.5463 \qquad (4 \text{ s.f.})$$

The approximation should be accurate because 0.5 is very close to $\frac{\pi}{6} \approx 0.5236$, so $\left|x - \frac{\pi}{6}\right| \approx 0.024$ is small. Higher powers of $\left(x - \frac{\pi}{6}\right)$ are then extremely small, so the terms left out of the series make a negligible difference to the value.

11. Given that $y = \sec^3 x$,

(a) show that

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 3\sec^3 x \tan x \left(p\sec^2 x + q\right)$$

where p and q are integers to be determined. **[5]**

(b) Hence find the Taylor series expansion of $\sec^3 x$ about $x = \frac{\pi}{4}$, in ascending powers of $\left(x - \frac{\pi}{4}\right)$, up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in exact simplest form. **[3]**

(c) Use your expansion to estimate the value of $\sec^3\left(\frac{\pi}{4} + 0.1\right)$, giving your answer to 3 decimal places. **[2]**

Solution

(a) Using the chain rule with $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sec^2 x \times \sec x \tan x = 3\sec^3 x \tan x$$

Differentiating again with the product rule, and using $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= 9\sec^2 x \times \sec x \tan x \times \tan x + 3\sec^3 x \sec^2 x \\ &= 9\sec^3 x \tan^2 x + 3\sec^5 x \\ &= 9\sec^3 x\left(\sec^2 x - 1\right) + 3\sec^5 x \\ &= 12\sec^5 x - 9\sec^3 x\end{aligned}$$

Differentiating once more:

$$\begin{aligned}\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} &= 60\sec^4 x \times \sec x \tan x - 27\sec^2 x \times \sec x \tan x \\ &= 60\sec^5 x \tan x - 27\sec^3 x \tan x \\ &= 3\sec^3 x \tan x\left(20\sec^2 x - 9\right)\end{aligned}$$

This is in the required form with $p = 20$ and $q = -9$.

(b) The Taylor series about $x = a$ is

$$y = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \cdots$$

At $x = \frac{\pi}{4}$, $\sec x = \sqrt{2}$, so $\sec^2 x = 2$, $\sec^3 x = 2\sqrt{2}$, $\sec^5 x = 4\sqrt{2}$, and $\tan x = 1$. Using part (a):

$$\begin{aligned}y\left(\tfrac{\pi}{4}\right) &= 2\sqrt{2} \\ y'\left(\tfrac{\pi}{4}\right) &= 3 \times 2\sqrt{2} \times 1 = 6\sqrt{2} \\ y''\left(\tfrac{\pi}{4}\right) &= 12 \times 4\sqrt{2} - 9 \times 2\sqrt{2} = 30\sqrt{2} \\ y'''\left(\tfrac{\pi}{4}\right) &= 3 \times 2\sqrt{2} \times 1 \times (20 \times 2 - 9) = 186\sqrt{2}\end{aligned}$$

Substituting into the series:

$$\sec^3 x \approx 2\sqrt{2} + \left(x - \frac{\pi}{4}\right) \times 6\sqrt{2} + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 \times 30\sqrt{2} + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 \times 186\sqrt{2}$$

$$= 2\sqrt{2} + 6\sqrt{2}\left(x - \frac{\pi}{4}\right) + 15\sqrt{2}\left(x - \frac{\pi}{4}\right)^2 + 31\sqrt{2}\left(x - \frac{\pi}{4}\right)^3$$

(c) Put $x - \frac{\pi}{4} = 0.1$ in the expansion from part (b):

$$\sec^3\left(\frac{\pi}{4} + 0.1\right) \approx \sqrt{2}(2 + 0.6 + 0.15 + 0.031)$$

$$= 2.781\sqrt{2}$$

$$\approx 3.933 \qquad \text{(3 d.p.)}$$

12. The function f is defined by $f(x) = \ln(1 + \tan x)$ for $0 \leq x < \frac{\pi}{2}$.

(a) Find the Taylor series expansion of $f(x)$ about $x = \frac{\pi}{4}$, in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in simplest form. **[7]**

(b) Hence show that

$$\ln\left(1 + \sqrt{3}\right) \approx \ln 2 + \frac{\pi}{12} + \frac{\pi^2}{A} + \frac{\pi^3}{B}$$

where A and B are integers to be determined. **[2]**

Solution

(a) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Here $a = \frac{\pi}{4}$, where $\tan\frac{\pi}{4} = 1$ and $\sec^2\frac{\pi}{4} = 2$, so

$$f\left(\tfrac{\pi}{4}\right) = \ln(1+1) = \ln 2$$

Differentiating with the chain rule, using $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$, gives

$$f'(x) = \frac{\sec^2 x}{1 + \tan x} \quad \Longrightarrow \quad f'\left(\tfrac{\pi}{4}\right) = \frac{2}{2} = 1$$

To find the higher derivatives, rearrange this as

$$(1 + \tan x)f'(x) = \sec^2 x$$

Differentiate with the product rule, using $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 x) = 2\sec^2 x \tan x$:

$$(1 + \tan x)f''(x) + \sec^2 x\, f'(x) = 2\sec^2 x \tan x$$

At $x = \frac{\pi}{4}$:

$$2f''\left(\tfrac{\pi}{4}\right) + 2 \times 1 = 2 \times 2 \times 1$$

$$f''\left(\tfrac{\pi}{4}\right) = 1$$

Differentiate again, using $\frac{\mathrm{d}}{\mathrm{d}x}(2\sec^2 x \tan x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$:

$$(1 + \tan x)f'''(x) + \sec^2 x\, f''(x) + \sec^2 x\, f''(x) + 2\sec^2 x \tan x\, f'(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$$(1 + \tan x)f'''(x) + 2\sec^2 x\, f''(x) + 2\sec^2 x \tan x\, f'(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

At $x = \frac{\pi}{4}$:

$$2f'''\left(\tfrac{\pi}{4}\right) + 2 \times 2 \times 1 + 2 \times 2 \times 1 \times 1 = 4 \times 2 \times 1 + 2 \times 4$$

$$2f'''\left(\tfrac{\pi}{4}\right) + 8 = 16$$

$$f'''\left(\tfrac{\pi}{4}\right) = 4$$

Substituting into the Taylor series:

$$f(x) \approx \ln 2 + \left(x - \frac{\pi}{4}\right) \times 1 + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 \times 1 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 \times 4$$
$$= \ln 2 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3$$

(b) Put $x = \frac{\pi}{3}$, which lies in $0 \leq x < \frac{\pi}{2}$. Then $\tan\frac{\pi}{3} = \sqrt{3}$ and $x - \frac{\pi}{4} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$, so part (a) gives

$$\ln\left(1 + \sqrt{3}\right) \approx \ln 2 + \frac{\pi}{12} + \frac{1}{2}\left(\frac{\pi}{12}\right)^2 + \frac{2}{3}\left(\frac{\pi}{12}\right)^3$$
$$= \ln 2 + \frac{\pi}{12} + \frac{\pi^2}{288} + \frac{2\pi^3}{5184}$$
$$= \ln 2 + \frac{\pi}{12} + \frac{\pi^2}{288} + \frac{\pi^3}{2592}$$

This is the required form with $A = 288$ and $B = 2592$.

13. **(a)** Using the substitution $t = x - \frac{\pi}{4}$ and the standard series for $\sin t$ and $\cos t$, show that the Taylor series of $\sin x$ about $x = \frac{\pi}{4}$, up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, is

$$\frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right] \quad \textbf{[3]}$$

(b) Hence find the value of the constant k for which the limit below exists, and find the value of this limit.

$$\lim_{x \to \frac{\pi}{4}} \frac{2\sqrt{2}\sin x + \cos 2x - 2 + k\left(x - \frac{\pi}{4}\right)^2}{\left(x - \frac{\pi}{4}\right)^3} \quad \textbf{[5]}$$

Solution

(a) Let $t = x - \frac{\pi}{4}$, so that $x = t + \frac{\pi}{4}$. Using the addition formulae:

$$\begin{aligned}\sin x &= \sin\left(t + \frac{\pi}{4}\right) \\ &= \sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4} \\ &= \frac{\sqrt{2}}{2}(\sin t + \cos t)\end{aligned}$$

The standard series are

$$\sin t = t - \frac{t^3}{3!} + \cdots, \qquad \cos t = 1 - \frac{t^2}{2!} + \cdots \qquad \text{(for all } t\text{)}$$

Keeping terms up to and including t^3:

$$\begin{aligned}\sin x &\approx \frac{\sqrt{2}}{2}\left[\left(t - \frac{1}{6}t^3\right) + \left(1 - \frac{1}{2}t^2\right)\right] \\ &= \frac{\sqrt{2}}{2}\left[1 + t - \frac{1}{2}t^2 - \frac{1}{6}t^3\right]\end{aligned}$$

Replacing t by $x - \frac{\pi}{4}$, the Taylor series of $\sin x$ about $x = \frac{\pi}{4}$ up to the term in $\left(x - \frac{\pi}{4}\right)^3$ is

$$\frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right]$$

as required.

(b) Again let $t = x - \frac{\pi}{4}$, so that $t \to 0$ as $x \to \frac{\pi}{4}$. In what follows, the dots stand for terms in t^4 and higher powers of t. Using part (a):

$$\begin{aligned}2\sqrt{2}\sin x &= 2\sqrt{2} \times \frac{\sqrt{2}}{2}\left[1 + t - \frac{1}{2}t^2 - \frac{1}{6}t^3\right] + \cdots \\ &= 2 + 2t - t^2 - \frac{1}{3}t^3 + \cdots\end{aligned}$$

For the cosine term, $2x = 2t + \frac{\pi}{2}$, so using the addition formulae:

$$\begin{aligned}\cos 2x &= \cos\left(2t + \frac{\pi}{2}\right) \\ &= \cos 2t \cos\frac{\pi}{2} - \sin 2t \sin\frac{\pi}{2} \\ &= -\sin 2t\end{aligned}$$

Using the series for $\sin t$ with t replaced by $2t$:

$$\begin{aligned}\cos 2x &= -\left(2t - \frac{(2t)^3}{3!} + \cdots\right) \\ &= -2t + \frac{4}{3}t^3 + \cdots\end{aligned}$$

The numerator is therefore

$$\begin{aligned}2\sqrt{2}\sin x + \cos 2x - 2 + kt^2 &= \left(2 + 2t - t^2 - \frac{1}{3}t^3\right) + \left(-2t + \frac{4}{3}t^3\right) - 2 + kt^2 + \cdots \\ &= (k-1)t^2 + t^3 + \cdots\end{aligned}$$

Dividing by t^3:

$$\frac{2\sqrt{2}\sin x + \cos 2x - 2 + kt^2}{t^3} = \frac{k-1}{t} + 1 + \cdots$$

where the remaining terms all contain positive powers of t, so they tend to 0 as $t \to 0$. If $k \neq 1$, then $\frac{k-1}{t}$ grows without bound as $t \to 0$, and the limit does not exist. So the limit exists only when $k - 1 = 0$, giving

$$k = 1$$

With $k = 1$, only the constant term survives as $t \to 0$, so

$$\lim_{x \to \frac{\pi}{4}} \frac{2\sqrt{2}\sin x + \cos 2x - 2 + \left(x - \frac{\pi}{4}\right)^2}{\left(x - \frac{\pi}{4}\right)^3} = 1$$

14. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

It is given that

$$y = x^x, \qquad x > 0$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^x\left[(1+\ln x)^2 + \frac{1}{x}\right] \quad \textbf{[4]}$$

(b) Hence find the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ when $x = 1$. **[3]**

(c) Find the Taylor series expansion of x^x about $x = 1$ in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$, giving each coefficient in simplest form. **[3]**

(d) Use your answer to part (c), explaining your reasoning clearly, to evaluate

$$\lim_{x\to 1}\frac{x^x - x^2 + x - 1}{(x-1)^2\ln x} \quad \textbf{[4]}$$

Solution

(a) Take natural logarithms of both sides:

$$\ln y = x\ln x$$

Differentiating implicitly with respect to x, using the product rule on the right-hand side:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + x\times\frac{1}{x} = 1 + \ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^x(1+\ln x)$$

Now differentiate again using the product rule with $u = x^x$ and $v = 1+\ln x$, so that $\frac{\mathrm{d}u}{\mathrm{d}x} = x^x(1+\ln x)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= x^x(1+\ln x)\times(1+\ln x) + x^x\times\frac{1}{x}\\ &= x^x\left[(1+\ln x)^2 + \frac{1}{x}\right]\end{aligned}$$

as required.

(b) Using part (a), write $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y\,g(x)$, where

$$g(x) = (1+\ln x)^2 + \frac{1}{x} \quad\Longrightarrow\quad g'(x) = \frac{2(1+\ln x)}{x} - \frac{1}{x^2}$$

Differentiating $y\,g(x)$ with the product rule gives

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}y}{\mathrm{d}x}\,g(x) + y\,g'(x)$$

When $x = 1$, $\ln 1 = 0$, so $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times (1 + 0) = 1$, $g(1) = 1 + 1 = 2$ and $g'(1) = 2 - 1 = 1$. Substituting these values:

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 1 \times 2 + 1 \times 1 = 3$$

(c) Let $f(x) = x^x$ and take $a = 1$. From parts (a) and (b):

$$f(1) = 1, \qquad f'(1) = 1, \qquad f''(1) = 1 \times \left[(1+0)^2 + 1\right] = 2, \qquad f'''(1) = 3$$

Substituting into the Taylor series about $x = 1$:

$$\begin{aligned} x^x &\approx 1 + (x-1) \times 1 + \frac{(x-1)^2}{2!} \times 2 + \frac{(x-1)^3}{3!} \times 3 \\ &= 1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3 \end{aligned}$$

(d) First write $x^2 - x + 1$ in powers of $(x-1)$, using $x = 1 + (x-1)$:

$$\begin{aligned} x^2 - x + 1 &= [1 + (x-1)]^2 - [1 + (x-1)] + 1 \\ &= 1 + 2(x-1) + (x-1)^2 - 1 - (x-1) + 1 \\ &= 1 + (x-1) + (x-1)^2 \end{aligned}$$

Using part (c), the numerator is

$$\begin{aligned} x^x - (x^2 - x + 1) &= \left[1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots\right] - \left[1 + (x-1) + (x-1)^2\right] \\ &= \frac{1}{2}(x-1)^3 + \text{terms in } (x-1)^4 \text{ and higher powers} \end{aligned}$$

For the denominator, the standard series is

$$\ln(1+t) = t - \frac{t^2}{2} + \cdots \qquad (-1 < t \leq 1)$$

With $t = x - 1$, which is small as $x \to 1$:

$$(x-1)^2 \ln x = (x-1)^2 \left[(x-1) - \frac{1}{2}(x-1)^2 + \cdots\right] = (x-1)^3 - \frac{1}{2}(x-1)^4 + \cdots$$

For $x \neq 1$, divide the numerator and denominator by $(x-1)^3$:

$$\frac{x^x - x^2 + x - 1}{(x-1)^2 \ln x} = \frac{\frac{1}{2} + \text{terms in } (x-1) \text{ and higher powers}}{1 - \frac{1}{2}(x-1) + \cdots}$$

As $x \to 1$, every term containing a positive power of $(x-1)$ tends to 0, so the numerator tends to $\frac{1}{2}$ and the denominator tends to 1. Therefore

$$\lim_{x \to 1} \frac{x^x - x^2 + x - 1}{(x-1)^2 \ln x} = \frac{1}{2}$$

15.

$$y = \ln(\cos x - \sin x) \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sec 2x - \tan 2x$$ **[3]**

(b) Find an expression for $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ in terms of $\sec 2x$ and $\tan 2x$. **[3]**

(c) Hence determine the first four non-zero terms, in ascending powers of x, of the Maclaurin series expansion of $\ln(\cos x - \sin x)$, giving each coefficient in simplest form. **[3]**

(d) Use the Maclaurin series expansion for $\ln(1+x)$ to write down the first three non-zero terms, in ascending powers of x, of the Maclaurin series expansion of $\ln(1+kx)$, where k is a non-zero constant. **[1]**

(e) Hence determine the value of k for which

$$\lim_{x\to 0}\left(\frac{1}{x^2}\ln\left((1+kx)(\cos x - \sin x)\right)\right)$$

exists, and find the value of this limit. **[4]**

Solution

(a) Differentiating using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x - \cos x}{\cos x - \sin x}$$

Multiply the numerator and denominator by $\cos x + \sin x$, then use $\sin^2 x + \cos^2 x = 1$ and the double angle formulae:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{(\cos x + \sin x)^2}{(\cos x - \sin x)(\cos x + \sin x)}\\ &= -\frac{\cos^2 x + 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x}\\ &= -\frac{1+\sin 2x}{\cos 2x}\\ &= -\sec 2x - \tan 2x\end{aligned}$$

as required.

(b) Use $\frac{\mathrm{d}}{\mathrm{d}x}(\sec 2x) = 2\sec 2x\tan 2x$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\tan 2x) = 2\sec^2 2x$, together with the chain and product rules. Differentiating the result of part (a):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\sec 2x\tan 2x - 2\sec^2 2x$$

Differentiating again:

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -2\left(2\sec 2x\tan^2 2x + 2\sec^3 2x\right) - 2\times 2\sec 2x\times 2\sec 2x\tan 2x\\ &= -4\sec^3 2x - 8\sec^2 2x\tan 2x - 4\sec 2x\tan^2 2x\end{aligned}$$

Differentiating each term once more:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(-4\sec^3 2x\right) &= -12\sec^2 2x \times 2\sec 2x\tan 2x \\ &= -24\sec^3 2x\tan 2x \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(-8\sec^2 2x\tan 2x\right) &= -8\left(4\sec^2 2x\tan^2 2x + 2\sec^4 2x\right) \\ &= -32\sec^2 2x\tan^2 2x - 16\sec^4 2x \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(-4\sec 2x\tan^2 2x\right) &= -4\left(2\sec 2x\tan^3 2x + 4\sec^3 2x\tan 2x\right) \\ &= -8\sec 2x\tan^3 2x - 16\sec^3 2x\tan 2x\end{aligned}$$

Adding these:

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -16\sec^4 2x - 40\sec^3 2x\tan 2x - 32\sec^2 2x\tan^2 2x - 8\sec 2x\tan^3 2x$$

(c) Maclaurin's series is

$$y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

When $x = 0$, $\sec 2x = 1$ and $\tan 2x = 0$, so using part (a) and part (b):

$$y(0) = \ln 1 = 0, \qquad y'(0) = -1, \qquad y''(0) = -2, \qquad y'''(0) = -4, \qquad y^{(4)}(0) = -16$$

Substituting into the series:

$$\begin{aligned}\ln(\cos x - \sin x) &\approx 0 + x \times (-1) + \frac{x^2}{2!} \times (-2) + \frac{x^3}{3!} \times (-4) + \frac{x^4}{4!} \times (-16) \\ &= -x - x^2 - \frac{2}{3}x^3 - \frac{2}{3}x^4\end{aligned}$$

(d) The Maclaurin series for $\ln(1 + x)$ is

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (-1 < x \leq 1)$$

Replacing x by kx:

$$\ln(1 + kx) \approx kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3}$$

(e) Using part (c) and part (d):

$$\begin{aligned}\ln((1 + kx)(\cos x - \sin x)) &= \ln(1 + kx) + \ln(\cos x - \sin x) \\ &= \left(kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3} - \cdots\right) + \left(-x - x^2 - \frac{2}{3}x^3 - \cdots\right) \\ &= (k - 1)x - \left(\frac{k^2}{2} + 1\right)x^2 + \frac{k^3 - 2}{3}x^3 + \cdots\end{aligned}$$

Dividing by x^2:

$$\frac{1}{x^2}\ln((1 + kx)(\cos x - \sin x)) = \frac{k - 1}{x} - \left(\frac{k^2}{2} + 1\right) + \frac{k^3 - 2}{3}x + \cdots$$

As $x \to 0$, the terms in x, x^2, … tend to 0, but $\frac{k-1}{x}$ grows without bound unless $k - 1 = 0$. So the limit exists only when $k = 1$, and then

$$\lim_{x \to 0} \left(\frac{1}{x^2} \ln((1 + x)(\cos x - \sin x)) \right) = -\left(\frac{1}{2} + 1 \right) = -\frac{3}{2}$$